Math 1271
Solutions for Fall 2005 Final Exam

1) Since the equation $x^2 + y^2 = e^{xy}$ cannot be rearranged algebraically in order to write y as an explicit function of x, we must instead differentiate this relation implicitly with respect to x, in order to find an expression for y' :

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(e^{xy}) \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = \underbrace{\frac{d}{du}(e^u) \cdot \frac{du}{dx}}_{\text{setting } u = xy}$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = e^u \cdot \frac{d}{dx}(xy) \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = e^u \cdot (x' \cdot y + x \cdot y')$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = e^{xy} \cdot (1 \cdot y + x \cdot \frac{dy}{dx}) \Rightarrow 2y \cdot \frac{dy}{dx} - x \cdot e^{xy} \cdot \frac{dy}{dx} = y \cdot e^{xy} - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \cdot e^{xy} - 2x}{2y - x \cdot e^{xy}} \quad . \qquad \textbf{(A)}$$

2) The two parts of the Fundamental Theorem of (Integral) Calculus can be put together to state that, for a function $h(x)$ continuous on $[a, b]$, the definite integral $\int_a^x h(t)\,dt$, with $a < x < b$, equals $f(x) - f(a)$, where $f'(x) = h(x)$. If we modify the upper limit of the integration to make it a function $u(x)$ in the same interval, we then have $\int_a^{u(x)} h(t)\,dt = f(u(x)) - f(a)$. The derivative of this integral is then

$$\frac{d}{dx}\int_a^{u(x)} h(t)\,dt = \frac{d}{dx}\left[f(u(x)) - f(a)\right] = \frac{d}{dx}f(u(x)) - \frac{d}{dx}f(a)$$

$$= \frac{d}{du}f(u) \cdot \frac{du}{dx} - \underbrace{0}_{f(a) \text{ is a constant}} = \left.\frac{df}{dx}\right|_{u(x)} \cdot \frac{du}{dx} \quad or \quad h(u(x)) \cdot u'(x) \quad .$$

The integrand given in the problem is continuous everywhere (since it is defined everywhere), so the Fundamental Theorem and its consequences can be applied here. For the integral function defined by $\int_{x^2}^{10} \frac{\sin(e^t)}{t^2 + 1}\,dt$, we must first make an alteration to accommodate the conditions for the Theorem: we move the constant limit of integration to become the lower limit by “reversing the direction” of integration:

$$\int_{x^2}^{10} \frac{\sin(e^t)}{t^2+1}\,dt = -\int_{10}^{x^2} \frac{\sin(e^t)}{t^2+1}\,dt \quad .$$

With $h(x) = \dfrac{\sin(e^x)}{x^2+1}$ *and* $u(x) = x^2$, the derivative of our integral function is

$$\frac{d}{dx}\int_{x^2}^{10} \frac{\sin(e^t)}{t^2+1}\,dt = -h(u(x))\cdot u'(x) = -\frac{\sin(e^{x^2})}{(x^2)^2+1}\cdot (x^2)' = -2x\cdot\frac{\sin(e^{x^2})}{x^4+1} \quad .$$

(C)

3) There are many possible ways to choose bounds for the value of a definite integral; in view of the choices given, which involve integer results, we will want to look at something fairly simple to calculate. With the interval being $[\,1\,,\,2\,]$, the smallest value the integrand itself, $\sqrt{x^4 + 9\sin^2 x}$, could *possibly* have would be if the second term within the radical were zero, reducing the function to $\sqrt{x^4} = x^2$, which is equal to 1 at $x = 1$ (in fact, the "sine-squared" term is *not* zero there). With the width of the interval of integration being $\Delta x = 2 - 1 = 1$, we can definitely say that

$\int_1^2 \sqrt{x^4 + 9\sin^2 x}\,dx \ \geq\ 1\cdot\Delta x = 1$. At the other extreme, the largest value the integrand could possibly have would be when $\sin^2 x$ is as large as possible, making the function equal to $\sqrt{x^4 + 9\cdot 1} = \sqrt{x^4 + 9}$; this takes on its largest value on the interval at $x = 2$, where $\sqrt{x^4 + 9} = \sqrt{2^4 + 9} = \sqrt{16 + 9} = 5$. This result provides an upper bound for our integral of $\int_1^2 \sqrt{x^4 + 9\sin^2 x}\,dx \ \leq\ 5\cdot\Delta x = 5$. **(B)**

4) In some situations, finding a limit involving a multiply-staged composition function could be dealt with by using substitutions. Here, however, a substitution based on, say, $u = \sin t$, would not make the expression less complicated, nor would it eliminate the troublesome zero in the denominator.

