Homework #6

5.1.1 $\quad A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0 = (1-\lambda)(4-\lambda)+2 = \lambda^2-5\lambda+6$$
$$= (\lambda-2)(\lambda-3)$$

$\lambda_1 = 2, \lambda_2 = 3$

$$\lambda_1 = 2: \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x+y=0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 3 \quad \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2x+y=0 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\operatorname{tr} A = 1+4 = 5 = \lambda_1 + \lambda_2$

$\det A = 4+2 = 6 = \lambda_1 \lambda_2$

5.1.2 $\quad A = S\Lambda S^{-1}, \quad \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad S = (v_1, v_2) = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

$S^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

$$u(t) = S e^{t\Lambda} S^{-1} u_0 = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}\begin{pmatrix} 6 \\ -6 \end{pmatrix} = \begin{pmatrix} 6e^{2t}-6e^{3t} \\ -6e^{2t}+12e^{3t} \end{pmatrix}$$

$$= 6e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 6e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

pure exponential solutions: $e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

5-1-3 The eigenvalues of B are $\lambda_1 - 7 = -5$, $\lambda_2 - 7 = -4$,
i.e., they are the eigenvalues of A shifted by -7.
The eigenvectors are unchanged: v_1, v_2.

5-2-2 $\lambda_1 = 1,\ v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\lambda_2 = 4,\ v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$$S = (v_1, v_2) = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix},\quad A = S\Lambda S^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 3 & 8 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 18 \\ -3 & 10 \end{pmatrix}$$

5-2-7 $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = 0 = (4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$

$\lambda_1 = 1,\ \lambda_2 = 5$

Eigenvectors: $\lambda_1 = 1$ $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = -y,\ v_1 = \begin{pmatrix} -1 \\ +1 \end{pmatrix}$

$\lambda_2 = 5$ $\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 3y,\ v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix},\ S = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix},\ S^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

$$A^{100} = S\Lambda^{100}S^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^{100} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -1 & 3\cdot 5^{100} \\ 1 & 5^{100} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 + 3\cdot 5^{100}, & -3 + 3\cdot 5^{100} \\ -1 + 5^{100}, & 3 + 5^{100} \end{pmatrix}$$

5-2-10 $\underset{3\times 3}{A}$: eigenvalues $\lambda_1=1, \lambda_2=2, \lambda_3=4$

A^2 : eigenvalues $\lambda_1^2=1, \lambda_2^2=4, \lambda_3^2=16$

$\text{trace}\, A^2 = \lambda_1^2+\lambda_2^2+\lambda_3^2 = 21$

$\det A = \det A^T = \lambda_1\lambda_2\lambda_3 = 8$

$\det A^{-1} = \dfrac{1}{\det A}, \quad \det(A^{-1})^T = \dfrac{1}{\det A^T} = \dfrac{1}{\det A} = \dfrac{1}{8}$

5-3-1 a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; A^2 = AA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \; A^3 = AA^2 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

$A^4 = AA^3 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},$

we recognize that $A^k = \begin{pmatrix} F_{k+1}, & F_k \\ F_k, & F_{k-1} \end{pmatrix}$

so $A^{100} = \begin{pmatrix} F_{101} & F_{100} \\ F_{100} & F_{99} \end{pmatrix}$

Note: $A^{k+1} = AA^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{k+1} & F_k \\ F_k, & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+1}+F_k, & F_k+F_{k-1} \\ F_{k+1}, & F_k \end{pmatrix}$

$= \begin{pmatrix} F_{k+2}, & F_{k+1} \\ F_{k+1}, & F_k \end{pmatrix}$

b) $B = \begin{pmatrix} 7 & 12 \\ -4 & -7 \end{pmatrix}$. Eigenvalues $\begin{vmatrix} 7-\lambda & 12 \\ -4 & -7-\lambda \end{vmatrix} = 0$

$\Rightarrow -(7-\lambda)(7+\lambda) + 48 = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \; \lambda_2 = -1$

Eigenvectors: $\lambda_1 = 1$: $\begin{pmatrix} 6 & 12 \\ -4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x = -2y, \; v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$: $\begin{pmatrix} 8 & 12 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3x+2y=0, \; v_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

