Wilson polynomials as expansion coefficients for the generic quantum superintegrable system on the 2-sphere

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Abstract

It has been known since 2007 that the Wilson and Racah polynomials can be characterized as basis functions for irreducible representations of the quadratic symmetry algebra of the quantum superintegrable system on the 2-sphere, $H\Psi = E\Psi$, with generic 3-parameter potential.

Clearly, the polynomials are expansion coefficients for one eigenbasis of a symmetry operator L_1 of H in terms of an eigenbasis of another symmetry operator L_2, but the exact relationship appears not to have been made explicit. We work out the details of the expansion to show, explicitly, how the polynomials arise and how the principal properties of these functions: the measure, 3-term recurrence relation, 2nd order difference equation, duality of these relations, permutation symmetry, and intertwining operators – follow from the symmetry of this quantum system.

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Outline

1. Introduction
2. The generic superintegrable system on the 2-sphere
3. Discussion and Conclusions

Superintegrable Systems: $H\Psi = E\Psi$

- A quantum superintegrable system is an integrable Hamiltonian system on an n-dimensional Riemannian/pseudo-Riemannian manifold with potential:

  $$H = \Delta_n + V$$

  that admits $2n-1$ algebraically independent partial differential operators commuting with H, the maximum possible.

  $$[H, L_j] = 0, \quad L_{2n-1} = H, \; n = 1, 2, \cdots, 2n-1.$$

- Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically.
- Typically the basis symmetries L_j generate an algebra under commutation, not usually a Lie algebra, that closes at finite order. It is this algebra that is responsible for the solvability of the quantum system.

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System $S9$ (in our listing)

Hamiltonian operator:

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \qquad a_j = \frac{1}{4} - k_j^2,$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

$$s_1^2 + s_2^2 + s_3^2 = 1.$$

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$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2},\ L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2},\ L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

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Structure equations:

$$R = [L_1, L_2], \quad \{A, B\} = AB + BA,$$

Closure:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

Casimir:

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 +$$

$$\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3$$

$$+ \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3.$$

Hypergeometric polynomials

$$ {}_4F_3\left(\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3 & \end{matrix} ; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k(a_4)_k}{(b_1)_k(b_2)_k(b_3)_k k!} x^k $$

$$ (a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1) \text{ if } n \geq 1. $$

Here $k! = (1)_k$. If $a_1 = -n$ for n a nonnegative integer then the sum is finite with $n+1$ terms.

Wilson polynomials (of order n in t^2):

$$ \Phi_n(\alpha,\beta,\gamma,\delta;t) = {}_4F_3\left(\begin{matrix} -n, & \alpha+\beta+\gamma+\delta+n-1, & \alpha-t, & \alpha+t \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta & \end{matrix} ; 1\right) $$

Racah polynomials: If $\alpha+\beta = -m$ for m a nonegative integer then only finite set $\Phi_0, \Phi_1, \cdots, \Phi_m$

Basis eigenfunctions of L_1, H:

$$\Psi_{N-n,n} = (s_1^2+s_2^2)^{\frac{1}{2}(2n+k_1+k_2+1)}(1-s_1^2-s_2^2)^{\frac{1}{2}(k_3+\frac{1}{2})}(\frac{s_2^2}{s_1^2+s_2^2})^{\frac{1}{2}(k_2+\frac{1}{2})}(\frac{s_1^2}{s_1^2+s_2^2})^{\frac{1}{2}(k_1+\frac{1}{2})}$$

$$\times P_n^{(k_2,k_1)}(\frac{s_1^2-s_2^2}{s_1^2+s_2^2})P_{N-n}^{(2n+k_1+k_2+1,k_3)}(1-2s_1^2-2s_2^2),$$

$$L_1\Psi_{N-n,n} = (k_1^2+k_2^2-\frac{1}{2}-(2n+1+k_1+k_2)^2\Psi_{N-n,n}, \quad n=0,1,\cdots,N,$$

$$H\Psi_{N-n,n} = E_N\Psi_{N-n,n}, \quad E_N = -[2N+k_1+k_2+k_3+2]^2+\frac{1}{4}, \quad N=0,1,\cdots.$$

Separable in spherical coordinates, orthogonal with respect to area measure on the 1st octant of the 2-sphere. Dimension of eigenspace E_N is $N+1$.