A more useful approach is to apply L'Hôpital's Rule, along with the Chain Rule:

$$\lim_{t\to 0} \frac{\sin(\sin(\sin t))}{t} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{t\to 0} \frac{[\,\sin(\sin(\sin t))\,]'}{t'} = \lim_{t\to 0} \frac{\frac{d}{du}\sin(u)\cdot\frac{du}{dt}}{1} = \lim_{t\to 0} \cos(u)\cdot\frac{d}{dt}(\sin(\sin t))$$

setting $u = \sin(\sin t)$

(continued)

$$= \lim_{t \to 0} \cos(u) \cdot \frac{d}{dv}(\sin v) \cdot \frac{d}{dt}(\sin t) = \lim_{t \to 0} \cos(u) \cdot \cos(v) \cdot \cos t$$

setting $v = \sin t$

$$= \lim_{t \to 0} \cos(\sin(\sin t)) \cdot \cos(\sin t) \cdot \cos t = \cos(\sin(\sin 0)) \cdot \cos(\sin 0) \cdot \cos 0$$

$$= \cos(\sin 0) \cdot \cos 0 \cdot 1 = \cos(0) \cdot 1 = 1 \quad .$$ **(C)**

5) A definite integral has a definition in terms of the so-called "Riemann sum", expressed as

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x = \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \cdot \sum_{i=1}^{n} f\left(a + i \cdot \left[\frac{b-a}{n}\right]\right) \quad ,$$

in which we have employed the "right-endpoint rule" for the n subintervals. If we put this expression into correspondence with the sum specified in the problem,

$\lim_{n \to \infty} \left(\frac{1}{n}\right) \cdot \sum_{i=1}^{n} \left(i \cdot \left[\frac{1}{n}\right]\right)^7$, we can see that $b - a = 1$ and that the function being used here is $f(x) = x^7$. Since there is no constant term added to the i/n in the argument of the function, we can surmise that $a = 0$, and therefore $b = 1$. The definite integral is thus $\int_0^1 x^7\, dx$. We can immediately evaluate this to find

$$\int_0^1 x^7\, dx = \frac{1}{8} x^8 \Big|_0^1 = \frac{1}{8} \cdot (1 - 0) = \frac{1}{8} \quad .$$ **(D)**

6) This problem is nearly the same as Problem 2 from the Spring 2005 final exam, so a discussion for most of the solution can be found in that solution set. As for the one revised choice, (d) , since the slope of the secant line connecting the endpoints of the curve for the function $f(x)$ on the interval [-1 , 2] , is equal to 4 , there must certainly be (a great many) places on the curve where the slope is positive ($f'(x) > 0$) . The other choices, (a), (b), and (c) are also true; consequently, the only false choice here is **(E)**

7) From the given equation for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, we can establish a relation between the rate of radius change and the rate of volume change for the (spherical) balloon by differentiating the volume implicitly with respect to time:

$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot \frac{d}{dt}(r^3) = \frac{4}{\cancel{3}}\pi \cdot \left(\cancel{3} r^2 \frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt} \quad .$$

Since we are told the rate of volume change (how fast air is being blown into the balloon), we can solve this equation for the rate of radius change to find:

(continued)

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{3 \text{ in.}^3/\text{min.}}{4\pi r_0^2} \quad .$$

In this problem, however, we are not given the radius at the current moment, but rather the *volume* of the balloon. If we apply the equation for the volume, we can determine that

$$V_0 = 36\pi \text{ in.}^3 = \frac{4\pi}{3} r_0^3 \Rightarrow r_0^3 = \frac{36 \cdot 3}{4} \text{ in.}^3 = 27 \text{ in.}^3 \Rightarrow r_0 = 3 \text{ in.}$$

This leads us to conclude that the rate of radius change is

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{3 \text{ in.}^3/\text{min.}}{4\pi (3 \text{ in.})^2} = \frac{3}{4\pi \cdot 9} \text{in./min.} = \frac{1}{12\pi} \text{in./min.} \qquad \textbf{(E)}$$

