Two-variable Wilson polynomials and the generic superintegrable system on the 3-sphere

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Special Functions and Orthogonal Polynomials of Lie Groups and their Applications.
Decin

Outline

Superintegrable Systems

In n dimensions, we call a classical or quantum Hamiltonian

$$\mathcal{H} = \sum_{i=0}^{n} p_i^2 + V(x_i), \qquad H = -\Delta + V(x_i)$$

(maximally, Nth-order) **Superintegrable** if it admits $2n-1$ symmetry operators, ie.

$$\{\mathcal{L}_i, \mathcal{H}\} = 0, \qquad [L_i, H] = 0, \qquad \forall i = 0, \ldots, 2n-1,$$

$L_1 = H$ such that $L_2, \cdots, L_{2n-1}$ are polynomial, degree at most N, in the momenta or as differential operators.

We also require that these operators be independent either functionally or algebraically.

Superintegrable systems can be solved algebraically as well as analytically and are associated with special functions and exact solvability.

Integrability and Superintegrability

An integrable system has n algebraically independent symmetry operators in involution. A superintegrable system has $2n - 1$ algebraically independent symmetry operators (the maximum possible).

The symmetries of a merely integrable system generate an abelian algebra, those of a superintegrable system generate an algebra that is necessarily nonabelian.

Claim: Superintegrability captures what it means for a Hamiltonian system to be explicitly solvable.

Some simple but important superintegrable systems:

- Kepler-Coulomb problem: Kepler's 3 laws, Hohmann transfer for celestial navigation.
- hydrogen atom: periodic table of the elements
- classical and quantum harmonic oscillator

A 2D Example 1

Here $n = 2$, so $2n - 1 = 3$. Hamiltonian:

$$H = \partial_x^2 + \partial_y^2 - \alpha^2(4x^2 + y^2) + bx + \frac{\frac{1}{4} - c^2}{y}.$$

Generating symmetry operators: H, L_1, L_2 where

$$L_1 = \partial_x^2 - 4\alpha^2 x^2 - bx$$

$$L_2 = \frac{1}{2}\{M, \partial_y\} + y^2(\frac{b}{4} - x\alpha^2) - (\frac{1}{4} - c^2)\frac{x}{y^2}.$$

Here $M = x\partial_y - y\partial_x$ and $\{A, B\} = AB + BA$.

A 2D Example 2

Add the commutator $R = [L_1, L_2]$ to the symmetry algebra. Then

$$[L_1, R] = 2bH + 16\alpha^2 L_2 - 2bL_1$$

$$[L_2, R] = 8L_1 H - 6L_1^2 - 2H^2 + 2bL_2 - 8\alpha^2(1 - c^2)$$

$$R^2 + 4L_1^3 + 4L_1 H^2 - 8L_1^2 H + 16\alpha^2 L_2^2 + 4bL_2 H - 2b\{L_1, L_2\}$$
$$+16\alpha^2(3 - c^2)L_1 - 32\alpha^2 H - b^2(1 - c^2) = 0$$

A 2D Example 3

- We have a second order superintegrable system whose symmetry algebra closes at level 6.
- If Ψ is an eigenvector of H with eigenvalue E, i.e., $H\Psi = E\Psi$, and L is a symmetry operator then also $H(L\Psi) = E(L\Psi)$, so the symmetry algebra preserves eigenspaces of H.
- Thus we can use the irreducible representations of the symmetry algebra to explain the "accidental" degeneracies of the eigenspaces of H.

A 2D Example 4

By classifying the finite dimensional irreducible representations of the symmetry algebra we can show that the possible bound state energy levels must take the form

$$E_m = 4\alpha(m - 2c) + \frac{b^2}{16\alpha^2}$$

where $m = 0, 1, \cdots$ and the multiplicity of the energy level E_m is $m + 1$.

General Associated Symmetry Algebras

In general, the integrals of the motion generate algebra relations via taking commutators

$$[L_i, L_j] = R_{ij}$$

where R_{ij} are (usually new) symmetries of the Hamiltonian.