$$P_n^{(\alpha,\beta)}(y) = \binom{n+\alpha}{n}\,{}_2F_1\left(\begin{matrix}-n & \alpha+\beta+n+1\\ \alpha+1 & \end{matrix};\frac{1-y}{2}\right),$$ Jacobi polynomials

Basis eigenfunctions of L_2, H:

Get immediately by permutation $1 \leftrightarrow 3$, $n \leftrightarrow q$, of L_1 basis:

$$L_2\Lambda_{N-q,q} = (k_3^2 + k_2^2 - \frac{1}{2} - (2q + 1 + k_3 + k_2)^2\Lambda_{N-q,q}, \quad q = 0, 1, \cdots, N,$$

$$H\Lambda_{N-q,q} = E_N\Lambda_{N-q,q}, \quad E_N = -[2N + k_1 + k_2 + k_3 + 2]^2 + \frac{1}{4}, \quad N = 0, 1, \cdots .$$

Separable in a different set of spherical coordinates, orthogonal with respect to area measure on the 1st octant of the 2-sphere. Dimension of eigenspace E_N is $N + 1$.

Structure equations

Using the method of Kalnins, kress and Miller (2012) we can derive the structure algebra of $S9$, just from the 1st order Gaussian differential recurrences for

$$ {}_2F_1\left(\begin{matrix} a, & b \\ c & \end{matrix} ; z\right) \to {}_2F_1\left(\begin{matrix} a\pm 1, & b \\ c & \end{matrix} ; z\right), $$

$$ {}_2F_1\left(\begin{matrix} a, & b \\ c & \end{matrix} ; z\right) \to {}_2F_1\left(\begin{matrix} a\pm 1, & b\mp 1 \\ c\mp 1 & \end{matrix} ; z\right). $$

One consequence is the action of L_2 on an L_1 eigenbasis:

$$ L_2\Psi_{m,n} = A_n\Psi_{m-1,n+1} + B_n\Psi_{m,n} + C_n\Psi_{m+1,n-1} = $$

$$ -\frac{4(N+k_3-n)(N+n+k_1+k_2+k_3+2)(n+1)(n+k_1+k_2+1)}{(2n+k_1+k_2+2)(2n+k_1+k_2+1)}\Psi_{m-1,n+1} $$

$$ [-\frac{(k_1^2-k_2^2)\left(k_3^2-(2N+k_1+k_2+k_3+2)^2\right)}{2(2n+k_1+k_2+2)(2n+k_1+k_2)} + \frac{1}{2}(2n+k_1+k_2+1)^2 $$

$$ +\frac{1}{4} - k_1^2 - \frac{1}{2}(2N+2+k_1+k_2+k_3)^2 + \frac{1}{2}k_3^2]\Psi_{m,n} $$

$$ -\frac{4(N-n+1)(N+n+k_1+k_2+1)(n+k_1)(n+k_2)}{(2n+k_1+k_2)(2n+k_1+k_2+1)}\Psi_{m+1,n-1}, $$

Interbasis expansion coefficients 1

The action of L_1 on and L_2 eigenbasis follows immediately from permutation symmetry. Now we expand the L_2 eigenbasis in terms of the L_1 eigenbasis:

$$\Lambda^{(k_1,k_2,k_3)}_{N-q,q} = \sum_{n=0}^{N} R^n_q(k_1,k_2,k_3)\,\Psi^{(k_1,k_2,k_3)}_{N-n,n}, \quad q = 0,\cdots,N$$

Using the self-adjoint properties of L_1, L_2 we can find recurrences to compute the norm of $\Psi_{N-n,n}$:

$$||\Psi_{N-n,n}||^2 =$$

$$\frac{1}{4n!\Gamma(N-n+1)}\frac{\Gamma(n+k_1+1)\Gamma(n+k_2+1)\Gamma(N-n+k_3+1)\Gamma(N+n+k_1+k_2+2)}{(2N+k_1+k_2+k_3+2)(2n+k_1+k_2+1)\Gamma(n+k_1+k_2+1)\Gamma(N+n+k_1+k_2+k_3+2)},$$

