Structure theory for extended Kepler-Coulomb 3D superintegrable systems

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Abstract 1

A classical superintegrable system is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent phase space functions, polynomial in the momenta, whose Poisson brackets with the Hamiltonian vanish.

Typically, the algebra generated by the symmetry operators and their commutators has been proven to close polynomally.

Abstract 2

However the degenerate 3-parameter potential for the 3D extended Kepler-Coulomb system (2nd order superintegrable) appeared to be an exception as Kalnins et al. (2007) showed that it didn't close polynomially.

The 3D 4-parameter potential for the extended Kepler-Coulomb system is not even 2nd order superintegrable. However, Verrier and Evans (2008) showed it was 4th order superintegrable, Tanoudis and Daskaloyannis (2011) showed it closed polynomially.

Abstract 3

We consider an infinite class of classical extended Kepler-Coulomb systems that we show to be superintegrable of arbitrarily high order and explain the (apparent) anomalies of these systems.

Joint work with Ernie Kalnins.

The 3 and 4 parameter systems

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2},$$

where x, y, z are the usual Cartesian coordinates with conjugate momenta p_x, p_y, p_z in phase space and $r = \sqrt{x^2 + y^2 + z^2}$.

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}.$$

Superintegrability (Classical)

Hamiltonian

$$\mathcal{H} = \sum g^{jk} p_j p_k + V(\mathbf{x})$$

on phase space with local coordinates x_j, p_j, where $ds^2 = \sum g^{jk} dx_j\, dx_k$. Superintegrable if there are $2n-1$ functionally independent functions polynomial in momenta

$$\mathcal{S}_j(\mathbf{p}, \mathbf{x}), \quad j = 1, \cdots, 2n-1$$

with $\mathcal{S}_1 = \mathcal{H}$, and globally defined such that $\{\mathcal{S}_j, \mathcal{H}\} = 0$, where $\{\cdot, \cdot\}$ is the Poisson bracket.

Order of a Classical System

- A classical superintegrable system is of order ℓ if ℓ is the maximum order of the generating symmetries other than the Hamiltonian as polynomials in the momenta.
  and
- Systems with $\ell = 1, 2$ have been the most tractible due to the association with Lie algebras and separation of variables.

Superintegrability (Quantum)

Schrödinger operator

$$H = \Delta + V(\mathbf{x})$$

where $\Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} g^{ij}) \partial_{x_j}$ is the Laplace-Beltrami operator on a Riemannian manifold, in local coordinates x_j. The system admits $2n-1$ algebraically independent globally defined differential symmetry operators

$$S_j, \quad j = 1, \cdots, 2n-1, \quad n \geq 2,$$

with $S_1 = H$ and $[H, S_j] \equiv HS_j - S_jH = 0$.

Order of a System

- A superintegrable system is of order ℓ if ℓ is the maximum order of the generating symmetries other than the Hamiltonian as a differential operator for the quantum system).
- Systems with $\ell = 1, 2$ have been the most tractible due to the association with Lie algebras and separation of variables.

Integrability and Superintegrability

- An integrable system has n algebraically independent commuting symmetry operators.
- A superintegrable system has $2n-1$ algebraically independent symmetry operators. For many researchers the definition of superintegrability also requires integrability.
- The symmetry algebra generated by the symmetries of a merely integrable system is abelian. The algebra corresponding to a superintegrable system is nonabelian.

3-parameter Kepler system 1

The extended Kepler-Coulomb Hamiltonian is

(1) $$\mathcal{H} = \mathcal{L}_1 = p_r^2 + \frac{\alpha}{r} + \frac{\mathcal{L}_2}{r^2},$$

where

$$\mathcal{L}_2 = p_{\theta_1}^2 + \frac{\mathcal{L}_3}{\sin^2(k_1\theta_1)}, \qquad \mathcal{L}_3 = p_{\theta_2}^2 + \frac{\beta}{\cos^2(k_2\theta_2)} + \frac{\gamma}{\sin^2(k_2\theta_2)}.$$

(2)

Here, $\mathcal{L}_2$, $\mathcal{L}_3$ are constants of the motion that determine additive separation of the Hamilton-Jacobi equation. Also $k_1 = p_1/q_1$, $k_2 = p_2/q_2$ are rational numbers.

