Verifying Quantum Superintegrability

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Superintegrable Systems

In n dimensions, a classical or quantum Hamiltonian

$$\mathcal{H} = \sum_{i,j=0}^{n} g^{ij} p_i p_j + V(x_i), \qquad H = \hbar^2 \Delta + V(x_i, \hbar)$$

is (maximally) Superintegrable if it admits $2n-1$ symmetry operators, i.e.

$$\{\mathcal{L}_i, \mathcal{H}\} = 0, \quad \text{or} \quad [L_i, H] = 0, \quad \forall i = 0, \ldots, 2n-1$$

which are polynomial in the momenta or finite order differential operators. Symmetries chosen lowest order possible.

Superintegrable Systems 1

We also require that these operators be independent in a sense based on the functional independence of the classical symmetries. Maximum order of generating symmetries (other than Hamiltonian) $=$ order of superintegrable system.

Polynomial Symmetry Algebras

The generating symmetries of a superintegrable system generate an algebra under sums, products and Poisson bracket (or commutator) of symmetries, and typically this algebra closes at finite order:

- 1st order superintegrable systems: Lie algebra
- 2nd order superintegrable systems: quadratic algebra
- 3rd order superintegrable systems: cubic or quadratic algebra
- . . .

Hidden symmetry algebra

The structure of this hidden symmetry algebra provides important information about the classical trajectories, and in the quantum case its representation theory yields vital information about the multiplicities of the energy levels and the spectra of the symmetry operators. Functional realizations of of the quantum representations have deep connections with special functions and multivariable orthogonal polynomials, involving both differential and difference equations.

Classical structure and classification theory 1

- 1st order systems classified. Lie symmetry algebras. Helmholtz equation, free particle Schrödinger equation, etc.
- 2nd order systems. Lots of classification and structure results known. Complete classification for $n = 2$. Relation with separation of variables. (Kalnins, Miller, Kress, Pogosyan, Daskaloyannis).

Classical structure and classification theory 1

- 3rd order systems. Some classification theory for 2D Euclidean space with separation. (Winternitz, Gravel, Marquette, Tremblay). Some structure theory. (Miller, Kalnins, Post, Winternitz). Passage to quantum systems nontrivial and sometimes impossible.
- Higher order systems. Almost no structure theory except CCM- Stäckel transform. Recent breakthroughs in generating many examples (Evans et al., Winternitz and Turbiner et al., Kalnins et al.)

The quantization problem

Higher order superintegrable systems are rare, but we now see that they are also ubiquitous! How can we verify that quantum systems associated with these newly discovered classical systems are also quantum superintegrable? We must produce the higher order quantum symmetries, a highly nontrivial task!

The quantization problem 1

Three methods developed within the past year and a half:

- Use of a Dunkl algebra (Quesne)
- Canonical form for higher order symmetries (Miller et al.)
- Use of special function recurrence relations (Kalnins et al.)

Canonical form for a quantum symmetry

Start with a Schrödinger equation on a 2D real or complex Riemannian manifold with Laplace-Beltrami operator Δ_2 and potential V:

$$H\Psi \equiv (\Delta_2 + V)\Psi = E\Psi$$

that also admits separation in orthogonal coordinates $\{u_1, u_2\}$:

$$H = L_1 = \Delta_2 + V(u_1, u_2) = \frac{1}{f_1(u_1) + f_2(u_2)} \times$$

$$\left(\partial_{u_1}^2 + \partial_{u_2}^2 + v_1(u_1) + v_2(u_2)\right).$$

Canonical form 2

Due to the separability, there is a second-order symmetry operator

$$L_2 = \frac{f_2(u_2)}{f_1(u_1) + f_2(u_2)} \left(\partial_{u_1}^2 + v_1(u_1)\right)$$

$$- \frac{f_1(u_1)}{f_1(u_1) + f_2(u_2)} \left(\partial_{u_2}^2 + v_2(u_2)\right),$$

i.e., $[L_2, H] = 0$, and the operator identities

$$f_1(u_1)H + L_2 - v_1(u_1) = \partial_{u_1}^2, \quad f_2(u_2)H - L_2 - v_2(u_2) = \partial_{u_2}^2.$$

