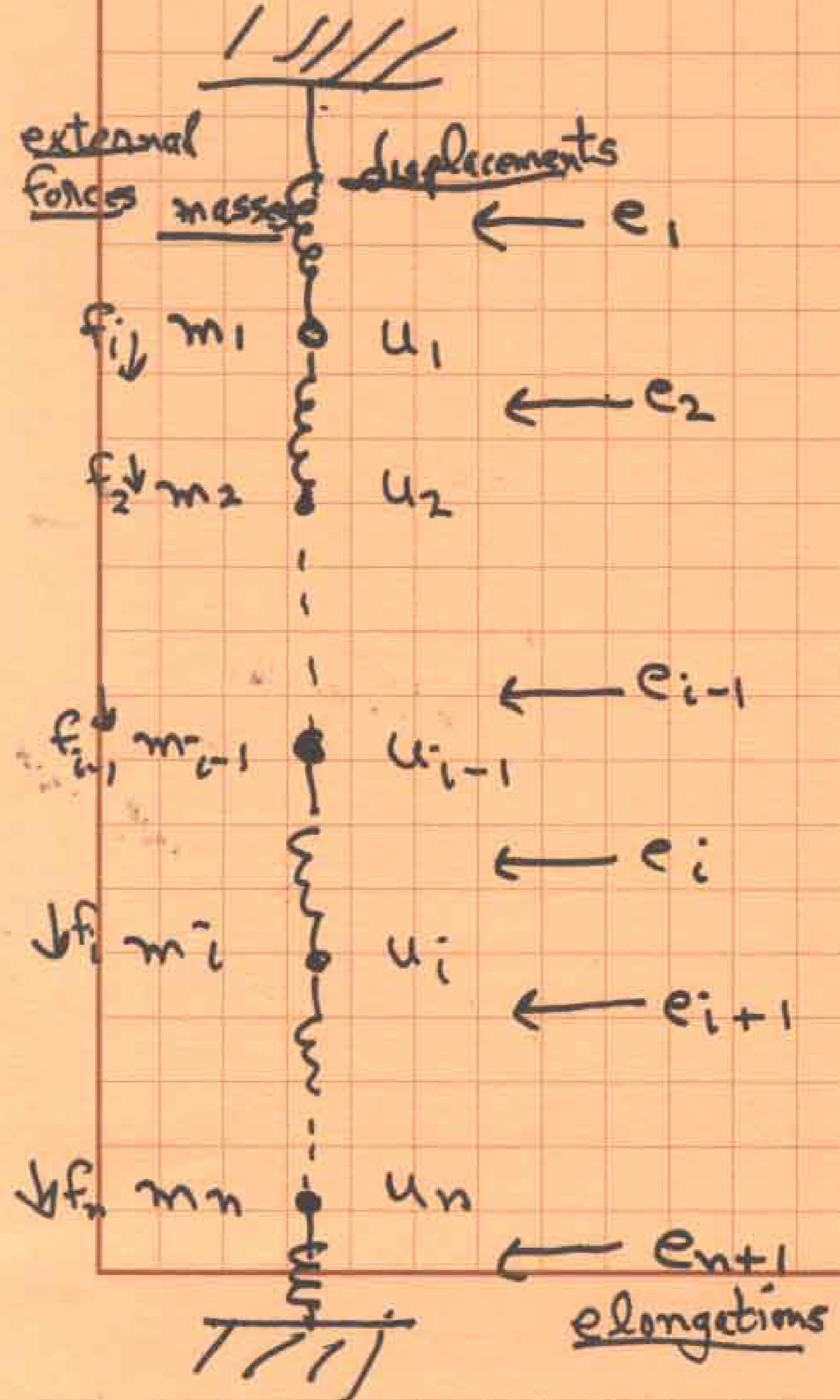


internal force on ~~spring~~ mass exerted by spring e_j is y_j

$$e_j = u_j - u_{j-1}, \quad j = 2, \ldots, n$$
$$e_1 = u_1, \quad e_{n+1} = -u_n$$

$$y_j = c_j e_j, \quad j = 1, \ldots, n+1$$

c_j ↑ stiffness coef.

