2D 2nd order Laplace superintegrable systems, Heun equations, QES and Bôcher contractions

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Abstract

Second-order conformal quantum superintegrable systems in 2 dimensions are Laplace equations with scalar potential and 3 independent 2nd order conformal symmetry operators. They encode all the information about 2D Helmholtz superintegrable systems in an efficient manner: Each of these systems admits a quadratic symmetry algebra (not usually a Lie algebra) and is multiseparable. The separation equations comprise all of the various types of hypergeometric and Heun equations in full generality. In particular, they yield all of the 1D Schrödinger exactly solvable (ES) and quasi-exactly solvable (QES) systems related to the Heun operator.

The separable solutions of these equations are the special functions of mathematical physics. The different systems are related by Stäckel transforms, by the symmetry algebras and by Bôcher contractions of the conformal algebra $so(4, \mathbb{C})$ to itself, which enables all systems to be derived from a single one: the generic potential on the complex 2-sphere.

Distinct separable bases for a single Laplace system are related by interbasis expansion coefficients which are themselves special functions, such as the Wilson polynomials. This approach facilitates a unified view of special function theory, incorporating hypergeometric and Heun functions in full generality.

Outline

Purpose

The purpose of this talk is to make clear how superintegrable systems theory unifies and simplifies the study of the special functions of mathematical physics, hypergeometric and Heun equations, and exactly solvable and quasi-exactly solvable systems. We consider here one of the simplest classes of such systems: 2nd order superintegrable systems in 2 complex variables.

This is an integrable Hamiltonian system on an 2-dimensional manifold with potential:

$$H = \Delta_2 + V,$$

that admits 3 algebraically independent 2nd order partial differential operators L_1, L_2, H commuting with H, the maximum possible,

$$[H, L_j] = 0, \qquad j = 1, 2.$$

Here $[A, B] = AB - BA$ is the operator commutator.

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Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically.

The 2nd order 2D systems have been classified. There are 44 nondegenerate (3 linear parameter potential) systems, on a variety of manifolds,

Under the Stäckel transform, an invertible structure preserving mapping, they divide into 6 equivalence classes with representatives on flat space and the 2-sphere.

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Properties of 2nd order superintegrable systems with potential

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of H and their multiplicity
- 2nd order superintegrable systems are multiseparable.
- Smorodinsky, Winternitz and collaborators inaugurated this study in 1965 by pointing out the multiseparability of systems such as the Smorodinsky-Winternitz system

$$H = \partial_{xx} + \partial_{yy} + \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}.$$

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Example: $S9$ - a nondegenerate system

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

Basis symmetries: $(s_1^2 + s_2^2 + s_3^2 = 1)$

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \; L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \; L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

Structure equations:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 +$$

$$\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3$$

$$+ \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3, \quad R = [L_1, L_2].$$

Here, $H = L_1 + L_2 + L_3 + a - 1 + a_2 + a_3$.

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Example: Higgs oscillator - a degenerate system

It is the same as $S9$ with $a_1 = a_2 = 0$, $a_3 = a$, but admits additional symmetry.

Basis symmetries:

$$X = J_3, \qquad L_1 = J_1^2 + \frac{as_2^2}{s_3^2}, \qquad L_2 = \frac{1}{2}(J_1J_2 + J_2J_1) - \frac{as_1s_2}{s_3^2}.$$

Structure equations:

$$[L_1, X] = 2L_2, \qquad [L_2, X] = -X^2 - 2L_1 + H - a,$$

$$[L_1, L_2] = -(L_1X + XL_1) - (\frac{1}{2} + 2a)X, \qquad R = [L_1, L_2]$$

$$0 = \{L_1, X^2\} + 2L_1^2 + 2L_2^2 - 2L_1H + \frac{5+4a}{2}X^2 - 2aL_1 - a.$$

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Nondegenerate flat space systems: $H\Psi = (\partial_x^2 + \partial_y^2 + V)\Psi = E\Psi.$

1. $E1$: $V = \alpha(x^2+y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}$, (2) $E2$: $V = \alpha(4x^2+y^2) + \beta x + \frac{\gamma}{y^2}$,
3. $E3'$: $V = \alpha(x^2+y^2) + \beta x + \gamma y$,
4. $E7$: $V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-b}} + \frac{\beta(x-iy)}{\sqrt{(x+iy)^2-b}\,\left(x+iy+\sqrt{(x+iy)^2-b}\,\right)^2} + \gamma(x^2+y^2)$,
5. $E8$ $V = \frac{\alpha(x-iy)}{(x+iy)^3} + \frac{\beta}{(x+iy)^2} + \gamma(x^2+y^2)$,
6. $E9$: $V = \frac{\alpha}{\sqrt{x+iy}} + \beta y + \frac{\gamma(x+2iy)}{\sqrt{x+iy}}$,
7. $E10$: $V = \alpha(x-iy) + \beta(x+iy-\frac{3}{2}(x-iy)^2) + \gamma(x^2+y^2-\frac{1}{2}(x-iy)^3)$,
8. $E11$: $V = \alpha(x-iy) + \frac{\beta(x-iy)}{\sqrt{x+iy}} + \frac{\gamma}{\sqrt{x+iy}}$,
9. $E15$: $V = f(x-iy)$,
10. $E16$: $V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \frac{\beta}{y+\sqrt{x^2+y^2}} + \frac{\gamma}{y-\sqrt{x^2+y^2}}\right)$,
11. $E17$: $V = \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta}{(x+iy)^2} + \frac{\gamma}{(x+iy)\sqrt{x^2+y^2}}$,
12. $E19$: $V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-4}} + \frac{\beta}{\sqrt{(x-iy)(x+iy+2)}} + \frac{\gamma}{\sqrt{(x-iy)(x+iy-2)}}$.
13. $E20$: $V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \beta\sqrt{x+\sqrt{x^2+y^2}} + \gamma\sqrt{x-\sqrt{x^2+y^2}}\right)$,