Often, if we include these new symmetries in our algebra, our algebra will close, i.e.

$$[L_i, R_{ij}] = P(L_i), \qquad P \text{ is a polynomial.}$$

Also, since $2n-1$ is maximal number of algebraically independent symmetries, there will be functional relations between the R_{ij}'s the L_i's.

These relations determine our symmetry algebra, usually not a Lie algebra.

The eigenspaces of the Hamiltonian (Schrödinger operator) are invariant under the action of the symmetry algebra, so the irreducible representations of the symmetry algebra enable us to determine the possible multiplicities of the eigenspaces and the spectrum.

Function Space Representations

How do special functions and orthogonal polynomials arise?

- By solving the original quantum eigenvalue problem for the Hamiltonian via separation of variables.
- By finding other function space representations of the symmetry algebra action and solving those.
- In either case the symmetry algebra structure yields important information about the special functions.

The quantum wave functions, along with the integrals, give a reducible representation of the algebra. We look for irreducible representations (so that the Hamiltonian is a constant) and look for operators which reproduce the structure equations.

Taking a guide from classical mechanics, where such models are guaranteed to exist, we look for operators on a Hilbert space with $n-1$ dimensions.

For the case of $n = 2$, all maximally superintegrable systems are known and their algebras close to form quadratic algebras. Each such algebra has been represented via function space realizations.

Plan of the remainder of the talk

- To introduce the superintegrable system on the n-sphere, which is in some sense the most general, and which admits a quadratic algebra structure.
- The representation theory of the 2-d system coincides with the theory of Wilson polynomials.
- The representation theory of the 3-d system coincides with the theory of a two variable generalization of the Wilson polynomials introduced by Tratnik [Tratnik 1991a, Tratnik 1991b].
- These representations give information about the original system, i.e. eigenvalues of symmetry operators and inter-basis expansion coefficients.
- The algebra gives the structure of recurrence operators which act on the Wilson polynomials.

The 2D System:Definition

For $n = 2$ we define the generic sphere system by embedding of the unit 2-sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in three dimensional flat space. Then the Hamiltonian operator is

$$H = \sum_{1 \le i < j \le 3} (x_i \partial_j - x_j \partial_i)^2 + \sum_{k=1}^{3} \frac{a_k}{x_k^2}, \quad \partial_i \equiv \partial_{x_i}.$$

The 3 operators that generate the symmetries are $L_1 = L_{12}, L_2 = L_{13}, L_3 = L_{23}$ where

$$L_{ij} \equiv L_{ji} = (x_i \partial_j - x_j \partial_i)^2 + \frac{a_i x_j^2}{x_i^2} + \frac{a_j x_i^2}{x_j^2},$$

for $1 \le i < j \le 4$. Here,

$$H = \sum_{1 \le i < j \le 3} L_{ij} + \sum_{k=1}^{3} a_k = H_0 + V, \quad V = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2}.$$

The 2D System:Algebra

For the generic 2-sphere quantum system the structure equations can be put in the symmetric form [Kalnins, Miller, and Post 2007]

$$\epsilon_{ijk}[L_{jk}, R] = 4\{L_{jk}, L_{ij}\} - 4\{L_{jk}, L_{ik}\} - (8 + 16a_j)L_{ik} + (8 + 16a_k)L_{ij} + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_{23}, L_{13}, L_{12}\} - (16a_1 + 12)L_{23}^2 - (16a_2 + 12)L_{13}^2 - (16a_3 + 12)L_{12}^2$$
$$+\frac{52}{3}(\{L_{23}, L_{13}\} + \{L_{13}, L_{12}\} + \{L_{12}, L_{23}\}) + \frac{1}{3}(16 + 176a_1)L_{23}$$
$$+\frac{1}{3}(16 + 176a_2)L_{13} + \frac{1}{3}(16 + 176a_3)L_{12} + \frac{32}{3}(a_1 + a_2 + a_3)$$
$$+48(a_1 a_2 + a_2 a_3 + a_3 a_1) + 64a_1 a_2 a_3.$$

Here ϵ_{ijk} is the pure skew-symmetric tensor, $R = [L_{23}, L_{13}]$ and $\{A, B\} = AB + BA$ with an analogous definition of $\{A, B, C\}$ as a symmetrized sum of 6 terms. Also, recall $L_3 = H - L_1 - L_2 - a_1 - a_2 - a_3$.