To better exploit the symmetry of our system we rescale the basis:

$$\Psi'_{N-n,n}(k_1,k_2,k_3) = \frac{(-1)^n n!\Gamma(N-n+1)}{\Gamma(N-n+k_3+1)\Gamma(n+k_2+1)}\Psi_{N-n,n}(k_1,k_2,k_3),$$

$$(2N+k_1+k_2+k_3+2)(2n+k_1+k_2+1)\,||\Psi'_{N-n,n}(k_1,k_2,k_3)||^2 =$$

$$\frac{n!\,\Gamma(N-n+1)\Gamma(n+k_1+1)\Gamma(N+n+k_1+k_2+2)}{4\Gamma(n+k_2+1)\Gamma(N-n+k_3+1)\Gamma(k_1+k_2+n+1)\Gamma(N+n+k_1+k_2+k_3+2)}$$

The norms of $\Lambda_{N-q,q}$ follow from the permutation $1 \leftrightarrow 3$, $n \leftrightarrow q$.

Interbasis expansion coefficients 2

The two sets of $N+1$ basis vectors $\{\frac{\Psi'_{m,n}}{||\Psi'_{m,n}||}\}$ and $\{\frac{\Lambda'_{p,q}}{||\Lambda'_{p,q}||}\}$ are each orthonormal, which implies that the $(N+1)\times(N+1)$ matrix

$$\left(\frac{||\Psi'_{m,n}||\,{R'}^{n}_{q}}{||\Lambda'_{p,q}||}\right), \quad 0\le n,q\le N,$$

is orthogonal. Thus we can derive the identities

$$\sum_{\ell=0}^{N}\frac{{R'}^{n_1}_{\ell}\,{R'}^{n_2}_{\ell}}{||\Lambda'_{N-\ell,\ell}||^2}=\frac{\delta_{n_1,n_2}}{||\Psi'_{m_1,n_1}||^2},\quad \sum_{\ell=0}^{N}\frac{{R'}^{\ell}_{q_1}\,{R'}^{\ell}_{q_2}}{||\Lambda_{p_1,q_1}||^2}=\frac{\delta_{q_1,q_2}}{||\Psi'_{N-\ell,\ell}||^2}.$$

Interbasis expansion coefficients 3

By permutation symmetry:

$$\Lambda'_{p,q}(k_1,k_2,k_3)=\Psi'_{p,q}(k_3,k_2,k_1).$$

We set

$$R'^{n}_{q}(k_1,k_2,k_3)_N\cdot||\Psi'_{N-n,n}(k_1,k_2,k_3)||^2\equiv\ \Xi'\begin{pmatrix}k_1 & k_2 & k_3\\ n & N & q\end{pmatrix}.$$

Note that Ξ' satisfies the orthogonality relations

$$\sum_{q=0}^{N}\frac{\Xi'\begin{pmatrix}k_1 & k_2 & k_3\\ n_1 & N & q\end{pmatrix}\Xi'\begin{pmatrix}k_1 & k_2 & k_3\\ n_2 & N & q\end{pmatrix}}{||\Lambda_{N-q,q}(k_1,k_2,k_3)||^2}=||\Psi'_{N-n_1,n_1}(k_1,k_2,k_3)||^2\delta_{n_1,n_2},$$

$$\sum_{\ell=0}^{N}\frac{\Xi'\begin{pmatrix}k_1 & k_2 & k_3\\ \ell & N & q_1\end{pmatrix}\Xi'\begin{pmatrix}k_1 & k_2 & k_3\\ \ell & N & q_2\end{pmatrix}}{||\Psi'_{N-\ell,\ell}(k_1,k_2,k_3)||^2}=||\Lambda'_{N-q_1,q_1}(k_1,k_2,k_3)||^2\delta_{q_1,q_2}.$$

3-term recurrence formula 1

Applying L_2 to both sides of the expansion formula we have

$$L_2\Lambda'_{q,p} = \sum_{n=0}^{N} {R'}_q^n \left({A'}_n\Psi'_{m-1,n+1}{B'}_n\Psi'_{m,n} + {C'}_n\Psi'_{m+1,n-1}\right),$$

$$L_2\Lambda'_{q,p} = (-(2q+k_2+k_3+1)^2 - \frac{1}{2} + k_2^2 + k_3^2)\sum_{n=0}^{N} {R'}_q^n\Psi'_{m,n}.$$

Thus, equating coefficients of $\Psi'_{m,n}$, we find a 3-term recurrence formula for ${R'}_q^n$, hence for Ξ'. If we make the identifications

$$k_1 = \delta+\beta-1,\ k_2 = \alpha+\gamma-1,\ k_3 = \alpha-\gamma,\ N = -\alpha-\beta, t = q + \frac{k_2+k_3+1}{2},$$

this formula for Ξ' agrees exactly with the 3-term recurrence formula for the Racah polynomials, so that

$$\Xi'\begin{pmatrix} k_1 & k_2 & k_3 \\ n & N & q \end{pmatrix}$$

is proportional to a Racah polynomial in t^2 of order n.

