MATH 2243: LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
SAMPLE FINAL EXAM

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You may not use a calculator, notes, books, etc. Only the exam paper and a pencil or pen may be kept on your desk during the test. You must show all work.
Good luck!

Problem 1. Suppose that some exotic nuclear waste contains five times as much dysprosium-159 (^{159}Dy, half-life = 144 days) as gold-195 (^{195}Au, half-life = 186 days). When will there be equal amounts of these two isotopes?

Problem 2. Solve the initial value problem

$$y'' + y' = e^{-t}, \qquad y(0) = 0, y'(0) = 0.$$

Problem 3. Use Gauss-Jordan elimination to solve the system

$$
\begin{aligned}
2x + 8y + 4z &= 2, \\
2x + 10y - z &= 5, \\
4x + 10y - z &= 1.
\end{aligned}
$$

Problem 4.

$$\text{Let} \quad A = \begin{bmatrix} 1 & 2 & k \\ 2 & 1 & 3 \\ 1 & 3 & k+1 \end{bmatrix}.$$

For which values of k does the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Justify your answer.

Problem 5. Let $f(t) = t$, $g(t) = e^t$, and $h(t) = t^3$. Are these three functions linearly independent? Is $a(t) = t^2$ in the subspace generated by f, g, and h? Justify your answers.

Problem 6. Let $\mathbf{v}_0 = (1, 0, -1)$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined as

$$T(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0.$$

What is the matrix A of T? Find the nullspace and the rank of A.

Problem 7.

$$\text{Let} \quad A = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}.$$

(1) Find a matrix diagonalizing A.
(2) Solve the equation $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = (1, 3)$.

Problem 8. In the mixing problem depicted in Figure 8.8.7 on p. 501 of the text, there is no inflow from or outflow to the outside, and the exchange rate between the two tanks is 2 L/min. Suppose that tank 1 contains 6 L of solution and tank 2 contains 12 L of solution, and that initially tank 1 contains 5g of chemical and tank 2 contains 25 g of chemical.

(1) Determine the amount of chemical in each tank at time t.
(2) Eventually, what is the amount of chemical in each tank?