Nondegenerate systems on the complex 2-sphere:

$H\Psi = (J_{23}^2 + J_{13}^2 + J_{12}^2 + V)\Psi = E\Psi, \quad J_{k\ell} = s_k\partial_{s_\ell} - s_\ell\partial_{s_k}, \quad s_1^2 + s_2^2 + s_3^2 = 1.$

Here,

1. $S1$: $V = \frac{\alpha}{(s_1+is_2)^2} + \frac{\beta s_3}{(s_1+is_2)^2} + \frac{\gamma(1-4s_3^2)}{(s_1+is_2)^4}$,
2. $S2$: $V = \frac{\alpha}{s_3^2} + \frac{\beta}{(s_1+is_2)^2} + \frac{\gamma(s_1-is_2)}{(s_1+is_2)^3}$,
3. $S4$: $V = \frac{\alpha}{(s_1+is_2)^2} + \frac{\beta s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\gamma}{(s_1+is_2)\sqrt{s_1^2+s_2^2}}$,
4. $S7$: $V = \frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\beta s_1}{s_2^2\sqrt{s_1^2+s_2^2}} + \frac{\gamma}{s_2^2}$,
5. $S8$: $V = \frac{\alpha s_2}{\sqrt{s_1^2+s_3^2}} + \frac{\beta(s_2+is_1+s_3)}{\sqrt{(s_2+is_1)(s_3+is_1)}} + \frac{\gamma(s_2+is_1-s_3)}{\sqrt{(s_2+is_1)(s_3-is_1)}}$,
6. $S9$: $V = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2}$,

Darboux 1 systems: $H\Psi = \left(\frac{1}{4x}(\partial_x^2+\partial_y^2)+V\right)\Psi = E\Psi$. (Winternitz et. al., 2002)

1. $D1A$: $V = \frac{b_1(2x-2b+iy)}{x\sqrt{x-b+iy}} + \frac{b_2}{x\sqrt{x-b+iy}} + \frac{b_3}{x} + b_4,$
2. $D1B$: $V = \frac{b_1(4x^2+y^2)}{x} + \frac{b_2}{x} + \frac{b_3}{xy^2} + b_4,$
3. $D1C$ $V = \frac{b_1(x^2+y^2)}{x} + \frac{b_2}{x} + \frac{b_3 y}{x} + b_4.$

Darboux 2 systems: $H\Psi = \left(\frac{x^2}{x^2+1}(\partial_x^2+\partial_y^2)+V\right)\Psi = E\Psi.$

1. $D2A$: $V = \frac{x^2}{x^2+1}\left(b_1(x^2+4y^2)+\frac{b_2}{x^2}+b_3 y\right)+b_4.$
2. $D2B$: $V = \frac{x^2}{x^2+1}\left(b_1(x^2+y^2)+\frac{b_2}{x^2}+\frac{b_3}{y^2}\right)+b_4,$
3. $D2C$: $V = \frac{x^2}{\sqrt{x^2+y^2}(x^2+1)}\left(b_1+\frac{b_2}{y+\sqrt{x^2+y^2}}+\frac{b_3}{y-\sqrt{x^2+y^2}}\right)+b_4,$

Darboux 3 systems: $H\Psi = \left(\frac{1}{2}\frac{e^{2x}}{e^x+1}(\partial_x^2+\partial_y^2)+V\right)\Psi = E\Psi.$

1. $D3A$: $V = \frac{b_1}{1+e^x} + \frac{b_2 e^x}{\sqrt{1+2e^{x+iy}}(1+e^x)} + \frac{b_3 e^{x+iy}}{\sqrt{1+2e^{x+iy}}(1+e^x)} + b_4,$
2. $D3B$: $V = \frac{e^x}{e^x+1}\left(b_1 + e^{-\frac{x}{2}}(b_2\cos\frac{y}{2} + b_3\sin\frac{y}{2})\right) + b_4,$
3. $D3C$: $V(= \frac{e^x}{e^x+1}\left(b_1 + e^x\left(\frac{b_2}{\cos^2\frac{y}{2}} + \frac{b_3}{\sin^2\frac{y}{2}}\right)\right) + b_4,.$
4. $D3D$: $V = \frac{e^{2x}}{1+e^x}(b_1 e^{-iy} + b_2 e^{-2iy}) + \frac{b_3}{1+e^x} + b_4.$

Darboux 4 systems: $H\Psi = \left(-\frac{\sin^2 2x}{2\cos 2x+b}(\partial_x^2 + \partial_y^2) + V\right)\Psi = E\Psi.$

