# Homework Problem Set #2

Math 5467

February 15, 2006

**Exercise 1** *Find the Fourier series of the function $f$ on $[-\pi, \pi)$ given by*

$$f(t) = \begin{cases} 1, & \text{for } -1 \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

*Show that none of the even terms in the series are zero at $t = \pi/2$, although $f(\pi/2) = 0$, and $f$ vanishes in a neighborhood of this point. Does this contradict the localization theorem?*

**Exercise 2** *Expand $f(t) = |\sin t|$ in a Fourier series on the interval $-\pi \le t \le \pi$. Plot both $f$ and the partial sums*

$$S_k(t) = \frac{a_0}{2} + \sum_{n=0}^{k} (a_n \cos nt + b_n \sin nt)$$

*for $k = 1, 2, 5, 7$. Observe how the partial sums approximate $f$.*

**Exercise 3** *Let*

$$\Pi(t) = \begin{cases} 1 & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

*be the box function on the real line. We will often have the occasion to express the Fourier transform of $f(at+b)$ in terms of the Fourier transform $\hat{f}(\lambda)$ of $f(t)$, where $a, b$ are real parameters. This exercise will give you practice in correct application of the transform.*

1. *Sketch the graphs of $\Pi(t)$, $\Pi(t+4)$, and $\Pi(2t+4) = \Pi(2(t+2))$.*

2. *Sketch the graphs of* $\Pi(t)$, $\Pi(2t)$, *and* $\Pi(2(t+4))$. *Note: In the first part a left 4-translate is followed by a 2-dilate; but in the second part a 2-dilate is followed by a left 4-translate. The results are not the same.*

3. *Find the Fourier transforms of* $g_1(t) = \Pi(2(t+2))$ *and* $g_2(t) = \Pi(2(t+4))$ *from parts 1 and 2.*

4. *Set* $g(t) = \Pi(2t)$ *and check your answers to part 3 by applying the translation rule to*

$$g_1(t) = g(t+2), \qquad g_2(t) = g(t+4), \qquad \text{noting } \; g_2(t) = g_1(t+2).$$

**Exercise 4** *Let* $f$ *be the function on* $L^2(R)$ *defined by*

$$f(t) = \begin{cases} \cos(5t) & \text{for } -\pi \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$$

**a.** *Use the addition formulas for the cosine to show that*

$$\int_{-\pi}^{\pi} \cos(mt)\cos(\lambda t)\ dt = -2(-1)^m \frac{\lambda \sin(\pi\lambda)}{m^2 - \lambda^2}$$

*for* $m$ *an integer and* $\lambda \neq m$.

**b.** *Show that*

$$\hat{f}(\lambda) = -2\frac{\lambda \sin(\lambda\pi)}{\lambda^2 - 25}.$$

*Determine the decay rate of the transform as* $\lambda \to \infty$.

**c.** *Write down the Plancherel formula for this function.*

**Exercise 5** *Let* $g$ *be the function on* $L^2(R)$ *defined by*

$$g(t) = \begin{cases} \sin(5t) & \text{for } -\pi \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$$

**a.** *Compute the transform function* $\hat{g}(\lambda)$. *Determine the decay rate of the transform as* $\lambda \to \infty$. *Can you account for the difference in the decay rate for this problem as opposed to the rate in the last problem?*

**Exercise 6** *(Haar wavelets on [0,1]) Let $\phi(t)$ be the Haar scaling function*

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

*and*

$$\psi(t) = \phi(2t) - \phi(2t-1)$$

*be the Haar wavelet. Show directly that the system*

$$\{\phi(t), \quad 2^{n/2}\psi(2^n t - m); \quad n = 0, 1, 2, \cdots; \quad m = 0, 1, \cdots, 2^n - 1\}$$

*is an ON set in $L^2[0,1]$. (In fact, it is an ON basis for $L^2[0,1]$.).*

**Exercise 7** *Let $\phi(t)$ be the Haar scaling function defined above. Show that the set*

$$\phi_{m,n}(t) = \exp(2\pi imt)\phi(t-n), \quad m, n = 0, \pm 1, \pm 2 \cdots$$

*is an ON basis for $L^2(R)$. This basis was proposed by D. Gabor in 1946 for use in communications theory.*

**Exercise 8** *(Another Haar-type set) Let $n$ be a fixed positive integer and $h_k(t) = \sqrt{n}\phi(nt-k), \quad k = 0, 1, \cdots, n-1$. These are just rescaled and translated versions of the basic Haar scaling function.*

1. *Show that $\{h_0, \cdots, h_{n-1}\}$ is an ON set in $L^2[0,1]$.*
2. *Let $f(t)$ be a continuous function on $[0,1]$ and form the projection $f_n(t)$ on the subspace $S_n$ of $L^2[0,1]$ spanned by $\{h_0, \cdots, h_{n-1}\}$:*

   $$f_n = \sum_{k=0}^{n-1} (f, h_k) h_k.$$

   *Show that $f_n(t) \to f(t)$ pointwise in $t$ as $n \to \infty$.*
3. *For $f(t) = 1/(1+t^2)$, compute explicitly the Haar scaling function decomposition for $n = 4$ and $n = 8$. Plot the results.*