3-term recurrence formula 2

Moreover,

$$||\Lambda'_{N-q,q}(k_1,k_2,k_3)||^2 \sim \frac{\Gamma(t-\alpha+1)\Gamma(t-\beta+1)\Gamma(t-\gamma+1)\Gamma(t-\delta+1)\Gamma(t)}{\Gamma(t+\alpha)\Gamma(t+\beta)\Gamma(t+\gamma)\Gamma(t+\delta)\Gamma(t+1)},$$

so the measure defined by $||\Lambda'_{N-q,q}(k_1,k_2,k_3)||^{-2}$ is a scalar times a function of t that is symmetric with respect to all permutations of $\alpha,\beta,\gamma,\delta$. This implies that the family of orthogonal polynomials determined by this measure must admit this symmetry up to a multiplicative factor. Further, the left hand side of the orthogonality relation is proportional to

$$\sum_{q=0}^{N} \frac{(2\alpha)_q(\alpha+1)_q(\alpha+\beta)_q(\alpha+\gamma)_q(\alpha+\delta)_q}{(\alpha)_q(\alpha-\beta+1)_q(\alpha-\gamma+1)_q(\alpha-\delta+1)_q q!} \equiv' \begin{pmatrix} k_1 & k_2 & k_3 \\ n_1 & N & q \end{pmatrix} \equiv' \begin{pmatrix} k_1 & k_2 & k_3 \\ n_2 & N & q \end{pmatrix}$$

precisely the measure for orthogonality of the Racah polynomials $\Phi_n^{(\alpha,\beta\gamma,\delta)}(t^2)$ where $t = q+\alpha$.

Duality

By making the transpositions $k_1 \leftrightarrow k_3,\ n \leftrightarrow q$ we obtain the result of applying L_1 to the expansion of $\Psi'_{N-n,n}$ in an L_2 eigenbasis. This gives gives a 3-term recurrence relation for Ξ, defining a family of orthogonal polynomials $p'_q(n)$ in the variable n. It is a 2nd order difference equation in q, hence t, for the Racah polynomials as eigenfunctions.

This action induces a function space model of an irreducible representation of the structure algbra of $S9$ in which the basis functions are Racah polynomials in t^2 and the symmetry operators map to difference operators.

Intertwining operators 1

Let $\mathcal{W}_{k_1,k_2,k_3}$ be the space of functions on the first octant of the 2-sphere and with Hamiltonian $H^{(k_1,k_2,k_3)}$, symmetry operators $L_j^{(k_1,k_2,k_3)}$ and inner product $<\Phi,\Psi>_{k_1,k_2,k_3}$. Let $\mathcal{W}_{k_1',k_2',k_3'}$ be another such space. An *intertwining operator* is a mapping $X^{(k_1,k_2,k_3)}:\mathcal{W}_{k_1,k_2,k_3}\to\mathcal{W}_{k_1',k_2',k_3'}$ such that

$$X^{(k_1,k_2,k_3)}H^{(k_1,k_2,k_3)}=H^{(k_1',k_2',k_3')}X^{(k_1,k_2,k_3)}.$$

Note that $X^{(k_1,k_2,k_3)}$ maps eigenfunctions of $H^{(k_1,k_2,k_3)}$ to eigenfunctions of $H^{(k_1',k_2',k_3')}$ and its adjoint $X^{*(k_1,k_2,k_3)}$ reverses the action.
Such energy shifting transformations are induced by the basic differential recurrence relations obeyed by Gaussian hypergeometric functions. For example the standard recurrence

$$\left[z(1-z)\frac{d}{dz}-(b+a-1)z+c-1\right]{}_2F_1\left(\begin{array}{cc}a & b\\ c & \end{array};z\right)=(c-1){}_2F_1\left(\begin{array}{cc}a-1 & b-1\\ c-1 & \end{array};z\right)$$

induces a 1st order differential operator

$$T^{(k_1,k_2,k_3)}:\mathcal{W}_{k_1,k_2,k_3}\to\mathcal{W}_{k_1-1,k_2-1,k_3},\quad T^{(k_1,k_2,k_3)}\Psi_{m,n}^{(k_1,k_2,k_3)}=-(n+1)\Psi_{m,n+1}^{(k_1-1,k_2-1,k_3)}.$$

