Homework Problem Set #1

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Exercise 1 *For $n > 0$, define the functions $f_n \in L^2[0,\infty]$ by*

$$f_n(t) = \begin{cases} \exp\frac{t-n}{2}, & \text{for } n \le t \le \ n+\ln 2 \\ 0, & \text{otherwise.} \end{cases}$$

1. *Compute $||f_n - f_m||$. Does the sequence $\{f_n\}$ converge in the L^2 norm?*
2. *Show that $f_n(t)$ converges pointwise in $[0,\infty)$ and find the limit.*
3. *Does the sequence converge pointwise uniformly? Justify your answer.*
4. *Show that $\{f_n\}$ is ON. Is it a basis?*

Exercise 2 *Use Lemma 5, page 29 of the notes, and the functions $u(t) = 1$ and*

$$v(t) = \begin{cases} 1, & \text{for } 0 \le t \le \ \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} < t \le 1 \end{cases}$$

to show that the norm $||f|| = \int_0^1 |f(t)|\, dt$ cannot arise in the form $||f||^2 = (f,f)$ for some inner product on $L^1[0,1]$.

Exercise 3 *For $n > 0$, let*

$$f_n(t) = \begin{cases} nt, & \text{for } 0 \le t \le \frac{1}{n} \\ 1, & \text{for } \frac{1}{n} \le t \le \pi \\ n(\pi - t) + 1, & \text{for } \pi < t \le \pi + \frac{1}{n} \\ 0, & \text{for } \pi + \frac{1}{n} < t \le \ 2\pi. \end{cases}$$

This sequence belongs to $C[0,2\pi]$, i.e., the space of real-valued continuous functions on the interval $[0,2\pi]$ with the usual L^2 inner product and norm.

1. *Show that* $f_n \to \chi_{(0,\pi]}$ *in the* L^2 *norm, where*

$$\chi_{(0,\pi]}(t) = \begin{cases} 1, & \text{for } 0 < t \leq \pi \\ 0, & \text{for } \pi < t \leq \ 2\pi \end{cases}$$

   so that $\{f_n\}$ *is Cauchy in* L^2.

2. *Show that* $||\chi - h|| > 0$ *for every* $h \in C[0, 2\pi]$. *Conclude that* $C[0, 2\pi]$ *is not a Hilbert space.*

Exercise 4 *Show that the functions* $f_n(t) = \sin[(n + 1/2)t]$, $n = 0, 1, 2, \cdots$ *form an orthogonal sequence on* $L^2(0, \pi)$, *i.e., show that these nonzero functions satisfy* $(f_n, f_m) = 0$ *for* $n \neq m$. *Find the associated ON set.*

Exercise 5 *Show that the four functions* $\phi(t), \psi(t), \sqrt{2}\psi(2t), \sqrt{2}\psi(2t - 1)$, *where*

$$\phi(t) = \begin{cases} 1, & \text{for } 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \qquad \psi(t) = \begin{cases} 1, & \text{for } 0 \leq t < \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} \leq t < 1 \\ 0, & \text{otherwise,} \end{cases}$$

form an ON set in $L^2[0, 1]$. *Project the function* $f(t) = t^3$ *onto the subspace of* $L^2[0, 1]$ *spanned by these four functions. (This is related to the Haar wavelet expansion for* f.*)*

Exercise 6 *Let* $L^2[0, \infty, \omega(t)]$ *be the space of square integrable functions on the positive real line, with respect to the weight function* $\omega(t) = e^{-t}$. *The inner product on this space is thus*

$$(f, g) = \int_0^\infty f(t)\overline{g(t)}\omega(t)dt.$$

The Laguerre polynomials $L_n(t)$, $n = 0, 1, \cdots$ *are the ON set of polynomials on* $L^2[0, \infty, \omega(t)]$, *obtained by applying the Gram-Schmidt process to the monomials* $1, t, t^2, t^3, \cdots$ *and defined uniquely by the requirement that the coefficient of* t^n *in* $L_n(t)$ *is positive. (In fact they form an ON basis for* $L^2[0, \infty, \omega(t)]$.*) Compute the first 3 of these polynomials.*

Exercise 7 *Find the $L^2[0, 2\pi]$ projection of the function $f_1(t) = |t - \pi|$ onto the $(2n + 1)$-dimensional subspace spanned by the ON set*

$$\{\frac{1}{\sqrt{2\pi}}, \frac{\cos kt}{\sqrt{\pi}}, \frac{\sin kt}{\sqrt{\pi}} \ : \ k = 1, \cdots, n\}$$

for $n = 1$. Repeat for $n = 2, 3$. Plot these projections along with f_1. (You can use MATLAB, a computer algebra system, a calculator, etc.) repeat the whole exercise for $f_2(t) = t - \pi$. Do you see any marked differences between the graphs in the two cases?

Exercise 8 *This exercise makes clearer the connection between the projection theorem and least squares techniques in numerical analysis.*

a. *Consider the system of equations $Ta = b$ or*

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix} = b.$$

Find the least squares solution a to this problem. Compute $\tilde{b} = Ta$ and compare with b.

b. *The range or column space of T is the subspace $\mathcal{R}$ of V_4 spanned by the 3 column vectors of T:*

$$c_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Note that the equation $Ta = b$ says that $b = \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3$, which is not so, since b doesn't belong to the range in this example. Use the Gram-Schmidt process on c_1, c_2, c_3 to construct an ON basis e_1, e_2, e_3 for the range of T.

c. *Compute the projection of b on the range of T, in this case:*

$$\mathrm{proj}_{\mathcal{R}} b = (b, e_1)e_1 + (b, e_2)e_2 + (b, e_3)e_3.$$

where $(\cdot, \cdot)$ is the standard dot product for 4-tuples. Now compare $\tilde{b}$ with $\mathrm{proj}_{\mathcal{R}} b$

Exercise 9 *Use least squares to fit a parabola of the form* $y = ax^2 + bx + c$ *s to the data*

$$\begin{array}{rcl} x & = & 0 \quad 1 \quad 3 \quad 4 \\ y & = & 1 \quad 5 \quad 8 \quad 10 \end{array}$$

in order to estimate the value of y *when* $x = 2.0$.