Canonical form 3

We look for an operator $\tilde{L}(H, L_2, u_1, u_2)$ that satisfies

$$[H, \tilde{L}] = 0.$$

Require that it take the standard form

$$\tilde{L} = \sum_{j,k} \left(A^{j,k}(u_1, u_2)\partial_{u_1 u_2} + B^{j,k}(u_1, u_2)\partial_{u_1} \right.$$

$$\left. + C^{j,k}(u_1, u_2)\partial_{u_2} + D^{j,k}(u_1, u_2) \right) H^j L_2^k.$$

Idea: Consider H, L_2 as parameters and only order terms when we wish to apply $\tilde{L}$ to a vector.

Canonical form 4

We can write

$$\tilde{L}(H, L_2, u_1, u_2) = A(u_1, u_2, H, L_2)\partial_{12} + B(u_1, u_2, H, L_2)\partial_1$$
$$+ C(u_1, u_2, H, L_2)\partial_2 + D(u_1, u_2, H, L_2),$$

and consider $\tilde{L}$ as an at most second-order order differential operator in u_1, u_2, polynomial in parameters H, L_2. Now we require $[H, \tilde{L}] = 0$.

Canonical form 5

Symmetry conditions take compact form

$$A_{u_1u_1}+A_{u_2u_2}+2B_{u_2}+2C_{u_1}=0,$$

$$B_{u_1u_1}+B_{u_2u_2}-2A_{u_2}v_2-Av_2'+(2A_{u_2}f_2+Af_2')H$$
$$-2A_{u_2}L_2=-2D_{u_1},$$

$$C_{u_1u_1}+C_{u_2u_2}-2A_{u_1}v_1-Av_1'+(2A_{u_1}f_1+Af_1')H$$
$$+2A_{u_1}L_2=-2D_{u_2},$$

$$D_{u_1u_1}+D_{u_2u_2}-2B_{u_1}v_1-2C_{u_2}v_2-Bv_1'-Cv_2'$$
$$+(2B_{u_1}f_1+2C_{u_2}f_2+Bf_1'+Cf_2')H+(2B_{u_1}-2C_{u_2})L_2=0.$$

Canonical form 6

Simplify by noting that there are two functions $F(u_1, u_2, H, L_2)$, $G(u_1, u_2, H, L_2)$ such first eqn, is satisfied by

$$A = F, \qquad B = -\frac{1}{2}\partial_2 F - \partial_1 G, \qquad C = -\frac{1}{2}\partial_1 F + \partial_2 G,$$

Canonical form 7

Integrability condition for next two eqns.:

$$2G_{1222}+\frac{1}{2}F_{2222}+2F_{22}(v_2-f_2H+L_2)+3F_2(v_2'-f_2'H)$$

$$+F(v_2''-f_2''H)=-2G_{1112}+\frac{1}{2}F_{1111}$$

$$+2F_{11}(v_1-f_1H-L_2)+3F_1(v_1'-f_1'H)+F(v_1''-f_1''H),$$

Canonical form 8

Last eqn. becomes

$$\frac{1}{2}F_{1112} + 2F_{12}(v_1 - f_1 H) + F_1(v_2' - f_2' H) + \frac{1}{2}G_{1111} +$$

$$2G_{11}(v_1 - f_1 H - L_2) + G_1(v_1' - f_1' H) =$$

$$-\frac{1}{2}F_{1222} - 2F_{12}(v_2 - f_2 H) - F_2(v_1' - f_1' H) + \frac{1}{2}G_{2222} +$$

$$2G_{22}(v_2 - f_2 H + L_2) + G_2(v_2' - f_2' H).$$

Canonical form 9

Any soln with A, B, C not identically 0 corresponds to a symmetry operator that does not commute with L_2, hence is algebraically independent of the symmetries H, L_2.

Canonical form 10

Strategy: Using the known symmetries of a classical superintegrable system as a guide, postulate a basis of functions $E_{ab}(u_1, u_2)$ such that

$$F = \sum_{a,b} A_{a,b} E_{a,b}(u_1, u_2), \;\; G = \sum_{a,b} B_{a,b} E_{a,b}(u_1, u_2),$$

and substitution into the symmetry equations above leads to a family of coupled recurrence relations for the coefficients $A_{a,b}, B_{a,b}$.