1. $D4(b)A$: $V = \frac{\sin^2 2x}{2\cos 2x+b}\left(\frac{b_1}{\sinh^2 y} + \frac{b_2}{\sinh^2 2y}\right) + \frac{b_3}{2\cos 2x+b} + b_4,$
2. $D4(b)B$: $V = \frac{\sin^2 2x}{2\cos 2x+b}\left(\frac{b_1}{\sin^2 2x} + b_2 e^{4y} + b_3 e^{2y}\right) + b_4.$
3. $D4(b)C$: $V = \frac{e^{2y}}{\frac{b+2}{\sin^2 x}+\frac{b-2}{\cos^2 x}}\left(\frac{b_1}{Z+(1-e^{2y})\sqrt{Z}} + \frac{b_2}{Z+(1+e^{2y})\sqrt{Z}} + \frac{b_3\ e^{-2y}}{\cos^2 x}\right) + b_4.$

Generic Koenigs spaces:

1. $K[1,1,1,1]$: $H\Psi = \frac{1}{V(b_1,b_2,b_3,b_4)}\left(\partial_x^2 + \partial_y^2 + V(a_1,a_2,a_3,a_4)\right)\Psi = E\Psi,$

   $V(a_1,a_2,a_3,a_4) = \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2+y^2-1)^2} - \frac{4a_4}{(x^2+y^2+1)^2},$
2. $K[2,1,1]$: $H\Psi = \frac{1}{V(b_1,b_2,b_3,b_4)}\left(\partial_x^2 + \partial_y^2 + V(a_1,a_2,a_3,a_4)\right)\Psi = E\Psi,$

   $V(a_1,a_2,a_3,a_4) = \frac{a_1}{x^2} + \frac{a_2}{y^2} - a_3(x^2+y^2) + a_4,$
3. $K[2,2]$: $H\Psi = \frac{1}{V(b_1,b_2,b_3,b_4)}\left(\partial_x^2 + \partial_y^2 + V(a_1,a_2,a_3,a_4)\right)\Psi = E\Psi,$

   $V(a_1,a_2,a_3,a_4) = \frac{a_1}{(x+iy)^2} + \frac{a_2(x-iy)}{(x+iy)^3} + a_3 - a_4(x^2+y^2),$

Generic Koenigs spaces:

1. $K[3,1]$: $H\Psi = \frac{1}{V(b_1,b_2,b_3,b_4)}\left(\partial_x^2 + \partial_y^2 + V(a_1,a_2,a_3,a_4)\right)\Psi = E\Psi$,

   $V(a_1,a_2,a_3,a_4) = a_1 - a_2 x + a_3(4x^2 + y^2) + \frac{a_4}{y^2}$,
2. $K[4]$: $H\Psi = \frac{1}{V(b_1,b_2,b_3,b_4)}\left(\partial_x^2 + \partial_y^2 + V(a_1,a_2,a_3,a_4)\right)\Psi = E\Psi$,

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   $a_1 - a_2(x+iy) + a_3\left(3(x+iy)^2 + 2(x-iy)\right) - a_4\left(4(x^2+y^2) + 2(x+iy)^3\right)$,
3. $K[0]$: $H\Psi = \frac{1}{V(b_1,b_2,b_3,b_4)}\left(\partial_x^2 + \partial_y^2 + V(a_1,a_2,a_3,a_4)\right)\Psi = E\Psi$,

   $V(a_1,a_2,a_3,a_4) = a_1 - (a_2 x + a_3 y) + a_4(x^2 + y^2)$,

Laplace systems 1

All these systems can be treated more conveniently as Laplace equations. Since every 2D manifold is conformally flat, there always exist "Cartesian-like" coordinates x, y such that $H = \frac{1}{\lambda(x,y)}(\partial_x^2 + \partial_y^2) + V(x, y)$.

Thus the Helmholtz equation $H\Psi = E\Psi$ on some conformally flat space is equivalent to the Laplace equation (with potential)

$$(\partial_x^2 + \partial_y^2 + \tilde{V}(x, y))\Psi = 0$$

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Laplace systems 2

More generally, we consider Laplace systems of the form

$$H\,\Psi(\mathbf{x}) \;\equiv\; (\,\Delta_2 + V(\mathbf{x})\,)\,\Psi(\mathbf{x}) \;=\; 0\,,$$

where Δ_2 is the Laplace-Beltrami operator on a $2D$ Riemannian or pseudo-Riemannian manifold. All variables can be complex. A *conformal symmetry* of this equation is a partial differential operator L such that $[L, H] \equiv LH - HL = R_L H$ for some differential operator R_L. A conformal symmetry maps any solution Ψ to another solution. Two conformal symmetries L, L' are identified if $L = L' + S\,H$ for some differential operator S, since they agree on the solution space.

The system is *conformally superintegrable* if there exist three algebraically independent conformal symmetries, $L_1,\ L_2,\ L_3$ with $L_3 = H$. It is second order conformally superintegrable if each L_2 can be chosen to be a 2nd order differential operator, and L_1 of at most 2nd order.

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Laplace systems and Stäckel transform

The mapping of a Helmholtz superintegrable system $H\Psi = E\Psi$ to the Laplace equation preserves superintegrability, i. e., the Laplace system is conformally superintegrable.