3-parameter Kepler system 2

In Cartesian coordinates with $k_1 = k_2 = 1$ the Hamiltonian is

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2}, \tag{3}$$

Here

$$x = r\sin(\theta_1)\cos(\theta_2),\; y = r\sin(\theta_1)\sin(\theta_2),\; z = r\cos(\theta_1).$$

It is 2nd order superintegrable.

3-parameter Kepler system 3

There is a standard action angle construction to get two independent (nonpolynomial) constants of the motion:

$$\mathcal{M}_1 = q_1 A_1 - p_1 B_1, \qquad \mathcal{M}_2 = p_1 q_2 A_2 - p_2 q_1 B_2.$$

$$\sinh A_1 = -\frac{i\sqrt{\mathcal{L}_2}\cos(k_1\theta_1)}{\sqrt{\mathcal{L}_2 - \mathcal{L}_3}}, \qquad \cosh A_1 = \frac{\sin(k_1\theta_1)p_{\theta_1}}{\sqrt{\mathcal{L}_2 - \mathcal{L}_3}}.$$

$$\sinh B_1 = -\frac{i(\alpha + 2\mathcal{L}_2/r)}{\sqrt{\alpha^2 + 4\mathcal{H}\mathcal{L}_2}}, \qquad \cosh B_1 = \frac{2\sqrt{\mathcal{L}_2}p_r}{\sqrt{\alpha^2 + 4\mathcal{H}\mathcal{L}_2}},$$

3-parameter Kepler system 4

$$\sinh A_2 = \frac{i(\mathcal{L}_3 \cos(2k_2\theta_2) + \gamma - \beta)}{\sqrt{(\beta - \gamma - \mathcal{L}_3)^2 - 4\gamma\mathcal{L}_3}},$$

$$\cosh A_2 = -\frac{\sqrt{\mathcal{L}_3}\sin(2k_2\theta_2)p_{\theta_2}}{\sqrt{(\beta - \gamma - \mathcal{L}_3)^2 - 4\gamma\mathcal{L}_3}}.$$

$$\sinh B_2 = \frac{i(2\mathcal{L}_3 \csc^2(k_1\theta_1) - \mathcal{L}_2 - \mathcal{L}_3)}{\mathcal{L}_3 - \mathcal{L}_2},$$

$$\cosh B_2 = -\frac{2\sqrt{\mathcal{L}_3}\cot(k_1\theta_1)p_{\theta_1}}{\mathcal{L}_3 - \mathcal{L}_2}.$$

3-parameter Kepler system 5

Then $\mathcal{M}_1, \mathcal{M}_2, \mathcal{H}, \mathcal{L}_2, \mathcal{L}_3$ are functionally independent constants of the motion. To get polynomials we form, for $j = 1, 2$,

$$
\begin{aligned}
e^{A_j} &= \cosh A_j + \sinh A_j = X_j/U_j,\\
e^{-A_j} &= \cosh A_j - \sinh A_j = \overline{X_j}/U_j,\\
e^{B_j} &= \cosh B_j + \sinh B_j = Y_j/S_j,\\
e^{-B_j} &= \cosh B_j - \sinh B_j = \overline{Y_j}/S_j.
\end{aligned}
$$

Here,
$U_1 = \sqrt{\mathcal{L}_2 - \mathcal{L}_3},\ U_2 = \sqrt{-(\beta - \gamma - \mathcal{L}_3)^2 + 4\gamma\mathcal{L}_3},$
$S_1 = \sqrt{\alpha^2 + 4\mathcal{H}\mathcal{L}_2},\ S_2 = \mathcal{L}_3 - \mathcal{L}_2.$