The Wilson Polynomials

Before we proceed to the model, we us present a basic overview of some of the characteristics of the Wilson polynomials [Wilson 1980]

$$w_n(t^2) \equiv w_n(t^2, \alpha, \beta, \gamma, \delta) = (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n \times$$

$$_4F_3\left(\begin{array}{llll} -n, & \alpha+\beta+\gamma+\delta+n-1, & \alpha-t, & \alpha+t \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta & \end{array} ; 1 \right)$$

$$= (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n \Phi_n^{(\alpha,\beta,\gamma,\delta)}(t^2).$$

The polynomial $w_n(t^2)$ is symmetric in $\alpha, \beta, \gamma, \delta$.
The Wilson polynomials are eigenfunctions of a divided difference operator given as

$$\tau^*\tau\Phi_n = n(n+\alpha+\beta+\gamma+\delta-1)\Phi_n$$

where

$$E^A F(t) = F(t+A), \ \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}),$$

$$\tau^* = \frac{1}{2t}\left[(\alpha+t)(\beta+t)(\gamma+t)(\delta+t)E^{1/2} - (\alpha-t)(\beta-t)(\gamma-t)(\delta-t)E^{-1/2}\right].$$

The 2D System:Representation via Wilson Polynomials

Using

$$a_1 = \frac{1}{4} - b_1^2,\ a_2 = \frac{1}{4} - b_2^2, \qquad a_3 = \frac{1}{4} - b_3^2,$$

and $\alpha = -\frac{b_1+b_2+1}{2} - \mu,\ \beta = \frac{b_1+b_2+1}{2},\ \gamma = \frac{b_2-b_1+1}{2},\ \delta = \frac{b_1+b_2-1}{2} + b + \mu + 2$, the algebra relations are realized by $H = E$ and

$$L_{12} = -4t^2 + b_1^2 + b_2^2,$$

$$L_{23} = -4\tau^*\tau - 2(b_2+1)(b_3+1) + \frac{1}{2},$$

$$E \equiv -\frac{4\mu + 2(b_1+b_2+b_3)+5)(4\mu + 2(b_1+b_2+b_3)+3}{4} + \frac{3}{2} - b_1^2 - b_2^2 - b_3^2.$$

The model realizes the algebra relations for arbitrary complex μ and restricts to a finite dimensional irreducible representation when $\mu = m \in \mathbb{N}$.

The 3D System:Definition

We define the Hamiltonian operator via the embedding of the unit 3-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ in four dimensional flat space.

$$H = \sum_{1 \le i < j \le 4} (x_i \partial_j - x_j \partial_i)^2 + \sum_{k=1}^{4} \frac{a_k}{x_k^2}, \quad \partial_i \equiv \partial_{x_i}.$$

A basis for the second order constants of the motion is

$$L_{ij} \equiv L_{ji} = (x_i \partial_j - x_j \partial_i)^2 + \frac{a_i x_j^2}{x_i^2} + \frac{a_j x_i^2}{x_j^2},$$

for $1 \le i < j \le 4$. Here,

$$H = \sum_{1 \le i < j \le 4} L_{ij} + \sum_{k=1}^{4} a_k.$$

The 3D System:Algebra

In the following i, j, k, ℓ are pairwise distinct integers such that $1 \leq i, j, k, \ell \leq 4$, and ϵ_{ijk} is the completely skew-symmetric tensor such that $\epsilon_{ijk} = 1$ if $i < j < k$.