Suppose we have a second order conformal superintegrable system

$$H = \partial_{xx} + \partial_{yy} + V(x,y) = 0, \quad H = H_0 + V$$

where $V(x,y) = W(x,y) - E\,U(x,y)$ for arbitrary parameter E. The potential U defines a *conformal Stäckel transform* to the (Helmholtz) system

$$\tilde{H}\Psi = E\Psi, \quad \tilde{H} = \frac{1}{U}(\partial_{xx} + \partial_{yy}) + \tilde{V}$$

where $\tilde{V} = \frac{W}{U}$. and this Helmholtz system is superintegrable.

There is a similar definition of Stäckel transforms of Helmholtz superintegrable systems $H\Psi = E\Psi$ which take superintegrable systems to superintegrable systems, essentially preserving the quadratic algebra structure. Thus any second order conformal Laplace superintegrable system admitting a nonconstant potential U can be Stäckel transformed to a Helmholtz superintegrable system.

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Suppose we have a second order conformal superintegrable system

$$H = \partial_{xx} + \partial_{yy} + V(x,y) = 0, \quad H = H_0 + V$$

where $V(x,y) = W(x,y) - E\,U(x,y)$ for arbitrary parameter E. The potential U defines a *conformal Stäckel transform* to the (Helmholtz) system

$$\tilde{H}\Psi = E\Psi, \quad \tilde{H} = \frac{1}{U}(\partial_{xx} + \partial_{yy}) + \tilde{V}$$

where $\tilde{V} = \frac{W}{U}$. and this Helmholtz system is superintegrable.

There is a similar definition of Stäckel transforms of Helmholtz superintegrable systems $H\Psi = E\Psi$ which take superintegrable systems to superintegrable systems, essentially preserving the quadratic algebra structure. Thus any second order conformal Laplace superintegrable system admitting a nonconstant potential U can be Stäckel transformed to a Helmholtz superintegrable system.

Laplace System	Non-degenerate potentials $V(x,\, y)$
[1111]	$\frac{a_1}{x^2}+\frac{a_2}{y^2}+\frac{4\,a_3}{(x^2+y^2-1)^2}-\frac{4\,a_4}{(x^2+y^2+1)^2}$
[211]	$\frac{a_1}{x^2}+\frac{a_2}{y^2}-a_3\,(x^2+y^2)+a_4$
[22]	$\frac{a_1}{(x+i\,y)^2}+\frac{a_2\,(x-i\,y)}{(x+i\,y)^3}+a_3-a_4\,(x^2+y^2)$
[31]	$a_1-a_2\,x+a_3\,(4\,x^2+y^2)+\frac{a_4}{y^2}$
[4]	$a_1-a_2\,(x+i\,y)+a_3\,(3(x+i\,y)^2+2(x-i\,y))-a_4\,(4(x^2+y^2)+2(x+i\,y)^3)$
[0]	$a_1-(a_2\,x+a_3\,y)+a_4\,(x^2+y^2)$

Each of the Helmholtz nondegenerate superintegrable (i.e. 3-parameter) eigenvalue systems is Stäckel equivalent to exactly one of these Laplace systems $H\Psi\equiv(\partial_x^2+\partial_y^2+V(x,y))\Psi=0$

System	Degenerate potentials $V(x,\ y)$
A	$\frac{4\,a_3}{(x^2+y^2-1)^2} - \frac{4\,a_4}{(x^2+y^2+1)^2}$
B	$\frac{a_1}{x^2} + a_4$
C	$a_3 - a_4\,(x^2+y^2)$
D	$a_1 - a_2\,x$
E	$\frac{a_1}{(x+i\,y)^2} + a_3$
F	$a_1 - a_2\,(x+i\,y)$

Table : Each of the Helmholtz degenerate superintegrable (i.e. 1-parameter) eigenvalue systems is Stäckel equivalent to exactly one of these Laplace systems $H\Psi \equiv (\partial_x^2 + \partial_y^2 + V(x,y))\Psi = 0$

Böcher contractions

All Laplace conformally superintegrable systems can be obtained as limits of the basic system [1111]. The conformal symmetry algebra of the underlying flat space free Laplace equation is $so(4, \mathbb{C})$, and these limits are described by Lie algebra contractions of this conformal algebra to itself, which can be classified. We call these Bôcher contractions since they are motivated by ideas of Bôcher,(1894), who used similar limits to construct separable coordinates of free Laplace, wave and Helmholtz equations from basic cyclidic coordinates.

There are 4 basic Bôcher contractions of 2d Laplace systems and each one when applied to a Laplace system yields another Laplace superintegrable system. These in turn induce contractions of the Helmholtz systems in each equivalence class to Helmholtz systems in other classes, over 200 contractions in all. However, we can summarize the basic results for Laplace systems in Figures 1 and 2. A system can be obtained from another superintegrable system via contraction provided it is connected to the other system by directed arrows.

