LIE ALGEBRAS AND GENERALIZATIONS OF HYPERGEOMETRIC FUNCTIONS

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We show how Lie algebras can be employed to study ${}_2F_1$ and its generalizations. We use the differential recurrence relations obeyed by a family of hypergeometric functions to generate a Lie algebra whose action determines basic properties of the corresponding functions.

For the ${}_pF_q$ we introduce functions and operators

$$\begin{aligned} &f_{\alpha_j\beta_k}(t_j, u_k, x) = {}_pF_q\left({\alpha_j \atop \beta_k} \,\middle|\, x\right) t_1^{\alpha_1}\cdots t_p^{\alpha_p} u_1^{\beta_1}\cdots u_q^{\beta_q}, \\ &E_{\alpha_l} = t_l(x\partial_x + t_l\partial_{t_l}), \qquad E_{-\beta_s} = u_s^{-1}(x\partial_x + u_s\partial_{u_s} - 1), \qquad 1 \leqq l \leqq p, \\ &E_{\alpha_1\cdots\beta_q} = t_1\cdots u_q\partial_x, \qquad T_l = t_l\partial_{t_l}, \qquad U_s = u_s\partial_{u_s}, \qquad 1 \leqq s \leqq q, \end{aligned}$$

obtained from the recurrence formulas for ${}_pF_q$. The operators generate a Lie algebra $\mathcal{G}_{p,q}$ of dimension $2(p+q)+1$ and the $f_{\alpha_j\beta_k}$ form bases for $\mathcal{G}_{p,q}$-representations. Let

$$L_{p,q} = E_{\alpha_1}\cdots E_{\alpha_p} - E_{\alpha_1\cdots\beta_q}E_{-\beta_1}\cdots E_{-\beta_q}.$$

THEOREM. *If* (1) $L_{p,q}f=0$, (2) $T_lf=\alpha_lf$, $1\leqq l\leqq p$, (3) $U_sf=\beta_sf$, $1\leqq s\leqq q$, *and* (4) *f analytic at $x=0$, then $f=cf_{\alpha_j\beta_k}$, c constant.*

THEOREM. *The null space of $L_{p,q}$ is invariant under $\mathcal{G}_{p,q}$.*

WEISNER'S PRINCIPLE. *If* (1) $L_{p,q}f=0$, (2) $f=\sum_{\alpha_j\beta_k} h_{\alpha_j\beta_k}(x)\, t_1^{\alpha_1}\cdots u_q^{\beta_q}$, (3) *$f$ analytic at $x=0$, and* (4) *$L_{p,q}$ can be applied term-by-term to the sum, then $h_{\alpha_j\beta_k}(x) = c_{\alpha_j\beta_k} f_{\alpha_j\beta_k}$, c a constant.*

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We can consider any analytic solution f of $L_{p,q}f=0$ as a generating function for the ${}_pF_q$ and use these theorems to determine the expansion coefficients. In practice f is characterized as a simultaneous eigenfunction of $p+q$ operators in the enveloping algebra of $\mathcal{G}_{p,q}$ [1].

By a simple transformation and change of variable we obtain $E_{\alpha_j}=\partial_{z_j}$, $E_{\beta_k}=\partial_{w_k}$, $E_{\alpha_1\cdots\beta_q}=\partial_{w_{q+1}}$,

$$(*) \qquad L_{p,q}f=(\partial_{z_1}\cdots\partial_{z_p}-\partial_{w_1}\cdots\partial_{w_{q+1}})f=0.$$

In addition to the $\mathcal{G}_{p,q}$ symmetries, permutation symmetries of equation $(*)$ are now apparent.

THEOREM. *If* $L_{p,q}f=0$, $L_{p',q'}f'=0$ *then* $L_{p+p',q+q'}(ff')=0$.

In special cases the symmetry algebra is larger:

	function	$Lf=0$	*algebra*	*dimension*	*reference*
1.	${}_2F_1$	$\Delta_4 f=0$	$sl(4)\cong o(6)$	15	[2], [3]
2.	${}_1F_1$	$\Delta_2 f=\partial_t f$		9	[2]
3.	D_ν	$\Delta_1 f=\partial_t f$		6	[4]
4.	${}_2F_1\left({-\alpha,\beta \atop 1-\alpha-\beta}\middle\vert\, x\right)$	$\Delta_3 f=0$	$o(5)$	10	[3].

Analogous results for Lauricella functions are ([2], [5]):

	function	$L_k f=0,\ 1\leq k\leq n$	*algebra*	*dimension*
5.	F_A	$(\partial u\partial u_k-\partial v_k\partial w_k)f=0$		$6n+2$
6.	F_B	$(\partial u_k\partial v_k-\partial w_k\partial w)f=0$		$6n+2$
7.	F_C	$(\partial u\partial v-\partial u_k\partial w_k)f=0$		$3n+4$
8.	F_D	$(\partial u\partial u_k-\partial v_k\partial v)f=0$	$sl(n+3)$	$(n+3)^2-1$

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