3-parameter Kepler system 6

$$X_1 = \sin(k_1\theta_1)p_{\theta_1} - i\sqrt{\mathcal{L}_2}\cos(k_1\theta_1),$$

$$\overline{X}_1 = \sin(k_1\theta_1)p_{\theta_1} + i\sqrt{\mathcal{L}_2}\cos(k_1\theta_1),$$

$$X_2 = -\sqrt{\mathcal{L}_3}\sin(2k_2\theta_2)p_{\theta_2} + i(\mathcal{L}_3\cos(2k_2\theta_2) + \gamma - \beta),$$

$$\overline{X}_2 = -\sqrt{\mathcal{L}_3}\sin(2k_2\theta_2)p_{\theta_2} - i(\mathcal{L}_3\cos(2k_2\theta_2) + \gamma - \beta),$$

$$Y_1 = 2\sqrt{\mathcal{L}_2}p_r - i(\alpha + 2\frac{\mathcal{L}_2}{r}),\ \overline{Y}_1 = 2\sqrt{\mathcal{L}_2}p_r + i(\alpha + 2\frac{\mathcal{L}_2}{r}),$$

$$Y_2 = -2\sqrt{\mathcal{L}_3}\cot(k_1\theta_1)p_{\theta_1} + i(2\mathcal{L}_3\csc^2(k_1\theta_1) - \mathcal{L}_2 - \mathcal{L}_3),$$

$$\overline{Y}_2 = -2\sqrt{\mathcal{L}_3}\cot(k_1\theta_1)p_{\theta_1} - i(2\mathcal{L}_3\csc^2(k_1\theta_1) - \mathcal{L}_2 - \mathcal{L}_3),$$

3-parameter Kepler system 7

The identity $e^{\mathcal{M}_1}e^{-\mathcal{M}_1}=1$ can be expressed as

$$X_1^{q_1}\overline{X_1}^{q_1}Y_1^{p_1}\overline{Y_1}^{p_1}=U_1^{2q_1}S_1^{2p_1}$$

$$=\mathcal{P}_1(\mathcal{H},\mathcal{L}_2,\mathcal{L}_3)=(\mathcal{L}_2-\mathcal{L}_3)^{q_1}(\alpha^2+4\mathcal{H}\mathcal{L}_2)^{p_1}$$

where $\mathcal{P}_1$ is a polynomial in $\mathcal{H}$, $\mathcal{L}_2$ and $\mathcal{L}_3$.
Similarly $e^{\mathcal{M}_2}e^{-\mathcal{M}_2}=1$ can be written

$$X_2^{p_1q_2}\overline{X_2}^{p_1q_2}Y_2^{p_2q_1}\overline{Y_2}^{p_2q_1}=U_2^{2p_1q_2}S_2^{2p_2q_1}$$

$$=\mathcal{P}_2(\mathcal{H},\mathcal{L}_2,\mathcal{L}_3)=(\mathcal{L}_2-\mathcal{L}_3)^{2p_2q_1}((\beta-\gamma-\mathcal{L}_3)^2-4\gamma\mathcal{L}_3)^{p_1q_2}$$

where $\mathcal{P}_2$ is a polynomial in $\mathcal{H}$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

3-parameter Kepler system 8

Define basic raising and lowering symmetries

$$\mathcal{J}^+ = X_1^{q_1}\overline{Y_1}^{p_1}, \; \mathcal{J}^- = \overline{X_1}^{q_1} Y_1^{p_1},$$

$$\mathcal{K}^+ = X_2^{p_1 q_2}\overline{Y_2}^{p_2 q_1}, \; \mathcal{K}^- = \overline{X_2}^{p_1 q_2} Y_2^{p_2 q_1}.$$

Take case where each of p_1, q_1, p_2, q_2 is an odd integer. (other cases similar) Let
$\mathcal{J}_1 = \frac{1}{\sqrt{\mathcal{L}_2}}(\mathcal{J}^- + \mathcal{J}^+), \; \mathcal{J}_2 = \frac{1}{i}(\mathcal{J}^- - \mathcal{J}^+),$
$\mathcal{K}_1 = \frac{1}{i\sqrt{\mathcal{L}_3}}(\mathcal{K}^- - \mathcal{K}^+), \; \mathcal{K}_2 = \mathcal{K}^- + \mathcal{K}^+,$
Then $\mathcal{J}_1, \mathcal{J}_2, \mathcal{K}_1, \mathcal{K}_2$ are constants of the motion, polynomial in the momenta.