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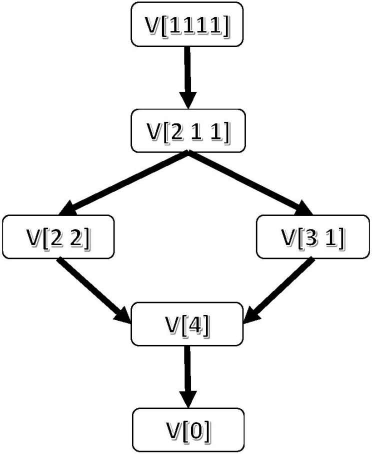

Figure : Contractions of nondegenerate Laplace systems

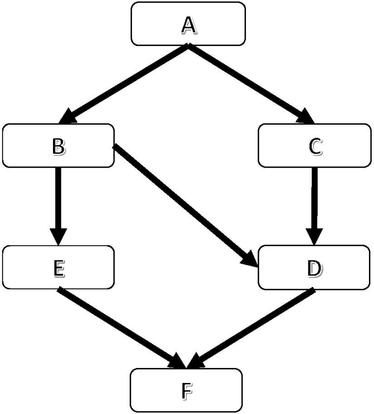

Figure : Contractions of degenerate Laplace systems

Multiseparability of Laplace equations

The last crucial bit of information about these Laplace and associated Helmholtz superintegrable systems is that they are multiseparable, Each family of separated solutions is characterized as the family of eigenfunctions of a 2nd order symmetry operator. Each family determines an eigenbasis of separated solutions of the 2D superintegrable system. An eigenbasis of one family can be expanded in terms of a eigenbasis for another family and the quadratic structure algebras help to derive the expansion coefficients.

The compete list of separation equations follows.. (The notation (2) means that the separation equations for the corresponding coordinates are both of the same, except that the separation constant occurs as c in one equation and $-c$ in the other.)

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Non-degenerate Systems

System	Separable coordinates and separation equations		
V[1111]		**Spherical** Hypergeometric eqn. (2)	**Ellipsoidal** Heun eqn. (2)
V[2 1 1]	**Cartesian** Confluent hypergeometric eqn. (2)	**Polar** Confluent hypergeometric eqn. Hypergeometric eqn.	**Elliptic** Confluent Heun eqn. (2)
V[2 2]		**Polar** Confluent hypergeometric eqn. (2)	**Hyperbolic** Double-confluent Heun eqn. (2)
V[3 1]		**Cartesian** Confluent hypergeometric eqn. Parabolic cylinder eqn.	**Parabolic** Bi-confluent Heun eqn. (2)
V[4]			**Semi-hyperbolic** Tri-confluent Heun eqn. (2)
V[0]		**Cartesian** Parabolic cylinder eqn. (2)	

Figure : Separation equations for nondegenerate Laplace systems

Degenerate Systems

System	Separable coordinates and separation equations			
A		**Spherical** Gegenbauer eqn. Hypergeometric eqn.		**Elliptic** Special Heun eqn. (2)
B	**Cartesian** Bessel's eqn. Exponential	**Polar** Bessel's eqn. Confluent hypergeometric eqn.	**Parabolic** Confluent hypergeometric eqn. (2)	**Elliptic** Spheroidal wave eqn. (2)
C	**Cartesian** Parabolic cylinder eqn. (2)	**Polar** Confluent hypergeometric eqn. Exponential		**Hyperbolic** Confluent Heun eqn. (2)
D		**Cartesian** Airy eqn. Exponential	**Parabolic** Special tri-confluent Heun eqn. (2)	**Elliptic** Special confluent Heun eqn. (2)
E	**Polar** Bessel's eqn. (2)	**Hyperbolic** Special double-confluent Heun eqn. (2)		**Elliptic** Special confluent Heun eqn. (2)
F	**Cartesian** Airy eqn. (2)		**Semi-hyperbolic** Parabolic cylinder eqn. (2)	

Figure : Separation equations for degenerate Laplace systems

Hypergeometric type separation equations

1. Hypergeometric equation: $z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\,\frac{dw}{dz} - abw = 0.$
2. Confluent hypergeometric equation: $z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0.$
3. Parabolic cylinder equation: $\frac{d^2w}{dz^2} + (az^2 + bz + c)w = 0.$
4. Gegenbauer equation: $(1-z^2)\frac{d^2w}{dz^2} - 2(\mu+1)z\frac{dw}{dz} + (\nu-\mu)(\nu+\mu+1)w = 0.$
5. Bessel's equation: $z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2-\nu^2)w = 0.$
6. Airy's equation: $\frac{d^2w}{dz^2} - zw = 0.$

Heun separation equations

1. Heun equation: $\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{dw}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} w = 0,$
$$\alpha + \beta + 1 = \gamma + \delta + \epsilon.$$
2. Confluent Heun equation: $\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{dw}{dz} + \frac{\alpha z - q}{z(z-1)} w = 0.$
3. Doubly-confluent Heun equation: $\frac{d^2w}{dz^2} + \left(\frac{\delta}{z^2} + \frac{\gamma}{z} + 1\right)\frac{dw}{dz} + \frac{\alpha z - q}{z^2} w = 0.$
4. Biconfluent Heun equation: $\frac{d^2w}{dz^2} - \left(\frac{\gamma}{z} + \delta + z\right)\frac{dw}{dz} + \frac{\alpha z - q}{z} w = 0.$
5. Triconfluent Heun equation: $\frac{d^2w}{dz^2} + (\gamma + z)z\frac{dw}{dz} + (\alpha z - q)w = 0.$
6. Spheroidal wave equation:
$\frac{d}{dz}\left((1-z^2)\frac{dw}{dz}\right) + \left(\lambda + \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\right) w = 0.$

Special functions and Bôcher contractions

Special functions associated with these systems arise in two distinct ways:

- As separable eigenfunctions of the quantum Hamiltonian. Second order superintegrable systems are multiseparable.
- As interbasis expansion coefficients relating distinct separable coordinate eigenbases. These are often solutions of difference equations.