CanonicaL form 11

1. Determine a finite box $(a_0 + n, b_0 + n)$, $0 \le n \le n_0$, $0 \le n \le n_0$ for which we can show that setting $A_{a,b} = B_{a,b} = 0$ for all indices outside the box is consistent with the recurrence relations.
2. Verify that there exist nonzero solutions of the recurrences with support contained entirely within the box.
3. Any such nonzero solution determines a symmetry $\tilde{L}$, so that $H, L_1, \tilde{L}$ defines a superintegrable system.

Example: Caged anisotropic oscillator

Here,

$$H\Psi = (\partial_{11} + \partial_{22} + V(u_1, u_2))\Psi,$$

where

$$V(u_1, u_2) = \omega^2(p^2u_1^2 + q^2u_2^2) + \frac{\alpha_1}{u_1^2} + \frac{\alpha_2}{u_2^2},$$

in Cartesian coordinates. We take p, q to be relatively prime positive integers. Thus

$$f_1 = 1,\ f_2 = 0,\ v_1 = \omega^2 p^2 u_1^2 + \frac{\alpha_1}{u_1^2},\ v_2 = \omega^2 q^2 u_2^2 + \frac{\alpha_2}{u_2^2}.$$

Caged oscillator 1

The 2nd order symmetry operator is

$$L_2 = -\left(\partial_2^2 + v_2(u_2)\right),$$

This system is known to be classically superintegrable for all relatively integers, p, q. Based on expressions for the forms of the classical symmetries, we postulate expansions of F, G in finite series

$$F = \sum_{a,b} A_{a,b} E_{a,b}(u_1, u_2), G = \sum_{a,b} B_{a,b} E_{a,b}(u_1, u_2),$$

$$E_{a,b}(u_1, u_2) = u_1^a u_2^b.$$

caged oscillator 2

Taking coefficients with respect to the basis in each symmetry equation we find recurrences for the vector

$$C_{a,b} = \begin{pmatrix} A_{a,b} \\ B_{a,b} \end{pmatrix}$$

in matrix form

$$M_{a,b}C_{a,b}+M_{a-1,b+1}C_{a-1,b+1}+M_{a,b+2}C_{a,b+2}+M_{a,b+4}C_{a,b+4}$$

$$+M_{a+1,b-1}C_{a+1,b-1}+M_{a+1,b+1}C_{a+1,b+1}+M_{a+1,b+3}C_{a+1,b+3}+$$

$$M_{a+2,b}C_{a+2,b} + M_{a+3,b+1}C_{a+3,b+1} + M_{a+4,b}C_{a+4,b} = 0,$$

where each $M_{i,j}$ is a 2×2 matrix.

Caged oscillator 3

Example:

$$M_{a,b} = \begin{pmatrix} 2\omega^2 \left([b+1]q - [a+1]p\right)\left([b+1]q + [a+1]p\right) \\ 0 \\ 0 \\ -2\omega^2 \left(bq - ap\right)\left(bq + ap\right) \end{pmatrix},$$

Visualize this as a template that relates 10 distinct points:

Caged oscillator 4

Figure 1: The template. Points contributing to the recurrence relation are marked with large dots (●, ○).

Caged oscillator 5

We can assume, without loss of generality, that p, q are even. Then we can verify that the minimum lattice rectangle has vertices at (q, p), $(q, 0)$, $(0, 0)$, and $(0, p)$. The proof that there are nonzero solutions with support contain within the minimum lattice rectangle is complicated, but once the rectangle is determined, numerical examples are easy to obtain via ordinary Gaussian elimination. See computer demo.