Let us define $\mathcal{A}$ to be the algebra generated by the L_{ij} for all $i, j = 1, .., 4$ and $\mathcal{I}$, the identity. The structure relations for the algebra are as follows:

There are 4 linearly independent commutators of the second order symmetries (no sum on repeated indices) [Kalnins, Miller and Post 2011]:

$$R_\ell = \epsilon_{ijk}[L_{ij}, L_{jk}]$$

The fourth order structure equations are

$$[L_{ij}, R_j] = 4\epsilon_{i\ell k}(\{L_{ik}, L_{j\ell}\} - \{L_{i\ell}, L_{jk}\} + L_{i\ell} - L_{ik} + L_{jk} - L_{j\ell})$$

$$[L_{ij}, R_k] = 4\epsilon_{ij\ell}(\{L_{ij}, L_{i\ell} - L_{j\ell}\} + (2 + 4a_j)L_{i\ell} - (2 + 4a_i)L_{j\ell} + 2a_i - 2a_j).$$

The fifth order structure equations are obtainable directly from the fourth order equations and the Jacobi identity. The sixth order structure equations are

$$R_\ell^2 = \frac{8}{3}\{L_{ij}, L_{ik}, L_{jk}\} - (12+16a_k)L_{ij}^2 - (12+16a_i)L_{jk}^2 - (12+16a_j)L_{ik}^2$$

$$+\frac{52}{3}(\{L_{ij}, L_{ik}+L_{jk}\} + \{L_{ik}, L_{jk}\}) + (\frac{16}{3} + \frac{176}{3}a_k)L_{ij} + (\frac{16}{3} + \frac{176}{3}a_i)L_{jk}$$

$$+(\frac{16}{3} + \frac{176}{3}a_j)L_{ij} + 64a_ia_ja_k + 48(a_ia_j + a_ja_k + a_ka_i) + \frac{32}{3}(a_i + a_j + a_k),$$

$$\frac{\epsilon_{ik\ell}\epsilon_{jk\ell}}{2}\{R_i, R_j\} = \frac{4}{3}(\{L_{i\ell}, L_{jk}, L_{k\ell}\} + \{L_{ik}, L_{j\ell}, L_{k\ell}\} - \{L_{ij}, L_{k\ell}, L_{k\ell}\})$$

$$+\frac{26}{3}\{L_{ik}, L_{j\ell}\} + \frac{26}{3}\{L_{i\ell}, L_{jk}\} + \frac{44}{3}\{L_{ij}, L_{k\ell}\} + 4L_{k\ell}^2$$

$$-2\{L_{j\ell} + L_{jk} + L_{i\ell} + L_{ik}, L_{k\ell}\} - (6+8a_\ell)\{L_{ik}, L_{jk}\} - (6+8a_k)\{L_{i\ell}, L_{j\ell}\}$$

$$-\frac{32}{3}L_{k\ell} - (\frac{8}{3} - 8a_\ell)(L_{jk} + L_{ik}) - (\frac{8}{3} - 8a_k)(L_{jl} + L_{i\ell})$$

$$+(\frac{16}{3} + 24a_k + 24a_\ell + 32a_ka_\ell)L_{ij} - 16(a_ka_\ell + a_k + a_\ell).$$

Finally, there is also an eighth order functional relation between the 6 integrals which is

$$\sum_{i,j,k,l}\Big[\frac{1}{8}L_{ij}^2L_{kl}^2-\frac{1}{92}\{L_{ik},L_{il},L_{jk},L_{jl}\}-\frac{1}{36}\{L_{ij},L_{ik},L_{kl}\}$$

$$-\frac{7}{62}\{L_{ij},L_{ij},L_{kl}\}+\frac{1}{6}(\frac{1}{2}+\frac{2}{3}a_l)\{L_{ij}L_{ik}L_{jk}\}$$

$$+\frac{2}{3}L_{ij}L_{kl}-(\frac{1}{3}-\frac{3}{4}a_k-\frac{3}{4}a_l-a_ka_l)L_{ij}^2+(\frac{1}{3}+\frac{1}{6}a_l)\{L_{ik},L_{jk}\}+(\frac{4}{3}a_k+\frac{4}{3}a_l+\frac{7}{3}a_ka_l)L_{ij}$$

$$+\frac{2}{3}a_ia_ja_ka_l+2a_ia_ja_k+\frac{4}{3}a_ia_j\Big]=0$$

Here, $\{A,B,C,D\}$ is the 24 term symmetrizer of 4 operators and the sum is taken over all pairwise distinct i,j,k,ℓ.