Most of the special functions in the DLMF appear one of these ways.

Example: Consider the Helmholtz system $S9$. The eigenfunctions of symmetry operator L_1 correspond to spherical coordinates lined up on the $y-z$ axis; those of operator L_2 also correspond to spherical coordinates but lined up on the $x-z$ axis. The expansion coefficients of L_2 eigenfunctions in terms of the L_1 eigenbasis are the Racah and Wilson polynomials in full generality.

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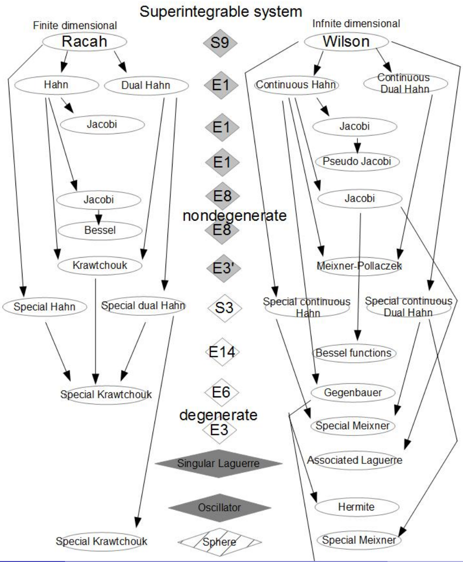
Special functions 2

Böcher contractions of $S9$ to other superintegrable systems induce limits of these expansion coefficients to expansion coefficients for the contracted superintegrable systems. Thus, a contraction of $S9$ to $E1$, (the Smorodinski-Winternitz superintegrable system), yields Hahn polynomials as limits of Wilson polynomials. The result of this is a reinterpretation of the Askey Scheme relating the possible hypergeometric orthogonal polynomials via limits.

Special functions 3



Exact and Quasi-exact solvability

Let $H = \frac{d^2}{dx^2} + V(x)$. We are concerned with the 1D eigenvalue problem $H\Psi = E\Psi$. The operator H is said to be **exactly solvable**, **(ES)** if there exists an infinite flag of subspaces of the domain of H: $\mathcal{P}_N$, $N = 1, 2, 3, \cdots$, such that $n_N = \dim\mathcal{P}_N \to \infty$ as $N \to \infty$ and $H\mathcal{P}_N \subseteq \mathcal{P}_N \subseteq \mathcal{P}_{N+1}$ for any N. In this case, for each subspace $\mathcal{P}_N$ the n_N eigenvalues and eigenfunctions of H can be obtained by pure algebraic means.

This is roughly equivalent to saying the eigenfunctions are hypergeometric.

The operator H is called **quasi-exactly solvable**, **(QES)** if there exist a *single* subspace $\mathcal{P}_k$ of dimension $n_k > 0$ such that $H\mathcal{P}_k \subseteq \mathcal{P}_k$. In this case, again we can find n_k eigenvalues and eigenfunctions of $\mathcal{H}$ by algebraic means, but we have no information about the remaining eigenvalues and eigenfunctions.

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Motivating example: A QES equation

Anharmonic oscillator with 6th order potential term:

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + [\frac{k_1^2}{8\omega^2} - (2n + \frac{3}{2})\omega]x^2 + \frac{k_1}{2}x^4 + \frac{\omega^2}{2}x^6.$$

For n a fixed positive integer, there are $n+1$ eigenfunctions

$$\Psi_i = P_n^{(i)}(x)e^{-\frac{k_1}{4\omega}x^2 - \frac{\omega}{2}x^6},$$

$i = 0, 1, \cdots, n$ where P is a polynomial of order at most n in x.

Similar examples studied by Turbiner, Schiffman, Ushveridze, Gonzales-Lopez, Olver, ...

Superintegrable explanation

The singular anisotropic oscillator potential, a Stäckel transform of system [31].

$$V_1(x,y) = \frac{1}{2}\omega^2(4x^2+y^2) + k_1 x + \frac{k_2^2 - \frac{1}{4}}{2y^2}$$

The Schrödinger equation has the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi + \left[2E - \omega^2(4x^2+y^2) - 2k_1 x - \frac{k_2^2 - \frac{1}{4}}{y^2}\right]\Psi = 0.$$

The Schrödinger equation separates in two systems: *Cartesian and parabolic coordinates*.

Cartesian separation

Separation of variables in Cartesian coordinates leads to the two *independent* one-dimensional Schrödinger equations

$$\frac{d^2\psi_1}{dx^2} + (2\lambda_1 - 4\omega^2 x^2 - 2k_1 x)\psi_1 = 0.$$

$$\frac{d^2\psi_2}{dy^2} + \left(2\lambda_2 - \omega^2 y^2 - \frac{k_2^2 - \frac{1}{4}}{y^2}\right)\psi_2 = 0.$$

where

$$\Psi(x, y; k_1, k_2) = \psi_1(x; k_1)\psi_2(y; k_2)$$

and λ_1, λ_2 are *Cartesian separation constants* with $\lambda_1 + \lambda_2 = E$.