8) To determine the intervals of concavity for a function, we will need to examine its second derivative:

$$f(x) = 2x^6 - 5x^4 + 20 \Rightarrow f'(x) = 12x^5 - 20x^3 \Rightarrow f''(x) = 60x^4 - 60x^2 \quad .$$

We are interesting in learning where the curve of this function is concave downward, which is to say where $f''(x) < 0$:

$$f''(x) = 60x^4 - 60x^2 = 60x^2 \cdot (x^2 - 1) < 0 \quad .$$

Since $60x^2$ is positive , this condition is only possible if $x^2 - 1 < 0 \Rightarrow x^2 < 1 \Rightarrow |x| < 1$.

(E)

9) To estimate the value of a function by the use of linear approximation, it is necessary to know the exact value of the function at a point as close as possible to the point at which we wish to make the estimate. Since we wish to estimate $\sqrt[3]{7.7}$, for the function in this problem, $f(x) = \sqrt[3]{x} = x^{1/3}$, the nearest value that will serve the purpose for making the estimation is $x = a = 8$. We will also need the first derivative of the function, which is $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$.

The estimate from the linear approximation is then

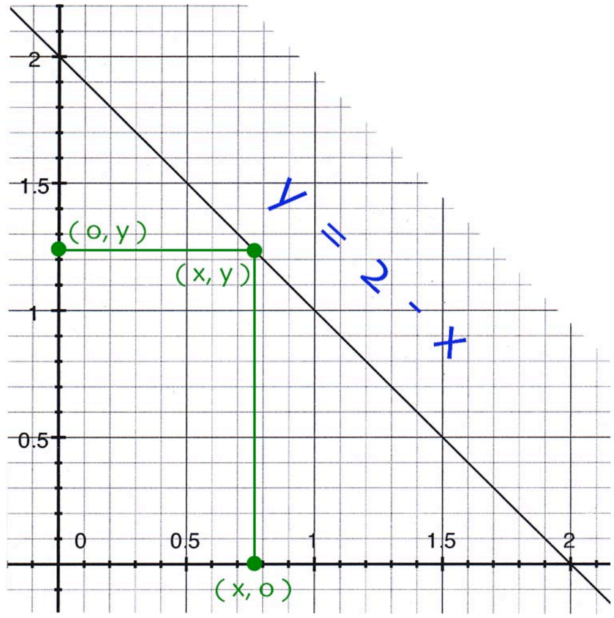
$$f(x) \approx f(a) + f'(a) \cdot (x - a)$$

$$\Rightarrow \quad f(7.7) \approx f(8) + f'(8) \cdot (7.7 - 8) = \sqrt[3]{8} + \frac{1}{3(\sqrt[3]{8})^2} \cdot (7.7 - 8)$$

$$= 2 + \frac{1}{3(2)^2} \cdot (-0.3) = 2 - \frac{0.3}{12} = 2 - 0.025 = 1.975 \quad . \qquad \textbf{(C)}$$

A more precise value from a calculator is $\sqrt[3]{7.7} \approx 1.974681$, so the linear approximation is quite good; this is because the slope of the cube-root function is not changing particularly quickly near $x = 8$.

10) We are asked to work out the maximization of the area of a rectangle that can be placed within a right isosceles (45-45-90) triangle having legs of length 2. In order for calculus to be useful here, we need to formulate this problem in terms of functions of a variable x.



We can arrange the triangle so that the two legs are places along the *x*- and *y*-axes; this places two of the vertices at $(2, 0)$ and $(0, 2)$. The hypotenuse then connects these points and thus lies along the line $y = 2 - x$. (The slope of the line containing these points is $m = \frac{0-2}{2-0} = -1$, and the y-intercept is at $(0, 2)$.) The rectangle would also have two of its sides along the coordinate axes; if we place one vertex (x, y) of the rectangle on the line we've determined, the other vertices are at $(0, 0)$, $(x, 0)$, and $(0, y)$. The area of the rectangle we wish to maximize is thus $A = xy = x \cdot (2 - x) = 2x - x^2$. If we set the derivative of this area function equal to zero, we find that

$$\frac{dA}{dx} = 2 - 2x = 0 \Rightarrow 2x = 2 \Rightarrow x = 1, \; y = 2 - x = 1 \quad .$$

The rectangle of greatest area that fits inside the isosceles triangle is thus a square of side 1, with area $A = xy = 1$. **(B)**

We can also quickly verify that this is the maximum area, since $\frac{d^2A}{dx^2} = -2 < 0$.