For the purposes of the representation, it is useful to redefine the constants as $a_i=b_i^2-\frac{1}{4}$.

Subalgebras

We note that the algebra described above contains several copies of 2-sphere algebra. Then, we can see that there exist subalgebras $\mathcal{A}_k$ generated by the set $\{L_{ij}, \mathcal{I}\}$ for $i, j \neq k$ and that these algebras are exactly those associated to the 2D analog of this system. Furthermore, if we define

$$H_k \equiv \sum_{i<j,i,j\neq k} L_{ij} - (\sum_{j\neq k} b_i^2 - \frac{3}{4})\mathcal{I}$$

then H_k will commute with all the elements of $\mathcal{A}_k$ and will represent the Hamiltonian for the associated system.

For example, $\mathcal{A}_4 \subset \mathcal{A}$ the algebra generated by L_{12}, L_{13}, L_{23} and the identity, $\mathcal{I}$ has as it center

$$H_4 = L_{12} + L_{13} + L_{23} + (3/4 - b_1^2 - b_2^2 - b_3^2)\mathcal{I}$$

which is the Hamiltonian for the associated 2-sphere system.

Creating the Model:A basis for $L_{13},\ H_4$

As described above, we seek to construct a representation of $\mathcal{A}$ by extending the representations obtained for the subalgebras $\mathcal{A}_k$. The most important difference for our new representation is that the operator H_4 is in the center of $\mathcal{A}_4$ but not $\mathcal{A}$. Hence, it can no longer be represented as a constant. We use the information about its eigenvalues to make an informed choice for its realization by choosing variables t and s, such that

$$H_4 = \frac{1}{4} - 4s^2, \qquad L_{13} = -4t^2 - \frac{1}{2} + b_1^2 + b_3^2.$$

In this basis, the eigenfunctions $d_{\ell,m}$ for a finite dimensional representation are given by delta functions

$$d_{\ell,m}(s,t) = \delta(t - t_\ell)\delta(s - s_m), \quad 0 \le \ell \le m \le M,$$

with the spectrum of s^2 is $\{(-s_m)^2 = (m + 1 + (b_1 + b_2 + b_3)/2)^2\}$ and the spectrum of t^2 is $\{t_\ell^2 = (\ell + (b_1 + b_2 + 1)/2)^2\}$.

Creating the Model: A basis for $L_{12},\ H_4$

Using the information from the representation of $\mathcal{A}_4$ we hypothesize that L_{12} take the form of an eigenvalue operator for Wilson polynomials in the variable t

$$L_{12} = -4\tau_t^*\tau_t - 2(b_1+1)(b_2+1) + 1/2$$

with parameters are given by:

$$\alpha = \frac{b_2+1}{2} + s,\ \beta = \frac{b_1+b_3+1}{2},\ \gamma = \frac{b_1-b_3+1}{2},\ \delta = \frac{b_2+1}{2} - s.$$

The basis functions corresponding to diagonalizing H_4 and L_{12} can be taken, essentially, as the Wilson polynomials

$$f_{n,m}(t,s) = w_n(t^2,\alpha,\beta,\gamma,\delta)\delta(s-s_m),$$

where $s_m = m+1+(b_1+b_2+b_3)/2$ as above. Note that $w_n(t^2)$ actually depends on m (or s^2) through the parameters α, δ.