1st Cartesian separation equation

The first equation represents the well-known linear singular oscillator system. The complete set of orthonormalized eigenfunctions, (on 1/2) in the interval $0 < y < \infty$ can be expressed in terms of finite confluent hypergeometric series or Laguerre polynomials

$$\psi_{n_2}(y;k_2) = \sqrt{\frac{2\omega^{(1+k_2)} n_2!}{\Gamma(n_2+k_2+1)}}\, y^{\frac{1}{2}+k_2}\, e^{-\frac{1}{2}\omega y^2}\, L_{n_2}^{k_2}(\omega y^2)$$

where $\lambda_2 = \omega(2n_2 + 1 + k_2)$.

2nd Cartesian separation equation

The second equation easily transforms to the ordinary one-dimensional oscillator problem. In terms of Hermite polynomials the orthonormal solutions are

$$\psi_{n_1}(x;k_1)=\left(\frac{2\omega}{\pi}\right)^{1/4}\frac{e^{-\omega z^2}}{\sqrt{2^{n_1}n_1!}}H_{n_1}(\sqrt{2\omega}z),$$

where $z=x+\frac{k_1}{4\omega^2}$, and $\lambda_1=\omega(2n_1+1)-\frac{k_1^2}{8\omega^2}$.

Energy spectrum

$$E = \lambda_1 + \lambda_2 = \omega[2n + 2 + k_2] - \frac{k_1^2}{8\omega^2}, \qquad n = n_1 + n_2 = 0, 1, 2, ...$$

The degree of degeneracy for fixed principal quantum number n is $(n + 1)$. The separation of variables in Cartesian coordinates leads to two **exactly solvable** one-dimensional Schrödinger equations.

Parabolic separation

Parabolic coordinates ξ and η are connected with the Cartesian x and y by

$$x = \frac{1}{2}(\xi^2 - \eta^2), \qquad y = \xi\eta, \qquad \xi \in \mathbf{R},\ \eta > 0.$$

The Schrödinger equation in parabolic coordinates is

$$\frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2}\right) +$$

$$\left[2E - \omega^2(\xi^4 - \xi^2\eta^2 + \eta^4) - k_1(\xi^2 - \eta^2) - \frac{k_2^2 - \frac{1}{4}}{\xi^2\eta^2}\right]\Psi = 0.$$

Parabolic separation equations

Upon substituting

$$\Psi(\xi,\eta) = X(\xi)Y(\eta)$$

and introducing the parabolic separation constant λ, we find the two separation equations:

$$\frac{d^2X}{d\xi^2} + \left(2E\xi^2 - \omega^2\xi^6 - k_1\xi^4 - \frac{k_2^2 - \frac{1}{4}}{\xi^2}\right)X = -\lambda X,$$

$$\frac{d^2Y}{d\eta^2} + \left(2E\eta^2 - \omega^2\eta^6 + k_1\eta^4 - \frac{k_2^2 - \frac{1}{4}}{\eta^2}\right)Y = +\lambda Y.$$

Substituting $E = \lambda_1 + \lambda_2 = \omega[2n + 2 + k_2] - \frac{k_1^2}{8\omega^2}$, in either of these equations we get the QES equation for the anharmonic oscillator with 6th order potential term, where now the energy is the separation constant, $\pm\lambda$.

QES $\Leftrightarrow$ 2nd order superintegrable systems

Kalnins, Miller and Pogosyan showed that there is a general relation between QES systems in 1D and 2nd order superintegrability in *n*D.

In two recent papers, Turbiner has studied and reported on the classification of QES systems in 1D. His emphasis is on QES systems that are special cases of the Heun equation and its confluent forms, and exactly solvable systems which are special cases of the hypergeometric equation.

We see now that all of these systems correspond to separation equations for the 2D 2nd order superintegrable systems as given here. Thus all of these solutions determine solutions of the 2D superintegrable systems.

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Another QES example. 1

Some special cases of Heun equations reduce to hypergeometric equations, see the impressive work of Maier. Moreover, in recent papers it has been shown that some special QES cases of the Heun equations have explicit solutions that are expressible in terms of derivatives of hypergeometric functions.

We can observe that all such special solutions lead to eigenfunctions of 2D superintegrable systems which also have separable ES hypergeometric eigenfunctions. The quadratic algebras of the 2D systems allow us to relate the QES and ES systems. Moreover a knowledge of the possible ES systems for a 2nd order superintegrable system gives important clues about the structure of the QES systems

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Another QES example. 2

Consider the inverse square root system written in the form

$$\frac{d^2 f(x)}{dx^2} + (\frac{a}{x^{1/2}} + \frac{b}{x} + \frac{c}{x^{3/2}} - E)f(x).$$

With $y = \sqrt{x}$ we have

$$y\frac{d^2 f(y)}{dy^2} - \frac{df(y)}{dy} + (4ay^2 + 4by + 4c - 4Ey^3)f(y).$$

The superintegrable system $E2$, in Cartesian coordinates y_1, y_2 is

$$\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + (-A(4y_1^2 + y_2^2) + By_1 + \frac{C}{y_2^2} - E')\right) f(y_1, y_2) = 0. \qquad (1)$$

It belongs to the [31] Laplace equivalence class. This system is separable in two coordinate systems: Cartesian and parabolic.