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induces a 1st order differential operator

$$T^{(k_1,k_2,k_3)}:\mathcal{W}_{k_1,k_2,k_3}\to\mathcal{W}_{k_1-1,k_2-1,k_3},\quad T^{(k_1,k_2,k_3)}\Psi_{m,n}^{(k_1,k_2,k_3)}=-(n+1)\Psi_{m,n+1}^{(k_1-1,k_2-1,k_3)}.$$

Intertwining operators 2

The adjoint is induced by $\frac{d}{dz}\,{}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix}; z\right) = \frac{ab}{c}\,{}_2F_1\left(\begin{matrix} a+1 & b+1 \\ c+1 & \end{matrix}; z\right),$

$$T^{*(k_1,k_2,k_3)} : \mathcal{W}_{k_1,k_2,k_3} \to \mathcal{W}_{k_1+1,k_2+1,k_3},$$

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The action of T and T^* on the Λ-basis can again be computed from 1st order relations obeyed by Gaussian hypergeometric functions. Transferring the T action to Ξ' we find

$$\tau^{(\alpha-\frac{1}{2},\beta-\frac{1}{2},\gamma-\frac{1}{2},\delta-\frac{1}{2})}\,\Xi'\begin{pmatrix} k_1-1 & k_2-1 & k_3 \\ n & N+1 & q \end{pmatrix} = n(k_1+k_2+n-1)\,\Xi'\begin{pmatrix} k_1 & k_2 & k_3 \\ n-1 & N & q \end{pmatrix}$$

$$\tau^{(\alpha-\frac{1}{2},\beta-\frac{1}{2},\gamma-\frac{1}{2},\delta-\frac{1}{2})}f(t) = \frac{1}{2t}\left[(f(t+\frac{1}{2}) - f(t-\frac{1}{2})\right], \quad t = q + \frac{k_2+k_3+1}{2}.$$

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Intertwining operators 3

Transferring the T^* action to Ξ' we find

$$\tau^{*(\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12)}\,\Xi'\begin{pmatrix} k_1+1 & k_2+1 & k_3 \\ n & N-1 & q \end{pmatrix} = \Xi'\begin{pmatrix} k_1 & k_2 & k_3 \\ n+1 & N & q \end{pmatrix}.$$

$$\tau^{*(\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12)} f(t) =$$

$$\frac{1}{2t}\left[(\alpha+t)(\beta+t)(\gamma+t)(\delta+t)f(t+\frac12)-(\alpha-t)(\beta-t)(\gamma-t)(\delta-t)f(t-\frac12)\right]$$

Note that (with $t = q + \frac{k_2+k_3+1}{2}$),

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a 2nd order difference equation for Ξ' as a polynomial in t^2. We will solve this equation.

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Calculation of Racah polynomials 1

Since $\Xi'_n(t)$ can be written as $\Xi' = G(n, N, k_1, k_2, k_3)\,\Phi_n(t)$

$$\Phi_n(t) = \sum_{k=0}^{n} w_k(\alpha+t)_k(\alpha-t)_k, \quad w_0 = 1,$$

Appliying τ to $(\alpha+t)_k(\alpha-t)_k$, we get,

$$\tau(\alpha+t)_k(\alpha-t)_k = -k\,(\alpha+1/2+t)_{k-1}\,(\alpha+1/2-t)_{k-1}$$

Applying τ^* to the shifted basis, we get

$$\tau^*(\alpha+1/2+t)_k(\alpha+1/2-t)_k = -(\alpha+\beta+\gamma+\delta+k)(\alpha+t)_{k+1}(\alpha-t)_{k+1}$$

$$+(\alpha+\beta+k)(\alpha+\gamma+k)(\alpha+\delta+k)(\alpha+t)_k(\alpha-t)_k,$$

Thus,

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Calculation of Racah polynomials 2

$$w_{k+1}=\frac{(-n+k)(n+\alpha+\beta+\gamma+\delta+k-1)}{(k+1)(\alpha+\beta+k)(\alpha+\gamma+k)(\alpha+\delta+k)}w_k,\ \ w_0=1.$$