Alternate proof for caged oscillator

Exploits separation of variables in Cartesian coordinates. We write the Hamiltonian as

$$H = \partial_x^2 + \partial_y^2 - \mu_1^2 x^2 - \mu_2^2 y^2 + \frac{\frac{1}{4} - a_1^2}{x^2} + \frac{\frac{1}{4} - a_2^2}{y^2}.$$

Look for separated eigenfunctions $\Psi = X(x)Y(y)$. Find solutions

$$X_n = e^{-\frac{1}{2}\mu_1 x^2} x^{a_1 + \frac{1}{2}} L_n^{a_1}(\mu_1 x^2), \quad Y_m = e^{-\frac{1}{2}\mu_2 y^2} y^{a_2 + \frac{1}{2}} L_m^{a_2}(\mu_2 y^2)$$

where the $L_n^{\alpha}(x)$ are associated Laguerre polynomials.

Alternate proof 1

Separation constants are

$$\lambda_x = -2\mu_1(2n + a_1 + 1), \quad \lambda_y = -2\mu_2(2m + a_2 + 1).$$

The total energy is

$$E = -2\mu(pn + qm + pa_1 + p + qa_2 + q).$$

where $\mu_1 = p\mu$ and $\mu_2 = q\mu$ and p and q are integers
Therefore, in order that E remain fixed we can admit values of integers m and n such that $pn + qm$ is a constant.

Alternate proof 2

One possibility that suggests itself is that $n \to n+q, m \to m-p$. To see that this is achievable via differential operators we need only consider

$$\Psi = L_n^{a_1}(z_1)L_m^{a_2}(z_2), \qquad z_1 = \mu_1 x^2,\ z_2 = \mu_2 y^2.$$

Note the recurrence formulas

$$x\frac{d}{dx}L_p^\alpha(x) = pL_p^\alpha(x) - (p+\alpha)L_{p-1}^\alpha(x)$$

$$= (p+1)L_{p+1}^\alpha(x) - (p+1+\alpha-x)L_p^\alpha(x).$$

Alternate proof 3

Because of the separation equations we can associate p with a differential operator for both m and n in the expression for Ψ. We do not change the coefficients of $L^{\alpha}_{p\pm1}(x)$. Therefore we can raise or lower the indices m and n in Ψ using differential operators. In particular we can perform the transformation $n \to n+q, m \to m-p$ and preserve the energy eigenvalue.

Alternate proof 4

Observe the formulas

$$D^+(\mu_1, x)X_n = (\partial_x^2 - 2x\mu_1\partial_x - \mu_1 + \mu_1^2 x^2 + \frac{\frac{1}{4} - \alpha_1^2}{x^2})X_n$$

$$= -4\mu_1(n+1)X_{n+1},$$

$$D^-(\mu_2, y)Y_m = (\partial_y^2 + 2y\mu_2\partial_y + \mu_2 + \mu_2^2 y^2 + \frac{\frac{1}{4} - \alpha_2^2}{y^2})Y_m$$

$$= -4\mu_2(m + \alpha_2)Y_{m-1}.$$

Alternate proof 5

In particular, if we make the choice $\mu_1 = 2\mu, \mu_2 = \mu$ then the operator $D^+(2\mu, x)D^-(\mu, y)^2$ transforms $X_n Y_m$ to $-128\mu^3(n+1)(m+\alpha_2)(m-1+\alpha_2)X_{n+1}Y_{m-2}$. This preserves the energy eigenspace. Thus can easily be extended to the case when $\mu_1 = p\mu$ and $\mu_2 = q\mu$ for p and q integers. A suitable operator is $D^+(p\mu, x)^q D^-(q\mu, y)^p$.

Alternate proof 6

Hence we have constructed a differential operator that commutes with H! The caged oscillator is quantum superintegrable. This works to prove superintegrability in all dimensions as we need only take coordinates pairwise.