3-parameter Kepler system 9

Moreover, $\mathcal{J}^+\mathcal{J}^- = \mathcal{P}_1$, an explicit polynomial of order $2(q_1 + p_1)$ in the momenta, and $\mathcal{K}^+\mathcal{K}^- = \mathcal{P}_2$, an explicit polynomial of order $4(p_2 q_1 + p_1 q_2)$ in the momenta.

Message: $\mathcal{J}^\pm, \mathcal{K}^\pm$ are nonpolynomial constants of the motion, depending on polynomials in $\sqrt{\mathcal{L}_2}, \sqrt{\mathcal{L}_3},$, but they form the building blocks of all true polynomial constants of the motion and Poisson brackets formed from them are explicitly computable.

3-parameter Kepler system 10

Examples:

$$\{\mathcal{L}_2, \mathcal{J}_2\} = -i\left(2ip_1\sqrt{\mathcal{L}_2}\mathcal{J}^- + 2ip_1\sqrt{\mathcal{L}_2}\mathcal{J}^+\right) = 2p_1\mathcal{L}_2\mathcal{J}_1,$$

$$\{\mathcal{L}_2, \mathcal{J}_1\} = \frac{1}{\sqrt{\mathcal{L}_2}}\left(2ip_1\sqrt{\mathcal{L}_2}\mathcal{J}^- - 2ip_1\sqrt{\mathcal{L}_2}\mathcal{L}^+\right) = -2p_1\mathcal{J}_2,$$

$$\{\mathcal{L}_3, \mathcal{J}_1\} = \{\mathcal{L}_3, \mathcal{J}_2\} = 0.$$

$$\{\mathcal{L}_3, \mathcal{K}_2\} = -4p_1p_2\mathcal{L}_3\mathcal{K}_1, \; \{\mathcal{L}_3, \mathcal{K}_1\} = 4p_1p_2\mathcal{K}_2,$$

$$\{\mathcal{L}_2, \mathcal{K}_2\} = \{\mathcal{L}_2, \mathcal{K}_1\} = 0.$$

3-parameter Kepler system 11

$$\mathcal{J}_2^2 = -\mathcal{L}_2\mathcal{J}_1^2 + 4\mathcal{P}_1(\mathcal{H}, \mathcal{L}_2, \mathcal{L}_3).$$

$$\{\mathcal{J}_2, \mathcal{J}_1\} = -p_1\mathcal{J}_1^2 - 4p_1\frac{\partial \mathcal{P}_1}{\partial \mathcal{L}_2}.$$

$$\mathcal{K}_2^2 = -\mathcal{L}_3\mathcal{K}_1^2 + 4\mathcal{P}_2(\mathcal{H}, \mathcal{L}_2, \mathcal{L}_3),$$

$$\{\mathcal{K}_2, \mathcal{K}_1\} = -2p_1p_2\mathcal{K}_1^2 + 8p_1p_2\frac{\partial \mathcal{P}_2}{\partial \mathcal{L}_3}.$$