It is easy to solve this recurrence to obtain

$$w_k=\frac{(-n)_k(n+\alpha+\beta+\gamma+\delta-1)_k}{k!(\alpha+\beta)_k(\alpha+\gamma)_k(\alpha+\delta)_k},\ \ k=0,1,\cdots.$$

Hence,

$$\Phi_n(t)={}_4F_3\left(\begin{matrix}-n & n+\alpha+\beta+\gamma+\delta-1 & \alpha+t & \alpha-t\\ \alpha+\beta & \alpha+\gamma & \alpha+\delta & \end{matrix};1\right),$$

unique up to a scalar multiple.

$$\Xi'\begin{pmatrix}k_1 & k_2 & k_3\\ n & N & q\end{pmatrix}=G(n,N,k_1,k_2,k_3)\,\Phi_n(t).$$

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Permutation symmetry

Since the measure for the Racah polynomials is invariant under all permutations of $\alpha, \beta, \gamma, \delta$, and the Racah polynomials can be obtained from the measure by the Gram-Schmidt process, each such permutation must take a Racah polynomial to a scalar multiple of itself.

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Intertwining operators 4

The recurrence

$$\left(z\frac{d}{dz}+c-1\right){}_2F_1\left(\begin{array}{cc} a & b \\ c & \end{array}; z\right) = (c-1){}_2F_1\left(\begin{array}{cc} a & b \\ c-1 & \end{array}; z\right)$$

leads to an intertwining operator

$$U^{(k_1,k_2,k_3)}_{(-,+,-,+)}:\ \mathcal{W}_{k_1,k_2,k_3} \longrightarrow \mathcal{W}_{k_1+1,k_2-1,k_3}.$$

$$\mu^{(\beta,\delta)}\,\Phi_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma-\frac{1}{2},\delta+\frac{1}{2})}(t) = \frac{(n+\beta+\delta)(n+\alpha+\gamma-1)}{(\alpha+\gamma-1)}\,\Phi_n^{(\alpha,\beta,\gamma,\delta)}(t)$$

Its action on the Λ' basis induces the recurrence

$$\mu^{(\beta,\delta)}f(t) = \frac{1}{2t}\left[(\beta+t)(\delta+t)f(t+\frac{1}{2}) - (\beta-t)(\delta-t)f(t-\frac{1}{2})\right],$$

$$\mu^{(\beta,\delta)}\,\Xi'\left(\begin{array}{ccc} k_1+1 & k_2-1 & k_3 \\ n & N & q \end{array}\right) = (n+k_1+1)(n+k_2)\,\Xi'\left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ n & N & q \end{array}\right).$$

The permutation invariance of the Racah polynomials leads to a familty of recurrences in the μ such that any pair of $\alpha, \beta, \gamma, \delta$ can be raised by $\frac{1}{2}$ and the other pair lowered by $\frac{1}{2}$. These also follow from intertwining operators induced by Gaussian hypergeometric differential recurrences.

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The expansion coefficients

Solving all of these recurrences for Ξ' we find

$$R'^{n}_{q}(k_1,k_2,k_3)_N \cdot ||\Psi'_{N-n,n}(k_1,k_2,k_3)||^2 = \Xi' \begin{pmatrix} k_1 & k_2 & k_3 \\ n & N & q \end{pmatrix} =$$

$$\frac{4c\,(2N+k_1+k_2+k_3+2)(2n+k_1+k_2+1)\Gamma(N+1)}{\Gamma(N+k_1+k_2+k_3+2)\Gamma(k_2+1)}\Phi_n^{(\alpha,\beta,\gamma,\delta)}(t^2),$$

$$t = q+\frac{k_2+k_3+1}{2}, \alpha = \frac{k_2+k_3+1}{2},$$

$$\beta = -N-\frac{k_2+k_3+1}{2},\ \gamma = \frac{k_2-k_3+1}{2},\ \delta = N+k_1+\frac{k_2+k_3+3}{2},$$

$$\Phi_n = {}_4F_3\left(\begin{matrix} -n, & k_1+k_2+n+1, & -q, & k_3+k_2+q+1 \\ -N, & k_2+1, & N+k_1+k_2+k_3+2 & \end{matrix} ;1\right)$$

