Math 2243, Final exam solutions

(1) **Problem 1.**

Find the general solution $y(t)$ to the following ODE: $y' + 3ty = t$.

There are several ways to do this problem since the equation is seperable and linear. Here is one solution:

The integrating factor is $\mu = e^{\int 3t} = C_1 e^{3t^2/2}$ and we can choose $C_1 = 1$ as usual. So the homogeneous solution is $Ce^{-3t^2/2}$. The particular solution can be calculated by $(\int \mu * t dt)/\mu = (e^{3t^2/2})/3\mu = 1/3$. So the final answer is $Ce^{-3t^2/2} + 1/3$.

(2) **Problem 2.**

Show that the functions $\{e^t, te^t, t^2e^t\}$ are linearly independent by using the Wronskian.

The Wronskian is

$$Det \begin{bmatrix} e^t & te^t & t^2e^t \\ e^t & e^t(t+1) & e^t(2t+t^2) \\ e^t & e^t(t+2) & e^t(2+4t+t^2) \end{bmatrix}$$

The determinant can be simplified by subtracting row 1 from rows 2 and 3. This gives us:

$$Det \begin{bmatrix} e^t & te^t & t^2e^t \\ 0 & e^t & e^t(2t) \\ 0 & 2e^t & e^t(2+4t) \end{bmatrix}$$

which is equal to $e^t(e^t * e^t(2+4t) - 2te^t * 2e^t) = 2e^{3t}$. Since $2e^{3t}$ is not identically zero, the functions are linearly independent.

(3) **Problem 3.**

Find the solution to the initial value problem $y'' - y = 4e^t$, $y(0) = -1$, $y'(0) = 1$.

First we find the homogeneous solution by considering the characteristic equation $r^2 - 1 = 0$. This has roots ± 1, so $y_h = C_1e^t + C_2e^{-t}$. To find a particular solution we can use undetermined coefficients. Since the inhomogeneous term $4e^t$ is in the span of the homogeneous solutions, we should consider functions of the form $y_p = Ate^t$. Then $y_p' = Ae^t + Ate^t$ and $y_p'' = 2Ae^t + Ate^t$. Plugging that into the ODE we find that

$$2Ae^t + Ate^t - Ate^t = 2Ae^t = 4e^t$$

so $A = 2$ and $y_p = 2te^t$. Now we know the general form of the solution is $y_h + y_p = C_1e^t + C_2e^{-t} + 2te^t$.

The initial conditions give us two linear equations for C_1 and C_2. To compute the second initial condition equation we need $y'(t) = C_1e^t - C_2e^t + 2e^t + 2te^t$.

$$y(0) = -1 = C_1 + C_2 + 0$$

and

$$y'(0) = 1 = C_1 - C_2 + 2 + 0.$$

These equations could be solved by row-reducing an augmented matrix for the system. Or we can simply add the two equations to see that $-2 = 2C_1$ and so $C_1 = -1$. Substituting that into the first equation we find $C_2 = 0$.

So the final answer to this IVP is $y(t) = 2te^t - e^t$.

(4) **Problem 4**.

Find the general form of the solution for the following linear system

$$x' = y \ ; y' = x$$

There are several ways to solve this problem. Considered as a matrix system, we can compute the characteristic polynomial $\lambda^2 - 1 = 0$ and we find that $\lambda = \pm 1$. The eigenvectors are solutions to the systems

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The respective matrices row reduce to

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

This implies that $v = (v_1, v_2) = (a, -a)$ and $w = (w_1, w_2) = (a, a)$. We can choose $a = 1$ in both cases. Now we know that solutions to the system are $C_1e^tv + C_2e^{-t}w$. Written out in (x, y) components, $x(t) = C_1e^t + C_2e^{-t}$ and $y(t) = C_1e^t - C_2e^{-t}$.

(5) **Problem 5**.

Let $K \subset \mathbb{R}^3$ be the subspace $K = \{(x, y, z) \in \mathbb{R}^3 | x = y\}$ and let $L \subset \mathbb{R}^3$ be the subspace $L = \{(x, y, z) \in \mathbb{R}^3 | x = -y \text{ and } z = 0\}$. I.e., K consists of vectors of the form (x, x, z) and L consists of vectors of the form $(x, -x, 0)$.

Find a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which has K as its kernel and L as its range.

Perhaps it is worth noting that very few people got this question right.

The kernel K is two dimensional and is spanned by vectors of the form $(c, c, 0)$ and $(0, 0, c)$ (other choices are possible but this is among the simplest). In particular let us choose a basis $v_1 = (1, 1, 0)$ and $v_2 = (0, 0, 1)$ for K. Let us write a matrix for our linear transformation as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Now the condition that K is the kernel is equivalent to $Av_1 = \vec{0}$ and $Av_2 = \vec{0}$. Expanding these equations gives us the conditions $a_{12} = -a_{11}$, $a_{22} = -a_{21}$, $a_{32} = -a_{31}$, and $a_{13} = 0$, $a_{23} = 0$, $a_{33} = 0$. So our matrix A must be of the form

$$A = \begin{bmatrix} a_{11} & -a_{11} & 0 \\ a_{21} & -a_{21} & 0 \\ a_{31} & -a_{31} & 0 \end{bmatrix}.$$

The fact that L must be the range space means that Av must be of the form $(b, -b, 0)^T$ for any vector v. Let us write $v = (X, Y, Z)$ so we must have

$$\begin{bmatrix} a_{11} & -a_{11} & 0 \\ a_{21} & -a_{21} & 0 \\ a_{31} & -a_{31} & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}.$$

Expanding this out and factoring gives us three equations: $a_{11}(X - Y) = b$, $a_{21}(X - Y) = -b$, and $a_{31}(X - Y) = 0$, which must hold for all X and Y. In particular they must hold for $X \neq Y$, so the last equation forces $a_{31} = 0$. Adding together the first two equations gives us $(a_{11} + a_{21})(X - Y) = 0$ so $a_{21} = -a_{11}$.

Thus any matrix of the form

$$A = \begin{bmatrix} a_{11} & -a_{11} & 0 \\ -a_{11} & a_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has the desired properties.

(6) **Problem 6**.

Find a 2x2 matrix A with non-zero entries which satisfies the equation

$$A^2 + A = 0.$$

There are several ways to do this problem. Perhaps the best is to use the Cayley-Hamilton theorem as follows:

First let us write our unknown matrix A as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The Cayley-Hamilton theorem says that a matrix will satisfy its own characteristic polynomial. So lets suppose that the characteristic polynomial of A is $\lambda^2 + \lambda = 0$. This would mean that the eigenvalues of A were 0 and -1. Recall that the trace of a matrix equals the sum of its eigenvalues and the determinant equals the product of the eigenvalues. This means that

$$tr(A) = a + d = -1$$

and

$$det(A) = ad - bc = 0.$$

So we can set $d = -1 - a$ and substitute that into the second equation to obtain $bc = -a(1 + a)$. Using a and b to parameterize our solution, the matrix A must be of the form

$$A = \begin{bmatrix} a & b \\ \frac{-a(a+1)}{b} & -(a+1) \end{bmatrix}.$$

As long as $a \neq -1$, $a \neq 0$, and $b \neq 0$, any A of the above form will have all non-zero entries and satisfy the required equation.