Creating the Model: A basis for L_{13}, L_{24}

A reasonable guess of the form of the operator L_{24} is as a difference operator in s, since it commutes with L_{13}. For L_{24} we take

$$L_{24} = -4\tilde{\tau}_s^* \tilde{\tau}_s - 2(b_2+1)(b_4+1) + \frac{1}{2}.$$

Here $\tilde{\tau}_s$ is a difference operator in s with parameters

$$\tilde{\alpha} = t + \frac{b_2+1}{2},\ \tilde{\beta} = -M - \frac{b_1+b_2+b_3}{2} - 1,$$

$$\tilde{\gamma} = M + b_4 + \frac{b_1+b_2+b_3}{2} + 2,\ \tilde{\delta} = -t + \frac{b_2+1}{2}.$$

With the operator L_{24} thus defined, the unnormalized eigenfunctions of the commuting operators L_{13}, L_{24} in the model take the form $g_{n,k}$ where $0 \le \ell \le M,\ 0 \le k \le M-\ell$, and

$$g_{\ell,k} = \delta(t-t_\ell) w_k(s^2, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}).$$

To complete the model, we will need parameter dependent raising and lowering operators for the Wilson polynomials,

$$R = \frac{1}{2y}[T^{1/2} - T^{-1/2}].$$

$$R\Phi_n = \frac{n(n+\alpha+\beta+\gamma+\delta-1)}{(\alpha+\beta)(\alpha+\gamma)(\alpha+\delta)}\Phi_{n-1}^{(\alpha+1/2,\beta+1/2,\gamma+1/2,\delta+1/2)}.$$

$$L = \frac{1}{2y}\Big[(\alpha-1/2+y)(\beta-1/2+y)(\gamma-1/2+y)(\delta-1/2+y)T^{1/2}$$
$$-(\alpha-1/2-y)(\beta-1/2-y)(\gamma-1/2-y)(\delta-1/2-y)T^{-1/2}\Big].$$

$$L\Phi_n = (\alpha+\beta-1)(\alpha+\gamma-1)(\alpha+\delta-1)\Phi_{n+1}^{(\alpha-1/2,\beta-1/2,\gamma-1/2,\delta-1/2)}.$$

$$L_{\alpha\beta} = \frac{1}{2y}\Big[-(\alpha-1/2+y)(\beta-1/2+y)T^{1/2} + (\alpha-1/2-y)(\beta-1/2-y)T^{-1/2}\Big].$$

$$L_{\alpha\beta}\Phi_n = -(\alpha+\beta-1)\Phi_n^{(\alpha-1/2,\beta-1/2,\gamma+1/2,\delta+1/2)}.$$

We finalize the construction of our model by realizing the operator L_{34}. The operator L_{34} must commute with L_{12}, so we hypothesize that it is of the form

$$L_{34} = A(s)S(L_{\alpha\beta}L_{\alpha\gamma})_t + B(s)S^{-1}(R^{\alpha\beta}R^{\alpha\gamma})_t + C(s)(LR)_t + D(s).$$

On the other hand, we can consider the action of L_{34} on the basis $g_{\ell,k}$. Considering L_{34} primarily as an operator on s we hypothesize that it must be of the form

$$L_{34} = \tilde{A}(t)T(L_{\tilde{\alpha}\tilde{\beta}}L_{\tilde{\alpha}\tilde{\gamma}})_s + \tilde{B}(t)T^{-1}(R^{\tilde{\alpha}\tilde{\beta}}R^{\tilde{\alpha}\tilde{\gamma}})_s + \tilde{C}(t)(LR)_s$$

$$+\tilde{D}(t)s^2 + \tilde{E}(t) + \kappa L_{12}.$$

By a long and tedious computation we can verify that the 3rd order structure equations are satisfied for certain choices of functions. We also obtain the quantization for E, for finite dimensional representations,

$$E = -(2M + \sum_{j=1}^{4} b_j + 3)^2) - 1.$$

The functional coefficients for L_{34} take the following form :