11) If we attempt to apply the Limit Laws directly, we find that this limit gives the indeterminate product $\lim_{x \to 0^+} x \ln x = "0 \cdot (-\infty)"$. In order to become able to apply L'Hôpital's Rule, we must modify a product of two functions, $f(x) \cdot g(x)$, into either of the indeterminate ratios, $\frac{f(x)}{1/g(x)}$ *or* $\frac{g(x)}{1/f(x)}$.

(continued)

When dealing with an indeterminate product involving a logarithmic factor, it is almost always advisable to put that factor into the numerator, as the required differentiation of 1/(ln x) in the denominator will generally make the situation worse for evaluating the limit. So saying, we can now apply L'Hôpital's Rule to obtain

$$\lim_{x \to 0^+} x \ln x \;=\; \lim_{x \to 0^+} \frac{\ln x}{1/x} \;=\; \text{"}\frac{-\infty}{\infty}\text{"}$$

$$=\; \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \;=\; \lim_{x \to 0^+} -\frac{x^2}{x} \;=\; \lim_{x \to 0^+} -x \;=\; 0 \qquad . \qquad \textbf{(A)}$$

12) This integral is not one for which we have a technique of solving it; it is simply necessary to recognize the integrand as the derivative of the arctangent function. We thus have

$$\int_0^1 \frac{1}{1+x^2}\,dx \;=\; \left(\tan^{-1} x\right)\Big|_0^1 \;=\; \tan^{-1} 1 - \tan^{-1} 0 \;=\; \frac{\pi}{4} - 0 \;=\; \frac{\pi}{4} \qquad . \qquad \textbf{(E)}$$

13) While it is true that $\lim\limits_{x \to +\infty} \sin x$ does not exist, it is possible that the limit of a product $\lim\limits_{x \to +\infty} f(x) \cdot \sin x$ could equal zero if $\lim\limits_{x \to +\infty} f(x) \;=\; 0$, as would be found by application of the "Squeeze Theorem". Here, we will need to separately investigate the limit $\lim\limits_{x \to +\infty} \dfrac{x}{\ln x}$, which gives us an indeterminate ratio:

$$\lim_{x \to +\infty} \frac{x}{\ln x} \;=\; \text{"}\frac{\infty}{\infty}\text{"} \quad\Rightarrow\quad \lim_{x \to +\infty} \frac{x}{\ln x} \;=\; \lim_{x \to +\infty} \frac{x'}{(\ln x)'} \;=\; \lim_{x \to +\infty} \frac{1}{1/x} \;=\; \lim_{x \to +\infty} x \;=\; +\infty \quad .$$

Hence, this factor grows without limit, so the product $\dfrac{x}{\ln x} \cdot \sin x$ grows without limit as x increases. **(D)**

14) The integrand in this definite integral is not one for which we know an anti-derivative function, so we will need to use another approach. The only other method we've learned at this stage is the *u*-substitution method, but there doesn't seem to be anything to base such a substitution upon.

Something to consider when working with the trigonometric functions is to see whether the integrand can be re-written in terms of factors of $\sin x$ and $\cos x$; the new form of the integrand may then suggest a substitution that will be useful. In our integral, we will get

$$\int_0^{\pi/4} \tan x \, dx \;=\; \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx \qquad\qquad u = \cos x \;\Rightarrow\; du = \sin x \; dx$$

(continued)

$\rightarrow \int_1^{(\sqrt{2})/2} \frac{-du}{u} = (-\ln\lvert u\rvert)\Big\rvert_1^{(\sqrt{2})/2}$	x :	0	$\pi/4$
	$u = \cos x$:	1	$(\sqrt{2})/2$

$$= (\ln\lvert u\rvert)\Big\rvert_{(\sqrt{2})/2}^{1} = (\ln 1) - (\ln\frac{\sqrt{2}}{2}) = 0 - (\ln\frac{\sqrt{2}}{2}) = \ln\frac{2}{\sqrt{2}} = \ln\sqrt{2} \;\; or \;\; \ln 2^{1/2} \; .$$

So the value of the integral is not $\ln\left(\frac{1}{\sqrt{2}}\right)$, but rather ½ ln 2 . **(A)**

15) This is an example of the sort of trigonometric limit problems presented in Section 3.3 . We can work out many of those very easily now that we are familiar with L'Hôpital's Rule, but that method would be very messy to apply here, since the numerator of the rational function is a three-term *product* which would have to be differentiated (more than once!).