Force Balance

$$f_i = c_i(u_i - u_{i-1}) - c_{i+1}(u_{i+1} - u_i)$$
$$i = 2, \ldots, n-1$$
$$f_1 = c_1 u_1 - c_2(u_2 - u_1)$$
$$f_n = c_n(u_n - u_{n-1}) + c_{n+1} u_n$$

$$\underset{n\times 1}{\underset{\sim}{u}} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \underset{\sim}{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

displacement vector — external force vector

Force Balance: $\underset{n\times n}{K}\,\underset{\sim}{u} = \underset{\sim}{f}$

$$K = \begin{pmatrix} c_1+c_2 & -c_2 & & & 0 \\ -c_2 & c_2+c_3 & -c_3 & & \\ & -c_3 & \ddots & \ddots & \\ & & \ddots & \ddots & -c_n \\ 0 & & & -c_n & c_n+c_{n+1} \end{pmatrix}$$

Claim K is pos. def. Soln. is $\underset{\sim}{u} = K^{-1}\underset{\sim}{f}$

Better way:

$$\underset{\substack{(n+1)\times 1\\ \text{internal}\\ \text{force}\\ \text{vector}}}{\underset{\sim}{y}} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix}, \quad \underset{\substack{(n+1)\times 1\\ \text{elongation}\\ \text{vector}}}{\underset{\sim}{e}} = \begin{pmatrix} e_1 \\ \vdots \\ e_{n+1} \end{pmatrix}, \quad \underset{\substack{(n+1)\times(n+1)\\ \text{stiffness}\\ \text{matrix}\\ \text{pos. def.}}}{C} = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_{n+1} \end{pmatrix}$$

Given $\underset{\sim}{f}$, C, find $\underset{\sim}{u}$.

$$\underset{\sim}{y} = C\underset{\sim}{e}, \quad \underset{(n+1)\times 1}{\underset{\sim}{e}} = A\underset{\sim}{u}, \quad \underset{\substack{(n+1)\times n\\ \text{reduced}\\ \text{incidence}\\ \text{matrix}}}{A} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & -1 & 1 \\ & & 0 & -1 \end{pmatrix}$$

Force balance: $\underset{\sim}{f} = A^T \underset{\sim}{y}$

$$\underset{n\times(n+1)}{A^T} = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix} \qquad f = \underbrace{A^T C A}_{\substack{K \leftarrow \text{Gram matrix}\\ \text{pos. def.}}} \underset{\sim}{u}$$

Example: 4 masses, equal spring const. $c_1 = \cdots = c_5 = 1$

$$\underset{4\times 4}{K} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad \underset{5\times 4}{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$K = A^T C A = LU = LDL^T, \quad K\underset{\sim}{u} = \underset{\sim}{f}$$

solve for K^{-1}.

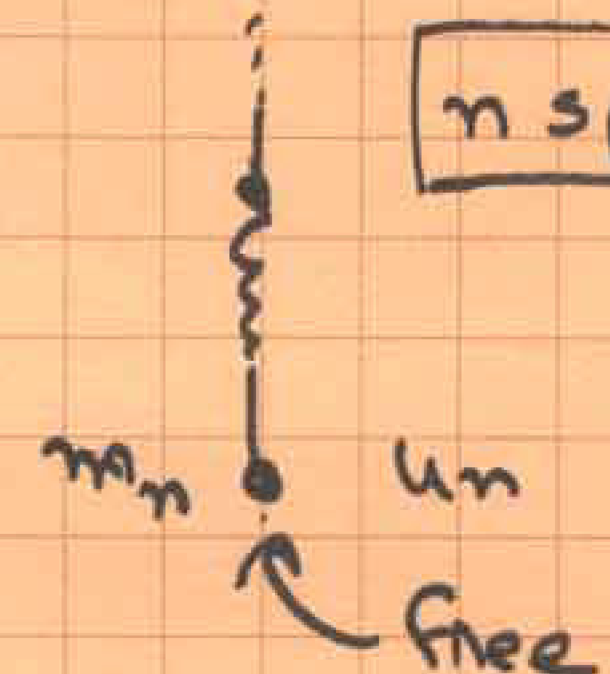
$$K^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\underset{\sim}{f} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \underset{\sim}{u} = \frac{1}{5}\begin{pmatrix} 2 \\ 4 \\ 6 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{f} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \underset{\sim}{u} = \frac{1}{5}\begin{pmatrix} 3 \\ 6 \\ 4 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{f} = mg\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \underset{\sim}{u} = \frac{mg}{5}\begin{pmatrix} 10 \\ 15 \\ 15 \\ 10 \end{pmatrix}$$

Change boundary conds



$\boxed{n \text{ springs}}$

$$\underset{n\times 1}{\underset{\sim}{e}} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \qquad e_j = u_j - u_{j-1},\ j = 2, \dots, n \qquad e_1 = u_1$$

$$\underset{\sim}{e} = A\underset{\sim}{u}, \quad \underset{n\times n}{A} = \begin{pmatrix} 1 & 0 & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & -1 & 1 \end{pmatrix}$$

$$\underset{n\times n}{A^T} = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix}$$

$$K = A^TCA = \begin{pmatrix} c_1+c_2 & -c_2 & & \\ -c_2 & c_2+c_3 & \ddots & \\ & \ddots & c_{n-1}+c_n & -c_n \\ & & -c_n & c_n \end{pmatrix}$$

can solve $\underset{\sim}{f} = \underset{\uparrow\ \text{square}}{A^T}\underset{\sim}{y}$ statically determinant

<u>Example:</u> 4 masses, equal spring constants

$c_1 = c_2 = c_3 = c_4 = 1$

$$\underset{4\times 4}{K} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$K^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\underset{\sim}{f} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{f} = mg \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \underset{\sim}{u} = mg \begin{pmatrix} 4 \\ 7 \\ 9 \\ 10 \end{pmatrix}$$

$$\underset{\sim}{f} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

In General: $K = A^T C A$, $\underset{\sim}{f}$ (ext), $\underset{\sim}{u}$ (disp.)