Another QES example. 3

In Cartesian coordinates the separable solutions $f(y_1, y_2) = g_1(y_1)g_2(y_2)$ are eigenfunctions $L_1 f = \lambda_1 f$ of the symmetry operator

$$L_1 = \partial_{y_1^2} + (-4Ay_1^2 + By_1),$$

and the separation equations are

$$\frac{d^2 g_1}{dy_1^2} + (-4Ay_1^2 + By_1 + \lambda_1 - E')g_1 = 0,$$

$$\frac{d^2 g_2}{dy_2^2} + (-Ay_2^2 + \frac{C}{y_2^2} - \lambda_1)g_2 = 0.$$

Here, λ_1 is the separation constant.

Another QES example. 4

In parabolic coordinates η, ξ with $y_1 = \xi + \eta, y_2 = 2i\sqrt{\xi\eta}$, and writing $f(\eta, \xi) = (\eta\xi)^{-3/4} f_1(\eta) f_2(\xi)$, we obtain the bi-confluent Heun separation equation

$$\eta \frac{d^2 f_1(\eta)}{d\eta^2} - \frac{df_1(\eta)}{d\eta} + \left(\frac{15}{16\eta} + \frac{C}{4\eta} - E'\eta + B\eta^2 - 4A\eta^3 + \lambda_2 \right) f_1(\eta) = 0,$$

for $f_1(\eta)$ with a similar equation for $f_2(\xi)$ Now note that, with the restriction to the superintegrable system $E2$ with $C = -15/4$ in the potential, this becomes the inverse square potential system, provided we make the identifications

$$y = \eta, \quad 4b = -E', \quad 4a = B, \quad 4c = \lambda_2, \quad E = A.$$

Thus this 1D inverse square root potential system corresponds to a special case of the separation equation for the 2D superintegrable system $E2$ in parabolic coordinates. This separation equation is QES, but we will show that it has explicit solutions which are a consequence of the exactly solvable $E2$ separation equation in Cartesian coordinates.

Another QES example. 5

We can solve the ES Cartesian coordinate system directly to get $g_1(y) = \exp(-y(-a + Ey)/\sqrt{E})\ G(y)$ where $G(y)$ is an arbitrary linear combination of

$$ {}_1F_1\left(\begin{matrix} \frac{1}{8}\left(\frac{2E^{3/2}+(4b-\lambda_1)E-a^2}{E^{3/2}}\right) \\ 1/2 \end{matrix} ;\ \frac{(2Ey-a)^2}{2E^{3/2}} \right) $$

and

$$ {}_1F_1\left(\begin{matrix} \frac{1}{8}\left(\frac{6E^{3/2}+(4b-\lambda_1)E-a^2}{E^{3/2}}\right) \\ 3/2 \end{matrix} ;\ \frac{(2Ey-a)^2}{2E^{3/2}} \right). $$

Here $G(y)$ is the general solution of the equation $S_1\ G(y) = 0$, equivalent to the above where

$$ S_1 = \frac{d^2}{dy^2} - \frac{2(2Ey-a)}{\sqrt{E}}\frac{d}{dy} - \frac{(2E^{3/2} + 4bE - \lambda_1 E - a^2)}{E}. $$

Another QES example. 6

Assuming $b=-4c^2, \lambda_1=-2\sqrt{E}-32c^2$, the equation for the inverse square root system is equivalent to $S_2 f=0$ where

$$S_2 = y\frac{d^2}{dy^2} - \frac{d}{dy} + 4(y^2 a - 4c^2 y + c - Ey^3).$$

We define operators K and Q by

$$K = \exp(\frac{ay}{\sqrt{E}} - y^2\sqrt{E})\left(\frac{d}{dy} - 4y\sqrt{E} + 4c + \frac{a}{\sqrt{E}}\right),$$

$$Q = \exp(\frac{ay}{\sqrt{E}} - y^2\sqrt{E})\left(y\frac{d}{dy} - 4y^2\sqrt{E} + 4cy + \frac{ay}{\sqrt{E}} - 1\right).$$

Then it is straightforward to verify the operator identity

$$S_2 K = QS_1.$$

This shows that K maps the solution space of the restricted Cartesian separation equation to the solution space of the restricted parabolic separation equation and provides explicit solutions for the 1D inverse square root potential.

Conclusions. 1

We have reviewed the theory of 2D 2nd order superintegrable Laplace systems and shown that they encode all the information about 2D Helmholtz or time-independent Schrödinger superintegrable systems in an efficient manner: there is a 1-1 correspondence between Laplace superintegrable systems and Stäckel equivalence classes of Helmholtz superintegrable systems.

The separation equations comprise all of the various types of hypergeometric and Heun equations in full generality. In particular, they coincide with all of the 1D Schrödinger exactly solvable (ES) and quasi-exactly solvable (QES) systems related to the Heun operator.

The separable solutions of these equations are the special functions of mathematical physics. The different systems are related by Stäckel transforms, by their symmetry algebras and by Bôcher contractions of the conformal algebra $so(4,\mathbb{C})$ to itself, which enables all of these systems to be derived from a single one: the generic potential on the complex 2-sphere.

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Distinct separable bases for a single Laplace system are related by interbasis expansion coefficients which are themselves special functions, such as the Wilson polynomials. Applying Böcher contractions to expansion coefficients for ES systems one can derive the Askey scheme for hypergeometric orthogonal polynomials.

This approach facilitates a unified view of special function theory, incorporating hypergeometric and Heun functions in full generality.

All of our considerations generalize to 2nd order superintegrable systems in 3D and higher dimensions.

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