$$A(s) = -\frac{(2M + b_1 + b_2 + b_3 - 2s + 2)(2M + b_1 + b_2 + b_3 + 2b_4 + 2s + 4)}{2s(2s + 1)},$$

$$B(s) = -\frac{(2M + b_1 + b_2 + b_3 + 2s + 2)(2M + b_1 + b_2 + b_3 + 2b_4 - 2s + 4)}{2s(2s - 1)},$$

$$C(s) = -2 + \frac{2(2M + b_1 + b_2 + b_3 + 3)(2M + b_1 + b_2 + b_3 + 2b_4 + 3)}{4s^2 - 1},$$

$$D(s) = 2s^2 - 2\left(\frac{2M + b_1 + b_2 + b_3 + b_4 + 4}{2}\right)^2 - \frac{(b_1 + b_2)^2 + b_3^2 + b_4^2}{2} + b_3 + b_4 + 2M + 3$$
$$+ \frac{((b_1 + b_2 + 1)^2 - b_3^2)(2M + b_1 + b_2 + b_3 + 3)(2M + b_1 + b_2 + b_3 + 2b_4 + 3)}{2(4s^2 - 1)}$$

$$\tilde{A}(t) = \frac{(b_1 - b_3 + 2t + 1)(b_1 + 1 + b_3 + 2t)}{2t(2t + 1)},$$

$$\tilde{B}(t) = \frac{(b_1 - b_3 - 2t + 1)(b_1 + 1 + b_3 - 2t)}{2t(2t - 1)},$$

$$\tilde{C}(t) = 2 + \frac{2(b_3^2 - b_1^2)}{4t^2 - 1},$$

$$\tilde{D}(t) = 2,$$

and $\kappa = -4$. The expression for $\tilde{E}(t)$ takes the form $\tilde{E}(t) = \mu_1 + \mu_2/(4t^2 - 1)$ where μ_1, μ_2 are constants, but we will not list it here in detail.

We shall now review what we have constructed, up to this point. We realize the algebra $\mathcal{A}$ by the following operators

$$\begin{aligned}
H &= -\left((2M + \sum_{j=1}^{4} b_j + 3)^2 + 1\right)\mathcal{I} \\
H_4 &= \frac{1}{4} - 4s^2 \\
L_{13} &= -4t^2 - \frac{1}{2} + b_1^2 + b_3^2 \\
L_{12} &= -4\tau_t^* \tau_t - 2(b_1 + 1)(b_2 + 1) + 1/2 \\
L_{24} &= -4\tilde{\tau}_s^* \tilde{\tau}_s - 2(b_2 + 1)(b_4 + 1) + \frac{1}{2} \\
L_{34} &= A(s)S(L_{\alpha\beta}L_{\alpha\gamma})_t + B(s)S^{-1}(R^{\alpha\beta}R^{\alpha\gamma})_t + C(s)(LR)_t + D(s)
\end{aligned}$$

The operators $L_{23},\ L_{14}$ can be obtained through linear combinations of this basis.

We have computed three sets of orthogonal basis vectors corresponding to diagonalizing three sets of commuting operators, $\{L_{13}, H_4\}$, $\{L_{12}, H_4\}$ and $\{L_{13}, L_{24}\}$, respectively,

$$\tilde{d}_{\ell,m}(s,t) = \delta(t - t_\ell)\delta(s - s_m), \qquad 0 \le \ell \le m \le M, \tag{1}$$

$$f_{n,m}(s,t) = w_n(t^2, \alpha, \beta, \gamma, \delta)\delta(s - s_m), \qquad 0 \le n \le m \le M, \tag{2}$$

$$g_{\ell,k}(s,t) = w_k(s^2, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})\delta(t - t_\ell) \qquad 0 \le \ell \le k + \ell \le M \tag{3}$$