$\rightarrow$ (*) $K \underset{\sim}{u}^* = \underset{\sim}{f}$

<u>Note:</u> (*) gives the unique soln of the minimization problem

$$\underset{\sim}{u}^T K \underset{\sim}{u} - 2\underset{\sim}{u}^T \underset{\sim}{f} = P(\underset{\sim}{u})$$

$\frac{1}{2} P(\underset{\sim}{u})$ is the potential energy of the system.

$K \underset{\sim}{u} = \underset{\sim}{f}$

Matlab: $u = K \backslash f$

100x100 identity matrix

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Matlab: `I = eye(100);` $\leftarrow$ use semi-colon to suppress output

$$L = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & 0 \\ & & 1 & 0 \end{pmatrix}$$ $\leftarrow$ 100x100 lower triangular matrix

Matlab: `L = eye(100); L(:,1) = [ ]`

$L(:,100) = \text{zeros}(100,1)$

$$U = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \leftarrow$$ 100x100 upper triangular matrix

Matlab: $U = L'$

$$K = \begin{pmatrix} 5 & -2 & & \\ -2 & 5 & \ddots & \\ & \ddots & \ddots & -2 \\ & & -2 & 5 \end{pmatrix} \leftarrow$$ 100x100 tridiagonal matrix

Matlab: $K = 5*I - 2*L - 2*U$

$$\underset{100\times 1}{\underset{\sim}{f}} = \begin{pmatrix} .3 \\ .6 \\ .9 \\ 1.2 \\ \vdots \\ 30 \end{pmatrix}$$

Matlab: f = .3*(linspace(1,100,100)')

f[1], ---, f[100]

$T = (1, 2, 3, \cdots, 100)$

Matlab: T = linspace(1, 100, 100)

$S = (1^2, 2^2, 3^2, \cdots, 100^2)$

Matlab: S = T.*T or S = T.^2

array multiplication

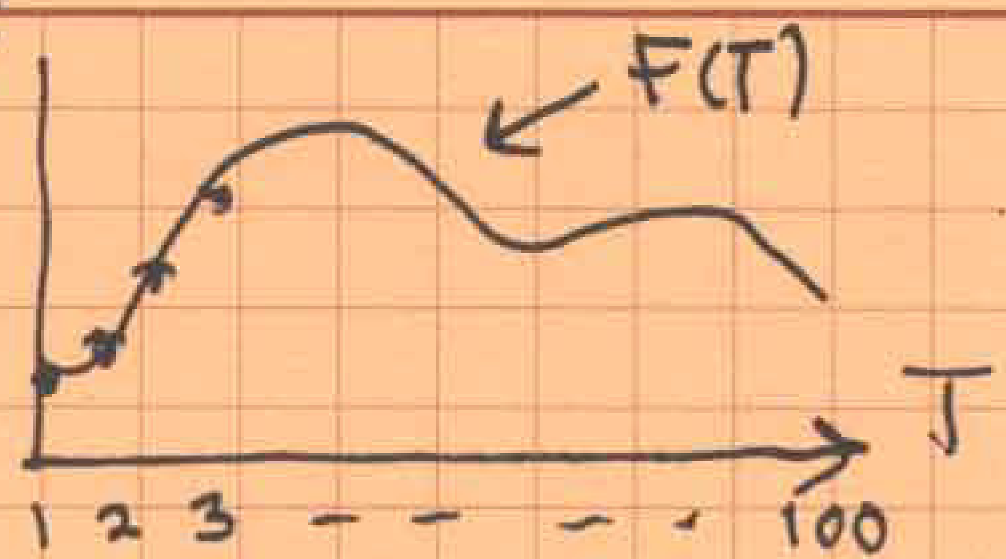
F = (F[1], F[2], ..., F[100])

where $F[T] = 13*10^{-4}(50T - T^2)$

<u>Matlab</u>: F = 13*10^(-4).*(50*T-T.^2)

array multiplication

Graph



Matlab: plot(T, F) row vectors

Set $k_{100,100} = 5$ in the 100×100 matrix $K = (k_{ij})$

Matlab: K(100,100) = 5