Example: Quantum TTW system

This is the 2-D Euclidean space quantum system with potential

$$V(r\theta) = \alpha r^2 + \frac{\beta}{r^2\cos^2(k\theta)} + \frac{\gamma}{r^2\sin^2(k\theta)}$$

expressed in polar coordinates We show that it is quantum superintegrable for all rational k. In terms of our notation and separable coordinates $u_1 = R = \ln r$, $u_2 = \theta$, we have

$$v_1 = \alpha e^{4R}, \ \ v_2 = \frac{\beta}{\cos^2(k\theta)} + \frac{\gamma}{\sin^2(k\theta)}$$

TTW system 1

Based on the proof of classical superintegrability of the TTW system, we postulate expansions of F, G in finite series

$$F = \sum_{a,b,c} A_{a,b,c} E_{a,b,c}(R, \theta), \quad G = \sum_{a,b,c} B_{a,b,c} E_{a,b,c}(R, \theta),$$

where

$$E_{a,b,0} = e^{2aR} \sin^b(2k\theta), \quad E_{a,b,1} = e^{2aR} \sin^b(2k\theta) \cos(2k\theta).$$

TTW system 2

In this case we can show that for $k = p/q$ with p, q relatively prime, the minimum lattice rectangle has left column 0, right column q or $q + 1$ (depending on which is odd), bottom row $-p$ and top row 0. Again, it is easy to find many examples via Gaussian elimination, although the proof of guaranteed existence of a nonzero solution is complicated.

Deformed Kepler system

Deformed Kepler-Coulomb system in Euclidean space

$$\left(\partial_r^2+\frac{1}{r}+\frac{1}{r^2}\partial_\phi^2-\frac{E}{4r}+\frac{\beta}{4r^2\cos^2(k\phi/2)}+\frac{\gamma}{4r^2\sin^2(k\phi/2)}\right)$$

$$\times\Psi=-\frac{\alpha}{4}\Psi.$$

Classically superintegrable for rational k (Post and Winternitz). The Stäckel transform induced by the potential $U=r^2$ transforms the TTW system to this system; the transform preserves classical superintegrability.

Deformed Kepler system 1

The effect of the Stäckel transform is to permute the parameters: E goes to $-\alpha$ and α goes to $-E$. However, we see that replacement of the TTW potential by the deformed Kepler-Coulomb potential in the canonical symmetry equations above, just corresponds to the exact same permutation. The proof of quantum superintegrability of the TTW potential carries over immediately to the deformed Kepler-Coulomb system! However, the actual symmetry operators differ in the two cases.

Caged oscillator on the hyperboloid

Using this idea we can find a superintegrable system on the 2-sheet hyperboloid by taking a Stäckel transformation of the quantum caged oscillator by multiplying with the potential $U = 1/u_1^2$. The transformed system is

$$\left(u_1^2(\partial_1^2 + \partial_2^2) + \omega^2 u_1^2 (p^2 u_1^2 + q^2 u_2^2) - E u_1^2 + \alpha_2 \frac{u_1^2}{u_2^2} \right) \Psi$$

$$= -\alpha_1 \Psi.$$

Caged oscillator on hyperboloid 1

Embed this system as the upper sheet $s_0 > 0$ of the 2-sheet hyperboloid $s_0^2 - s_1^2 - s_2^2 = 1$ in 3-dimensional Minkowski space with Minkowski metric $ds^2 = -ds_0^2 + ds_1^2 + ds_2^2$, via the coordinate transformation

$$s_0 = \frac{1 + u_1^2 + u_2^2}{2u_1}, \quad s_1 = \frac{1 - u_1^2 - u_2^2}{2u_1}, \quad s_2 = \frac{u_2}{u_1}, \; u_1 > 0.$$

Caged oscillator on the hyperboloid 2

Potential for the transformed system is

$$\tilde{V} = \frac{\alpha_2}{s_2^2} - \frac{E}{(s_0+s_1)^2} + \frac{\omega^2(p^2-q^2)}{(s_0+s_1)^4} + \frac{\omega^2 q^2(s_0-s_1)}{(s_0+s_1)^3}.$$

System is classically superintegrable and from the canonical equations, it is also quantum superintegrable. The equations are identical except for the switches $\alpha_1 \to -E$, $H \to -\alpha_1$. Thus our proof of quantum superintegrability for the caged oscillator shows that the system on the hyperboloid is also quantum superintegrable.

Recurrence relation proofs

We are working on recurrence relation proofs of quantum superintegrability. They are based on the observation that the symmetry operators of a system must preserve the energy eigenspaces, so that they map a separable eigenfunction basis into a a linear combination of the eigenfunctions corresponding to the same energy. Thus they can be realized as differential recurrence relations for the eigenfunctions.