Scaling factor

To determine the overall scaling factor $c(k_1, k_2, k_3)$ we can evaluate the double integral for Ξ' in the simplest case $n = q = N = 0$ where it factors into a product of beta integrals:

$$\Xi' \begin{pmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\frac{\Gamma(\frac{1}{2}k_1 + \frac{1}{2}k_3 + 1)^2 \, \Gamma(\frac{1}{2}k_1 + k_2 + \frac{5}{4}) \, \Gamma(\frac{1}{2}k_1 + k_2 + \frac{7}{4})}{4\Gamma(k_1 + 1) \, \Gamma(k_3 + 1) \, \Gamma(k_2 + 1)^2 \, \Gamma(k_1 + k_2 + \frac{1}{2}k_3 + \frac{9}{4}) \, \Gamma(k_1 + k_2 + \frac{1}{2}k_3 + \frac{11}{4})}.$$

This expression can be simplified by using the duplication formula for gamma functions.

Extension to Wilson polynomials 1

Racah polynomials as are $S9$ expansion coefficients for finite dimensional representations on the real 2-sphere. Wilson polynomials are expansion coefficients related to infinite dimensional representations for the Schrödinger eigenvalue equation of the generic potential on the upper sheet of the 2d hyperboloid. A Hilbert space structure is imposed on the eigenspace corresponding to a single continuous spectrum eigenvalue. where N is a negative real number, not an integer.

Extension to Wilson polynomials 2

We expand the L_2 basis vectors in terms of the L_1 basis:

$$\Lambda_q = \sum_{n=0}^{\infty} R_q^n \Psi_n, \quad \sum_{\ell=0}^{\infty} \frac{||\Psi_{n_1}||^2}{||\Lambda_\ell||^2} R_\ell^{n_1} R_\ell^{n_2} = \delta_{n_1,n_2},$$

Applying L_2 to both sides of the expansion we can show that the R_q^n satisfy a three term recurrence relation and a difference equation as before, and the orthogonality relation can be rewritten in the form

$$\sum_{q=0}^{\infty} \frac{(2\alpha)_q(\alpha+1)_q(\alpha+\beta)_q(\alpha+\gamma)_q(\alpha+\delta)_q}{(\alpha)_q(\alpha-\beta+1)_q(\alpha-\gamma+1)_q(\alpha-\delta+1)_q\, q!} R'^{\,n_1}_{q} R'^{\,n_2}_{q} = \delta_{n_1,n_2} h_{n_1},$$

where $R'^{\,n}_{q} \sim \Phi_n(t)$ is a Wilson polynomial. This is equivalent to a ${}_5F_4$ identity and can all be made rigorous. Wilson recast the orthogonality into the form of a contour integral which greatly extended its domain of validity

Wrap-up. 1

- We showed explicitly how Racah and Wilson polynomials arise as expansion coefficients for the generic superintegrable system on the complex 2-sphere, relating two different sets of spherical coordinate bases.
- We showed how the principal properties of these functions: the measure, 3-term recurrence relation, 2nd order difference equation, duality of these relations, permutation symmetry, and intertwining operators – follow from the symmetry of this quantum system.
- The duality between the 3-term recurrence fomula and the 2nd order difference equation for these polynomials is a consequence of the permutation symmetry of the quantum Hamiltonian.
- The parameter changing difference relations for the polynomials follow from intertwining operators for the quantum system. All of the properies of this system are induced by the fundamental differential recurrence relations of the gaussian hypergeometric functions.
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- The duality between the 3-term recurrence fomula and the 2nd order difference equation for these polynomials is a consequence of the permutation symmetry of the quantum Hamiltonian.
- The parameter changing difference relations for the polynomials follow from intertwining operators for the quantum system. All of the properies of this system are induced by the fundamental differential recurrence relations of the gaussian hypergeometric functions.
- The orthogonality measure for the polynomials and its symmetry follow from the symmetry of the norms of the quantum basis functions.

Wrap-up. 2

- There is active interest in the relation between multivariable Wilson polynomials and the quantum superintegrable system on the *n*-sphere with generic potential, and these results should aid in the generalization.
- We, together with collaborators, have shown that contracting function space realizations of irreducible representations of the $S9$ quadratic algebra to the other superintegrable systems we obtain the full Askey scheme of orthogonal hypergeometric polynomials. All of these contractions of superintegrable systems with potential are uniquely induced by Wigner Lie algebra contractions of $so(3,\mathbb{C})$ and $e(2,\mathbb{C})$. This work should be extended to multivariable orthogonal polynomials.

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