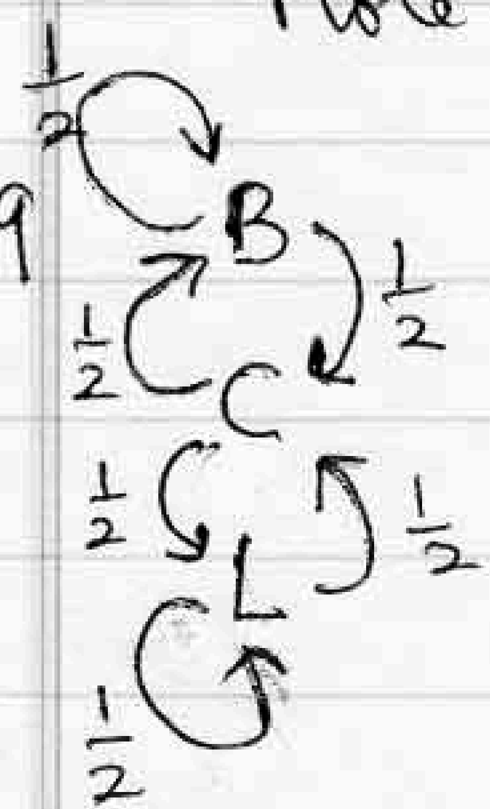
$$S=(v_1,v_2)=\begin{pmatrix}-2 & -2\\ 1 & 3\end{pmatrix},\quad S^{-1}=-\frac{1}{4}\begin{pmatrix}+3, & +2\\ 1, & -2\end{pmatrix}$$

$$\Lambda=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \Lambda^{101}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\Lambda=\Lambda^{-1}$$

$$B^{-101}=S\Lambda^{-101}S^{-1}=S\Lambda S^{-1}=B=\begin{pmatrix}7 & 12\\ -4 & -7\end{pmatrix}$$

Note: Not necessary to compute S since $\Lambda^{-101}=\Lambda$.

5-3-9



$$B(n+1)=\frac{1}{2}B(n)+\frac{1}{2}C(n)$$

$$C(n+1)=\frac{1}{2}B(n) \qquad +\frac{1}{2}L(n)$$

$$L(n+1) \qquad \frac{1}{2}C(n)+\frac{1}{2}L(n)$$

$$u_n=\begin{pmatrix}B(n)\\ C(n)\\ L(n)\end{pmatrix},\qquad u_{n+1}=Au_n=\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}u_n$$

eigenvector corresponding to $\lambda=1$:

$$\begin{pmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -1 & \frac{1}{2}\\ 0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \quad x=y=z$$

$$u_\infty=(B(0)+C(0)+L(0))\begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{pmatrix}$$

$\therefore$ Steady state is that the trucks are equally split between Boston, Chicago, and Los Angeles.

5.4.1 $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, Eigenvalues $\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0 = (\lambda+1)^2 - 1$

$\lambda^2 + 2\lambda = 0 = \lambda(\lambda+2)$, $\lambda_1 = 0$, $\lambda_2 = -2$

Eigenvectors: $\lambda_1 = 0$: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = y$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = -y$, $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

$$e^{At} = S e^{\Lambda t} S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -e^{-2t} \\ 1 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+e^{-2t}, & 1-e^{-2t} \\ 1-e^{-2t}, & 1+e^{-2t} \end{pmatrix}$$

5.4.2 $\frac{du}{dt} = Au$, $u(t) = e^{At} u_0 = \cancel{e^{At}\begin{pmatrix} 3 \\ 1 \end{pmatrix}}$ $u_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 1+e^{-2t}, & 1-e^{-2t} \\ 1-e^{-2t}, & 1+e^{-2t} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} x_0(1+e^{-2t}) + y_0(1-e^{-2t}) \\ x_0(1-e^{-2t}) + y_0(1+e^{-2t}) \end{pmatrix}$$

If $u_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ then $u(t) = \frac{1}{2}\begin{pmatrix} 4+2e^{-2t} \\ 4-2e^{-2t} \end{pmatrix}$

The steady state is $\lim_{t \to +\infty} u(t) = \frac{1}{2}\begin{pmatrix} x_0+y_0 \\ x_0+y_0 \end{pmatrix} = \frac{x_0+y_0}{2} v_1$