Theorem (Cauchy - Schwarz)

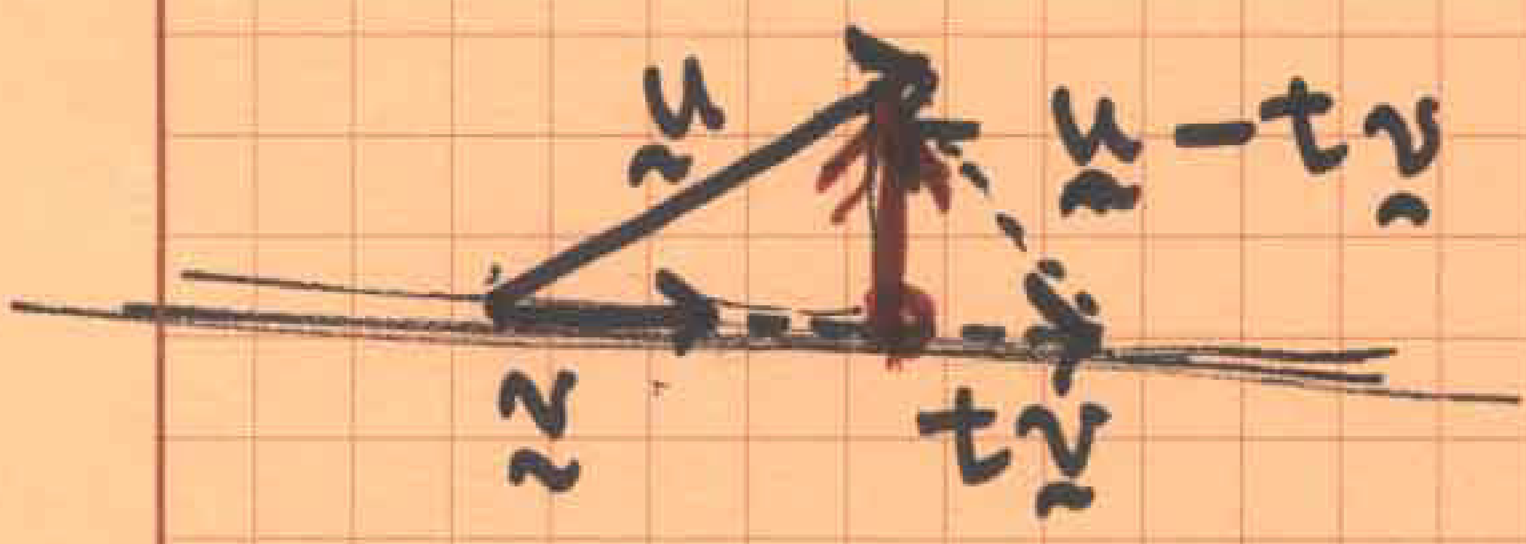
V vector space with inner product $(\mathbf{u}, \mathbf{v})$.

Then $|(\mathbf{u}, \mathbf{v})| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$.

Note: Could define $\cos\theta$ by $\dfrac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \cos\theta$

Proof: If $\mathbf{v} = \mathbf{0}$ then C-S is obviously true

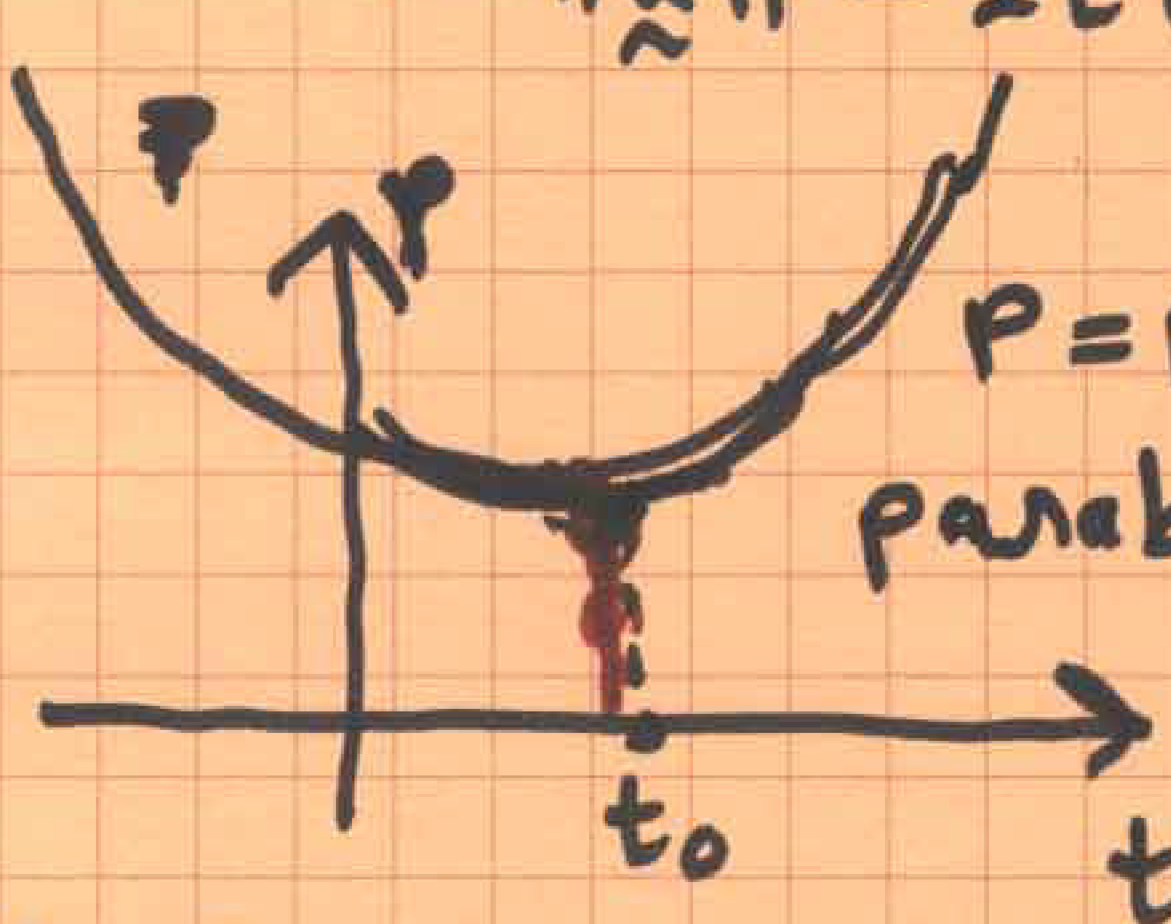
Assume $\mathbf{v} \neq \mathbf{0}$.