3-parameter Kepler system 12

Remaining brackets are polynomials in the momenta but rational in the generators:

$$\{\mathcal{J}_1, \mathcal{K}_1\} = \frac{2q_1p_1p_2}{\mathcal{L}_2 - \mathcal{L}_3}(-\mathcal{J}_1\mathcal{K}_2 + \mathcal{J}_2\mathcal{K}_1)$$

$$\{\mathcal{J}_2, \mathcal{K}_1\} = \frac{2q_1p_1p_2}{\mathcal{L}_2 - \mathcal{L}_3}(\mathcal{J}_2\mathcal{K}_2 + \mathcal{L}_2\mathcal{J}_1\mathcal{K}_1)$$

$$\{\mathcal{J}_1, \mathcal{K}_2\} = \frac{2q_1p_1p_2}{\mathcal{L}_2 - \mathcal{L}_3}(\mathcal{L}_3\mathcal{K}_1 + \mathcal{J}_2\mathcal{K}_2)$$

$$\{\mathcal{J}_2, \mathcal{K}_2\} = \frac{2q_1p_1p_2}{\mathcal{L}_2 - \mathcal{L}_3}(\mathcal{L}_3\mathcal{J}_2\mathcal{K}_1 - \mathcal{L}_2\mathcal{J}_1\mathcal{K}_2)$$

A complication 1

The generators for the polynomial symmetry algebra that we have produced so far are not of minimal order. Note that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are of order 2 and the orders of $\mathcal{J}_1, \mathcal{K}_1$ are one less than the orders of $\mathcal{J}_2, \mathcal{K}_2$, respectively. We will construct a symmetry $\mathcal{K}_0$ of order one less than $\mathcal{K}_1$.
Note that the symmetry $\mathcal{K}_2$ is a polynomial in $\mathcal{L}_3$. The constant term in this polynomial expansion is itself a constant of the motion:

$$\mathcal{D}_2(\mathcal{L}_2) = 2(-1)^{(p_1q_2+p_2q_1)/2+1}\mathcal{L}_2^{p_2q_1}(\gamma-\beta)^{p_1q_2}.$$

A complication 2

Thus

$$\mathcal{K}_0 = \frac{\mathcal{K}_2 - \mathcal{D}_2}{\mathcal{L}_3}$$

is a polynomial symmetry of order two less than $\mathcal{K}_2$. We have the identity

$$\mathcal{K}_2 = \mathcal{L}_3\mathcal{K}_0 + \mathcal{D}_2.$$

From this, $\{\mathcal{L}_3, \mathcal{K}_2\} = \mathcal{L}_3\{\mathcal{L}_3, \mathcal{K}_0\}$ and

$$\{\mathcal{L}_3, \mathcal{K}_0\} = -4p_1p_2\mathcal{K}_1, \quad \{\mathcal{L}_2, \mathcal{K}_0\} = 0.$$

The same construction fails for $\mathcal{J}_2$.

3-parameter Kepler system 13

Now we choose $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{J}_1, \mathcal{K}_0$ as the generators of our algebra. We define the basic nonzero commutators as

$$\mathcal{R}_1 = \{\mathcal{L}_2, \mathcal{J}_1\} = -2p_1\mathcal{J}_2, \quad \mathcal{R}_2 = \{\mathcal{L}_3, \mathcal{K}_0\} = -4p_1p_2\mathcal{K}_1,$$

Then we have

$$\frac{\mathcal{R}_1^2}{4p_1^2} = \mathcal{J}_2^2 = -\mathcal{L}_2\mathcal{J}_1^2 + 4\mathcal{P}_1,$$

$$\frac{\mathcal{R}_2^2}{16p_1^2p_2^2} = \mathcal{K}_1^2 = \frac{-(\mathcal{L}_3\mathcal{K}_0 + \mathcal{D}_2)^2 + 4\mathcal{P}_1}{\mathcal{L}_3}$$

polynomials in the generators.

3-parameter Kepler system 14

We conclude that the polynomial symmetry algebra generated by the 5 basic generators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{J}_1, \mathcal{K}_0$ and their 2 commutators closes rationally, but not polynomially.