Instead, we will break up the quotient into three factors first:

$$\lim_{x\to 0}\frac{(\sin 2x)(\sin 4x)(\sin 6x)}{x^3} = \lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{\sin 4x}{x}\cdot\frac{\sin 6x}{x} \; .$$

We could now use the "limit law" that was developed in that Section, $\lim_{x\to 0}\frac{\sin x}{x} = 1$, which can be extended to any function $u(x)$ which is continuous at $x = 0$, thus $\lim_{u\to 0}\frac{\sin u}{u} = 1$. The ratios need to have the same argument for the sine function as the term appearing in the denominator in order to apply this limit law, which can be easily arranged:

$$\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{\sin 4x}{x}\cdot\frac{\sin 6x}{x} = \lim_{x\to 0}\frac{\sin 2x}{x}\cdot\lim_{x\to 0}\frac{\sin 4x}{x}\cdot\lim_{x\to 0}\frac{\sin 6x}{x}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{2}{2}\cdot\lim_{x\to 0}\frac{\sin 4x}{x}\cdot\frac{4}{4}\cdot\lim_{x\to 0}\frac{\sin 6x}{x}\cdot\frac{6}{6}$$

$$= 2\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot 4\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot 6\lim_{x\to 0}\frac{\sin 6x}{6x} = 2\cdot 1\cdot 4\cdot 1\cdot 6\cdot 1 = 48 \; .$$ **(D)**

16) We are asked to find the derivative of the function $f(x) = x^3$, using the limit definition of differentiation. It will be easier to work this out using the "*h*-definition" of derivative:

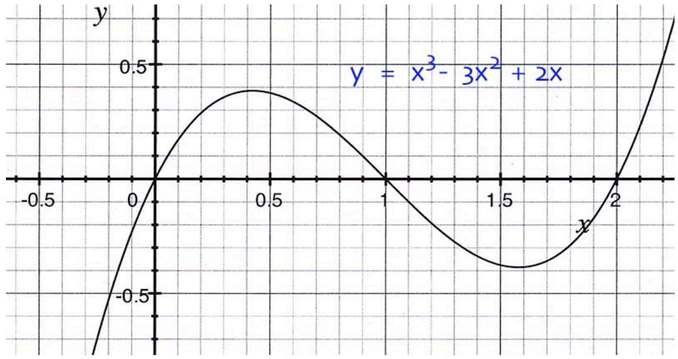
$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h}$$

multiply out the three factors
or use the Binomial Theorem

$$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2 + 3x\cdot 0 + 0^2 = 3x^2 \; .$$

17)



The function $f(x) = x \cdot (x-1) \cdot (x-2) = x^3 - 3x^2 + 2x$ is a polynomial with the indicated x-intercepts at $x = 0$, 1, and 2 and large-$|x|$ "tails" extending to negative or positive infinity beyond $x < 0$ or $x > 2$, respectively. So the regions bounded by this curve and the x-axis must lie in the intervals $(0, 1)$ and $(1, 2)$. Without actually drawing a graph, we can also find that $f(x)$ is negative if either one or all three factors are negative, which can only happen for the intervals $1 < x < 2$ or $x < 0$, respectively. So the total area for the bounded regions must be found from the sum of two integrals:

$$A = \int_0^2 |f(x)|\, dx = \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx + \int_1^2 -\left(x^3 - 3x^2 + 2x\right) dx$$

$$= \left(\frac{1}{4}x^4 - \frac{3}{3}x^3 + \frac{2}{2}x^2\right)\Bigg|_0^1 + \left(-\frac{1}{4}x^4 + \frac{3}{3}x^3 - \frac{2}{2}x^2\right)\Bigg|_1^2$$