We also have a nonorthogonal basis given by

$$h_{n,k}(s,t) = t^{2n}s^{2k}, \quad 0 \le n + k \le M.$$

Recall that the spectrum of the variables s, t is given by

$$t_\ell = \ell + \frac{b_1 + b_3 + 1}{2}, \quad s_m = -(m + 1 + \frac{b_1 + b_2 + b_3}{2}), \quad 0 \le \ell \le m \le M.$$

Weight Function

It is possible to write an inner product on the finite dimensional representation as

$$\langle f(t,s), g(t,s)\rangle = \int\int f(t,s)g(t,s)\omega(t,s)\delta(t-t_\ell)\delta(s-s_m)dsdt,$$

with

$$\omega(t_\ell, s_m) = \frac{M!(1+b_4)_{M-m}(M+b_1+b_2+b_3+b_4+3)_m(2m+2+b_1+b_2+b_3)}{(M-m)!(1+b_4)_M(M+b_1+b_2+b_3+3)_m(2+b_1+b_2+b_3)}$$
$$\times \frac{(1+b_1)_\ell(1+b_1+b_3)_\ell(1+b_2)_{m-\ell}(2+b_1+b_2+b_3)_{m+\ell}(2\ell+1+b_1+b_3)}{\ell!(m-\ell)!(1+b_3)_\ell(2+b_1+b_3)_{m+\ell}(1+b_1+b_3)c_{0,0}^2}$$

Th normalization of the identity $\langle 1, 1\rangle = 1$ gives

$$c_{0,0}^2 = \frac{(3+b_1+b_2+b_4)_M(3+b_1+b_2+b_3)_M}{(1+b_4)_M(1+b_3)_M},$$

Relation with Tratnik Polynomials

The two-variable extension of the Wilson polynomials defined by Tratnik [Tratnik 1991a, Tratnik 1991b] are given for $0 \le n_1 \le n_1 + n_2 \le M$ by

$$R_2(n_1, n_2; \beta_i; x_1, x_2; M) = r_{n_1}(\beta_1 - \beta_0 - 1, \beta_2 - \beta_1 - 1, -x_2 - 1, x_2 + \beta_1; x_1) \\ \times r_{n_2}(n_1 + \beta_2 - \beta_0 - 1, \beta_3 - \beta_2 - 1, n_1 - M - 1, n_1 + \beta_2 + M; -n_1 + x_2)$$

Recall, the Racah polynomials are related to the Wilson polynomials via

$$r_n(a, b, c, d, x) = w_n\left(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, (x + \tilde{a})^2\right)$$

The weight function for the 2-variable Tratnik polynomials, R_2 agree with $\omega(x, y)$ under the substitution

$$x_0 = 0, \quad x_1 = t - \frac{b_1 + b_3 + 1}{2}, \quad x_2 = -s - 1 - \frac{b_1 + b_2 + b_3}{2}, \quad x_3 = M,$$
$$\beta_0 = b_3, \quad \beta_1 = b_1 + b_3 + 1, \quad \beta_2 = b_1 + b_2 + b_3 + 2, \quad \beta_3 = b_1 + b_2 + b_3 + 3.$$

Further, it can be seen by direct computation that the operators which are diagonalized by the Tratnik polynomials [Geronimo and Iliev 2010] can be written in terms of the operators L_{12} and $L_{14} + L_{12} + L_{24}$ and so form a basis for the representation of $\mathcal{A}$.

Recap

- To introduce the superintegrable system on the n-sphere, which is in some sense the most general, and which admits a quadratic algebra structure.
- The representation theory of the 2-d system can be constructed in terms of Wilson polynomials.
- The representation theory of the 3-d system can be constructed in terms of a two variable generalization of the Wilson polynomials introduced by Tratnik.
- These representations give information about the original system, i.e. eigenvalues of symmetry operators and inter-basis expansion coefficients.
- The algebra gives the structure of recurrence operators which act on the Wilson polynomials.

Future Projects

- Study the system on the n-sphere
- Consider limiting processes on the system to get other representations in terms of orthogonal polynomials.
- Representations of higher-order algebras and relations with higher-order integrals.

Thanks for listening.

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