4-parameter Kepler system 1

Now we consider the Hamiltonian

$$\mathcal{H} = \mathcal{L}_1 = p_r^2 + \frac{\alpha}{r} + \frac{\mathcal{L}_2}{r^2}$$

$$\mathcal{L}_2 = p_{\theta_1}^2 + \frac{\mathcal{L}_3}{\sin^2(k_1\theta_1)} + \frac{\delta}{\cos^2(k_1\theta_1)},$$

$$\mathcal{L}_3 = p_{\theta_2}^2 + \frac{\beta}{\cos^2(k_2\theta_2)} + \frac{\gamma}{\sin^2(k_2\theta_2)}.$$

Here $\mathcal{L}_2,\ \mathcal{L}_3$ are constants of the motion, in involution. They determine additive separation in the variables r, θ, ϕ.

4-parameter Kepler system 2

In Cartesian coordinates with $k_1 = k_2 = 1$,

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}.$$

Here

$$x = r\sin(\theta_1)\cos(\theta_2),\; y = r\sin(\theta_1)\sin(\theta_2),\; z = r\cos(\theta_1).$$

It is NOT 2nd order superintegrable.

4-parameter Kepler system 3

For general rational k_1, k_2 we can construct nonpolynomial constants of the motion $\mathcal{J}^{\pm}, \mathcal{K}^{\pm}$ in analogy with the construction for the 3-parameter system and polynomial symmetries

$$\mathcal{J}_1 = \frac{1}{\sqrt{\mathcal{L}_2}}(\mathcal{J}^- + \mathcal{J}^+),\ \mathcal{J}_2 = \frac{1}{i}(\mathcal{J}^- - \mathcal{J}^+),$$

$$\mathcal{K}_1 = \frac{1}{\sqrt{\mathcal{L}_3}}(\mathcal{K}^- + \mathcal{K}^+),\ \mathcal{K}_2 = \frac{1}{i}(\mathcal{K}^- - \mathcal{K}^+).$$

The structure relations are similar to those derived in the 3-parameter case.

4-parameter Kepler system 4

Here $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are of order 2 and the orders of $\mathcal{J}_1, \mathcal{K}_1$ are one less than the orders of $\mathcal{J}_2, \mathcal{K}_2$, respectively. In analogy with the 3-parameter case we can construct symmetries $\mathcal{J}_0$, $\mathcal{K}_0$ of order one less than $\mathcal{J}_1$,$\mathcal{K}_1$, respectively.

The structure equations can be computed explicitly and, again, the symmetry algebra closes algebraically, but not polynomially.

Caged Kepler problem: $k_1 = k_2 = 1$. 1

For $k_1 = k_2 = 1$, $\mathcal{J}_1$ is of order 5 and $\mathcal{J}_2$ is of order 6, whereas $\mathcal{K}_1$ is of order 3 and $\mathcal{K}_2$ is of order 4. Then $\mathcal{J}_0$, $\mathcal{K}_0$ are of orders 4 and 2, respectively.

The symmetry algebra generators are $\mathcal{H} = \mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{K}_0$ and $\mathcal{J}_0$, so the system is 4th order superintegrable, as found by Verrier and Evans (2008). The system closes only rationally.

Caged Kepler problem: $k_1 = k_2 = 1$. 2

However, Tanoudis and Daskaloyannis (2011) showed that an analog of the Kalnins - Kress - Miller $5 \to 6$ Theorem applies here. There is another 4th order constant of the motion $\mathcal{J}_0'$ such that the set of 6 generators

$$\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{K}_0, \mathcal{J}_0, \mathcal{J}_0'\}$$

is linearly independent and generates a symmetry algebra that closes polynomially.

Caged Kepler problem: $k_1 = k_2 = 1$. 3

We are in Euclidean space and our system has additional symmetry. Here

$$x = r\sin\theta_1\cos\theta_2,\ \ y = r\sin\theta_1\sin\theta_2,\ \ z = r\cos\theta_1,$$

and

$$\mathcal{H} = p_x^2 + p_y^2 + p_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}.$$

Note that any permutation of the ordered pairs $(x,\beta),(y,\gamma),(z,\delta)$ leaves the Hamiltonian unchanged. This discrete S_3 symmetry leads to additional structure in the symmetry algebra.