$$= \left[\left(\frac{1}{4}\cdot 1^4 - 1^3 + 1^2\right) - (0 - 0 + 0)\right] + \left[\left(-\frac{1}{4}\cdot 2^4 + 2^3 - 2^2\right) - \left(-\frac{1}{4}\cdot 1^4 + 1^3 - 1^2\right)\right]$$

$$= \frac{1}{4} - \cancel{1} + \cancel{1} - \frac{\cancel{16}}{\cancel{4}} + \cancel{8} - \cancel{4} + \frac{1}{4} - \cancel{1} + \cancel{1} = \frac{1}{2} \quad .$$

18) The average value of a function $f(x)$ on an interval $[a, b]$ is calculated from the formula $f_{ave} = \dfrac{\int_a^b f(x)\, dx}{b - a}$. For the function $f(x) = (1 + \sqrt{x})^2$ over $[0, 4]$, we find

$$f_{ave} = \frac{\int_0^4 (1 + \sqrt{x})^2\, dx}{4 - 0} = \frac{1}{4}\int_0^4 (1 + 2\sqrt{x} + x)\, dx$$

$$= \frac{1}{4}\left[\left(x + 2\cdot\frac{x^{3/2}}{3/2} + \frac{1}{2}x^2\right)\Bigg|_0^4\right] = \frac{1}{4}\left[\left(4 + \frac{4}{3}\cdot 4^{3/2} + \frac{1}{2}\cdot 4^2\right) - (0 + 0 + 0)\right]$$

$$= \frac{1}{4}\cdot\left(4 + \frac{32}{3} + 8\right) = \frac{1}{4}\cdot\frac{12 + 32 + 24}{3} = \frac{1}{4}\cdot\frac{68}{3} = \frac{17}{3} \quad .$$

19) To determine the equation for the tangent line to a point $(x_0, f(x_0))$ on the curve for the function $y = f(x)$, we need to find the first derivative at that value of x, $f'(x_0)$.

For the given function, $f(x) = x^{x^2}$, this will require logarithmic differentiation:

$$\ln y = x^2 \ln x \quad \Rightarrow \quad \frac{d}{dx}(\ln y) = \frac{d}{dx}(x^2 \ln x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x^2)\cdot \ln x + x^2 \cdot \frac{d}{dx}(\ln x) = 2x \ln x + x^2 \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\cdot(2x \ln x + x) = x^{x^2}\cdot(2x \ln x + x) \quad .$$

The slope of the tangent line at $x = 1$ is then

$$m = \left.\frac{dy}{dx}\right|_{x=1} = 1^{1^2}\cdot(2\cdot 1\cdot \ln 1 + 1) = 1\cdot(2\cdot 1\cdot 0 + 1) = 1 \quad .$$

Since the value of the function at $x = 1$ is $y(1) = 1^{1^2} = 1$, the point-slope form of the equation for the tangent line is $(y - 1) = 1 \cdot (x - 1)$, which can be reduced to just $y = x$.

20) This problem is essentially the same as Problem 18 in the Spring 2005 final exam. A discussion of the calculation can be found in that solution set (in which you are also directed to Problem 16 of the Spring 2003 final exam solutions).

21) We are asked to investigate the function $y = f(x) = xe^{-2x^2}$.

i) To find where the extrema of this function and its intervals of increase and decrease, we must calculate its first derivative, which is

$$f'(x) = \frac{d}{dx}(xe^{-2x^2}) = \frac{d}{dx}(x)\cdot e^{-2x^2} + x\cdot\frac{d}{dx}(e^{-2x^2})$$

$$= 1\cdot e^{-2x^2} + x\cdot e^{-2x^2}\cdot\frac{d}{dx}(-2x^2) = e^{-2x^2} + x\cdot e^{-2x^2}\cdot(-4x)$$

$$= e^{-2x^2}\cdot(1 - 4x^2) \quad .$$

Since the exponential factor is always positive, the derivative function can only equal zero where $1 - 4x^2 = 0 \Rightarrow x^2 = ¼ \Rightarrow x = ±½$; these are the locations of the extrema of $f(x)$. The function is increasing where $1 - 4x^2 > 0 \Rightarrow 1 > 4x^2 \Rightarrow |x| < ½$ or $-½ < x < ½$; thus, the function is decreasing for $1 - 4x^2 < 0 \Rightarrow |x| > ½$.

