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Math 5467. Midterm II takehome problems

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Let $\phi(t)$ be the scaling function satisfying the dilation equation

$$\phi(t) = \sqrt{2}\sum_{k=0}^{3} \mathbf{c}[k]\phi(2t-k) \tag{1}$$

where the filter coefficients are those of the Daubechies 4-tap filter $db_2 = D_4$:

$$(\mathbf{c}[0], \cdots, \mathbf{c}[3]) = \frac{1}{4\sqrt{2}}\left((1+\sqrt{3}), (3+\sqrt{3}), (3-\sqrt{3}), (1-\sqrt{3})\right).$$

The support of the scaling function is contained in the interval $[0,3)$. For the problems to follow you can assume that $\phi(t)$ is continuous and that it satisfies the unit area condition $\int_{-\infty}^{\infty}\phi(x)\,dx = 1$ and the translation-orthogonality relations

$$\int_{-\infty}^{\infty}\phi(x)\phi(x-k)\,dx = \delta_{0k}.$$

(These facts will all be proved rigorously in this course.)

We define the moments μ_ℓ of the scaling function by

$$\mu_\ell = \int_{-\infty}^{\infty} x^\ell \phi(x)\,dx, \quad \ell = 0, 1, 2, \cdots.$$

Note that $\mu_0 = 1$. In the following problems you will compute μ_1 exactly and show how it relates to the expansion of polynomials in D_4 wavelets.

Problem 1 *Since $p = 2$ for D_4, it follows from results proved in the course notes that there are real constants A_k, B_k such that*

$$\sum_{k=-\infty}^{\infty} A_k\phi(t-k) = 1, \qquad \sum_{k=-\infty}^{\infty} B_k\phi(t-k) = t,$$

i.e., such that 1 *and* t *can be expressed as linear combinations of the integer translates of the scaling function. However, though we showed how to compute the* B_k *we didn't find an explicit expression for these coefficients. Use the unit area condition and the translation orthogonality alone to compute the following:*

a. *The coefficients* A_k.

b. *Find an explict expression for each of the coefficients* B_k *in terms of the moment* μ_1.

Problem 2 *The Fourier transform of the scaling function is*

$$\hat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(x) e^{-i\omega x}\, dx, \quad -\infty < \omega < \infty.$$

Verify that

$$\mu_\ell = (i)^k \frac{d^\ell}{d\omega^\ell} \hat{\phi}(0), \quad \ell = 0, 1, 2, \cdots.$$

Problem 3 *Take the Fourier transform of the dilation equation (1) for* D_4 *and show that in the frequency domain this equation takes the form*

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} C(\frac{\omega}{2}) \hat{\phi}(\frac{\omega}{2}), \tag{2}$$

where

$$C(\omega) = \sum_{k=0}^{3} c[k] e^{-i\omega k}.$$

Problem 4 *By differentiating both sides of equation (2) with respect to* ω *and then setting* $\omega = 0$, *show that*

$$\mu_1 = \frac{3 - \sqrt{3}}{2} \approx .63397.$$

Use this result to obtain the coefficients B_k *explicitly.*

Problem 5 *Compute the explicit expansion*

$$3t + 2 = \sum_{k=-\infty}^{\infty} C_k \phi(t - k),$$

i.e., determine the C_k *explicitly. For fixed* t *how many terms on the right-hand side of this equation are nonzero?*

Problem 6 *Describe qualitatively how you could compute all of the* μ_ℓ *recursively.*