Caged Kepler problem: $k_1 = k_2 = 1$. 4

Here

$$\mathcal{L}_3 = \mathcal{I}_{xy} = (xp_y - yp_x)^2 + \frac{\beta(x^2+y^2)}{x^2} + \frac{\gamma(x^2+y^2)}{y^2}.$$

Note that the permutation symmetry of the Hamiltonian shows that $\mathcal{I}_{xz}, \mathcal{I}_{yz}$ are also constants of the motion, and that

$$\mathcal{L}_2 = \mathcal{I}_{xy} + \mathcal{I}_{xz} + \mathcal{I}_{yz} - (\beta + \gamma + \delta).$$

$$\mathcal{K}_0 = 2(\mathcal{I}_{yz} - \mathcal{I}_{xz}),$$

Caged Kepler problem: $k_1 = k_2 = 1.5$

$\mathcal{J}_0$ is 4th order:

$$\mathcal{J}_0 = -16\left(\mathcal{M}_3^2 + \frac{\delta(xp_x + yp_y + zp_z)^2}{z^2}\right)$$

$$+8\mathcal{H}(\mathcal{I}_{xz} + \mathcal{I}_{yz} - \beta - \gamma - \delta) + 2\alpha^2,$$

where

$$\mathcal{M}_3 = (yp_z - zp_y)p_y - (zp_x - xp_z)p_x$$

$$-z\left(\frac{\alpha}{2r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}\right).$$

Caged Kepler problem: $k_1 = k_2 = 1$. 6

The symmetries $\mathcal{H}, \mathcal{L}_2, \mathcal{L}_3, \mathcal{J}_0, \mathcal{K}_0$ form a generating (rational) basis for the constants of the motion. Under the transposition $(x, \beta) \leftrightarrow (z, \delta)$ this basis is mapped to an alternate basis $\mathcal{H}, \mathcal{L}'_2, \mathcal{L}'_3, \mathcal{J}'_0, \mathcal{K}'_0$ where

$$\mathcal{L}'_2 = \mathcal{L}_2, \qquad \mathcal{L}'_3 = \frac{1}{4}\mathcal{K}_0 + \frac{1}{2}\mathcal{L}_2 - \frac{1}{2}\mathcal{L}_3 + \frac{\beta+\gamma+\delta}{2},$$

$$\mathcal{K}'_0 = \frac{1}{2}\mathcal{K}_0 - \mathcal{L}_2 + 3\mathcal{L}_3 - (\beta+\gamma+\delta),$$

Caged Kepler problem: $k_1 = k_2 = 1$. 7

All of the identities derived for the unprimed symmetries hold for the primed symmetries. It is easy to see that the $\mathcal{K}'$ symmetries are simple polynomials in the $\mathcal{L}, \mathcal{K}$ symmetries already constructed.
However, the $\mathcal{J}'$ symmetries are new.
Under the transposition symmetry $(x, \beta) \leftrightarrow (y, \gamma)$ the constants of the motion $\mathcal{L}_2, \mathcal{L}_3, \mathcal{J}_0, \mathcal{J}_1$ are invariant, whereas $\mathcal{K}_0, \mathcal{K}_1$ change sign. Similarly we can classify the various constants of the motion in terms of their behavior under transpositions.

Caged Kepler problem: $k_1 = k_2 = 1$. 8

Tanoudis and Daskaloyannis show that the quiantum symmetry algebra generated by the 6 functionally dependent symmetries $\mathcal{H}, \mathcal{L}_2, \mathcal{L}_3, \mathcal{J}_0, \mathcal{K}_0$ and $\mathcal{J}_0'$ closes polynomially, in the sense that all double commutators of the generators are again expressible as polynomials in the generators, strng evidence that the classical analog also closes polynomially. Keys to understanding this are the rational structure equations, transposition symmetries and addition of the new symmetry $\mathcal{J}_0'$.