Set $P(t) = \|\mathbf{u} - t\mathbf{v}\|^2$.

Find t_0 such that $P(t_0)$ is a minimum.

$$P(t) = (\mathbf{u} - t\mathbf{v}, \mathbf{u} - t\mathbf{v})$$
$$= \|\mathbf{u}\|^2 - 2t(\mathbf{u}, \mathbf{v}) + t^2 \|\mathbf{v}\|^2$$



$P = P(t)$ parabola

$$\boxed{\begin{array}{l}\text{Claim:} \\ t_0 = \dfrac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|^2} \\ \text{is min}\end{array}}$$

Proof: $P(t) - P(t_0) = \left[\|\mathbf{u}\|^2 - 2t(\mathbf{u}, \mathbf{v}) + t^2\|\mathbf{v}\|^2\right]$

$$- \left[\|\mathbf{u}\|^2 - 2\frac{(\mathbf{u}, \mathbf{v})^2}{\|\mathbf{v}\|^2} + \frac{(\mathbf{u}, \mathbf{u})^2}{\|\mathbf{v}\|^2}\right]$$

$$= \|\mathbf{v}\|^2 \left(t - \frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|^2}\right)^2 \geq 0$$

$$\therefore \quad P(t) - P(t_0) = 0 \iff t = \frac{(\underset{\sim}{u},\underset{\sim}{v})}{\|\underset{\sim}{v}\|^2} = t_0$$

$$\begin{aligned}
0 \le P(t_0) = \|\underset{\sim}{u} - t_0\underset{\sim}{v}\|^2 &= \left\|\underset{\sim}{u} - \frac{(\underset{\sim}{u},\underset{\sim}{v})}{\|\underset{\sim}{v}\|^2}\underset{\sim}{v}\right\|^2 \\
&= \left(\underset{\sim}{u} - \frac{(\underset{\sim}{u},\underset{\sim}{v})}{\|\underset{\sim}{v}\|^2}\underset{\sim}{v},\ \underset{\sim}{u} - \frac{(\underset{\sim}{u},\underset{\sim}{v})}{\|\underset{\sim}{v}\|^2}\underset{\sim}{v}\right) \\
&= \|\underset{\sim}{u}\|^2 - 2\frac{(\underset{\sim}{u},\underset{\sim}{v})^2}{\|\underset{\sim}{v}\|^2} + \frac{(\underset{\sim}{u},\underset{\sim}{v})^2}{\|\underset{\sim}{v}\|^2} \\
&= \|\underset{\sim}{u}\|^2 - \frac{(\underset{\sim}{u},\underset{\sim}{v})^2}{\|\underset{\sim}{v}\|^2} \ge 0
\end{aligned}$$

$$\Rightarrow \|u\|^2\|\underset{\sim}{v}\|^2 \ge (\underset{\sim}{u},\underset{\sim}{v})^2 \iff \|\underset{\sim}{u}\|\|\underset{\sim}{v}\| \ge |(\underset{\sim}{u},\underset{\sim}{v})|$$

<u>Note</u> $\quad P(t) = \|\underset{\sim}{u} - t\underset{\sim}{v}\|^2 = 0$

$$\iff \underset{\sim}{u} = t\underset{\sim}{v}$$

$$\therefore \quad |(\underset{\sim}{u},\underset{\sim}{v})| = \|\underset{\sim}{u}\|\cdot\|\underset{\sim}{v}\|$$

$\iff \underset{\sim}{u}$ + $\underset{\sim}{v}$ are lin. dep.