This tells us then that the interval of increase for $f(x)$ is $(-½ , ½)$, the intervals of decrease are $(-\infty , -½)$ and $(½ , \infty)$, a local minimum occurs at $x = -½$, and a local maximum occurs at $x = ½$.

ii) To study the concavity of the function, we will need its second derivative:

$$f''(x) = \frac{d}{dx}\left[e^{-2x^2}\cdot(1-4x^2)\right] = \frac{d}{dx}(e^{-2x^2})\cdot(1-4x^2) + e^{-2x^2}\cdot\frac{d}{dx}(1-4x^2)$$

$$= e^{-2x^2}\cdot(-4x)\cdot(1-4x^2) + e^{-2x^2}\cdot(-8x)$$

$$= e^{-2x^2}\cdot(-4x+16x^3-8x) = e^{-2x^2}\cdot(16x^3-12x) \quad .$$

Again, the exponential factor can only be positive, so the second derivative is zero for $16x^3 - 12x = 0 \Rightarrow 4x\cdot(4x^2 - 3) = 0$; this requires that either $x = 0$ or $x^2 = ¾$ $\Rightarrow x = \pm\frac{\sqrt{3}}{2}$. The first derivative is *not* zero at any of these values, so these all represent *points of inflection* in the curve for the function. These points divide the real numbers into four intervals: $(-\infty\,,\,-\frac{\sqrt{3}}{2})\,,(-\frac{\sqrt{3}}{2}\,,\,0)\,,(0\,,\,\frac{\sqrt{3}}{2})\,,$ *and* $(\frac{\sqrt{3}}{2}\,,\,\infty)$.

We could simply test values of x in each interval to find the sign of the second derivative there or solve the inequalities $4x\cdot(4x^2 - 3) > 0$ and $4x\cdot(4x^2 - 3) < 0$ directly (which involves some little work). We conclude that the intervals of upward concavity are $(-\frac{\sqrt{3}}{2}\,,\,0)$ *and* $(\frac{\sqrt{3}}{2}\,,\,\infty)$, while those of downward concavity are $(-\infty\,,\,-\frac{\sqrt{3}}{2})$ *and* $(0\,,\,\frac{\sqrt{3}}{2})$.

iii) The "limit at infinity" cannot be determined through use of the Limit Laws for this function, as this gives us $\lim_{x\to+\infty} xe^{-2x^2} = "\infty\cdot e^{-\infty} = \infty\cdot 0"$, which is an indeterminate product. We can bring L'Hôpital's Rule to bear on the problem by first re-writing the product as an indeterminate ratio:

$$\lim_{x\to+\infty} xe^{-2x^2} = \lim_{x\to+\infty}\frac{x}{e^{2x^2}} = "\frac{\infty}{\infty}"$$

$$= \lim_{x\to+\infty}\frac{x'}{\left(e^{2x^2}\right)'} = \lim_{x\to+\infty}\frac{1}{e^{2x^2}\cdot 4x} = "\frac{1}{\infty}" = 0 \quad .$$

(In the other direction, we find basically the same result: $\lim_{x\to+\infty} xe^{-2x^2} = "\frac{1}{-\infty}" = 0$.)
Thus, $y = 0$ is the horizontal asymptote for our function.

iv) Because the exponential factor e^{-2x^2} is always positive, the sign of $f(x)$ is entirely determined by the sign of x. Thus, $f(x) > 0$ for $x > 0$ and $f(x) > 0$ for $x > 0$.

Note that this also means that $f(x)$ is an odd function. Knowing this would allow us to find the properties of the function just for $x > 0$ and then just reverse them appropriately for $x < 0$. For instance, by finding that $x = ½$ is a local maximum, that (0 , ½) is an interval of increase, and that (½ , ∞) is an interval of decrease, we learn immediately that $x = -½$ is a local *minimum*, that (–½ , 0) is an interval of increase, and that (–∞ , –½) is an interval of decrease. This odd symmetry can also be applied to the intervals of concavity, locations of inflection points, the limit at negative infinity, and so forth.

v) A graph of this function is presented in the Answer Key.

G. Ruffa – 6/09