Caged Kepler problem: $k_1 = k_2 = 1$. 9

We find, for example,

$$\mathcal{J}_1\mathcal{K}_1 =$$

$$\frac{1}{2}[\mathcal{L}_2 + \mathcal{L}_3 - \delta]\mathcal{J}_0\mathcal{K}_0 + \alpha^2[\mathcal{L}_2 - 3\mathcal{L}_3 - \delta]\mathcal{K}_0 + (\beta - \gamma)[3\mathcal{L}_2 - \mathcal{L}_3 + \delta]\mathcal{J}_0 + 2\alpha^2(\gamma - \beta)[\mathcal{L}_2 + \mathcal{L}_3 - 5\delta] + \left(-\mathcal{J}_0 - 2\mathcal{J}_0' + 2\alpha^2\right)\left((\mathcal{L}_3 - \mathcal{L}_2 - \delta)^2 - 4\delta\mathcal{L}_2\right).$$

If the new symmetry $\mathcal{J}_0'$ were not allowed, the expression for $\mathcal{J}_1\mathcal{K}_1$ would have involved square roots.

Caged Kepler problem: $k_1 = k_2 = 1$. 10

From our construction for general k_1, k_2 we have

$$\mathcal{J}_1^2 = -\mathcal{L}_2 \mathcal{J}_0^2 + \mathcal{P}^{(1)} \mathcal{J}_0 + \mathcal{P}^{(2)}$$

$$\mathcal{K}_1^2 = -\mathcal{L}_3 \mathcal{K}_0^2 + \mathcal{P}^{(3)} \mathcal{K}_0 + \mathcal{P}^{(4)}$$

Where the $\mathcal{P}^{(j)}$ are given polynomials in the generators $\mathcal{H}, \mathcal{L}_2, \mathcal{L}_3$. Note that $\mathcal{J}_0'$ does not appear in these expressions.

Caged Kepler problem: $k_1 = k_2 = 1$. 11

To determine the functional relationship between the 6 generators we can use the identity $\mathcal{J}_1^2\mathcal{K}_1^2 - (\mathcal{J}_1\mathcal{K}_1)^2 = 0$, where $\mathcal{J}_1^2$, $\mathcal{K}_1^2$, and $\mathcal{J}_1\mathcal{K}_1$ are already known polynomials in the 6 generators. The result is a polynomial in the momenta of order 12 that is too lengthy to be written down here, but of the simple form

$$\mathcal{A}_1(\mathcal{J}_0')^2+\mathcal{A}_2\mathcal{J}_0'\mathcal{J}_0+\mathcal{A}_3(\mathcal{J}_0)^2+\mathcal{A}_4\mathcal{J}_0'+\mathcal{A}_5\mathcal{J}_0+\mathcal{A}_6 = 0,$$

where $\mathcal{A}_j = \mathcal{A}_j(\mathcal{H}, \mathcal{L}_2, \mathcal{L}_3, \mathcal{K}_0)$.

Caged Kepler problem: $k_1 = k_2 = 1$. 12

By similar computations we can compute all of the structure equations and prove polynomial closure of the symmetry algebra.

Due to the Stäckel equivalence of Kepler-Coulomb and caged isotropic oscillator systems, it can be seen that for the system $k_1 = k_2 = \frac{1}{2}$ the same S_3 transposition symmetry exists and polynomial closure is achieved by adding a 6th generator.

Conclusions and Outlook 1

- The raising and lowering symmetry method developed by Kalnins, Kress and Miller for proving superintegrability and dermining the structure equations for families of 2D Hamiltonian systems can be extended to the 3D case. The construction appears to be quite general and not restricted to Kepler-Coulomb analogs.
- For these higher order superintegrable systems it appears that algebraic closure is the norm. For polynomial closure, extra symmetry is needed.

Conclusions and Outlook 2

- Discrete symmetries need to be considered in the definition of superintegrability. How should this be done?
- These constructions been extended to the quantum case. That is the subject of another talk.
- We have generalized this approach to systems in $n > 3$ dimensions. The 2D and 3D constructions involved analogs of polar and spherical coordinates. In n dimensions polyspherical coordinates of various types can be employed.