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The wave equation, $O(2, 2)$, and separation of variables on hyperboloids

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Synopsis

We classify group-theoretically all separable coordinate systems for the eigenvalue equation of the Laplace-Beltrami operator on the hyperboloid $z_1^2+z_2^2-z_3^2-z_4^2=1$, finding 71 orthogonal and 3 non-orthogonal systems. For a number of cases the explicit spectral resolutions are worked out. We show that our results have application to the problem of separation of variables for the wave equation and to harmonic analysis on the hyperboloid and the group manifold $SL(2, R)$. In particular, most past studies of $SL(2, R)$ have employed only 6 of the 74 coordinate systems in which the Casimir eigenvalue equation separates.

Introduction

This paper is one in a series devoted to the relation between the separation of variables problem for the wave equation $\Box\Psi=0$ in four-dimensional space time and the $O(4, 2)$ symmetry of this equation, [**1–3**]. Here we consider a special case of the wave equation in which one sets $x_3=r\sin\varphi$, $x_4=r\cos\varphi$ and separates out the angular variable φ. The reduced equation so obtained is equivalent to the eigenvalue equation for the Laplace-Beltrami operator on the hyperboloid $z_1^2+z_2^2-z_3^2-z_4^2=1$ and admits the symmetry group $O(2, 2)$. Furthermore it is equivalent to the eigenvalue equation for the Casimir operator on the group $SL(2, R)$.

In [**2**] we solved the separation of variables problem for the complexification of our reduced equation. Here, we modify those results to show that the reduced equation separates in 74 coordinate systems, of which 71 are orthogonal. Each separable system is characterized by a pair of second-order commuting operators in the enveloping algebra of $O(2, 2)$. We work out the spectral resolution for many of these operators and show explicitly the relation between our results and harmonic analysis on the hyperboloid.

Section 1. Models of $O(2, 2)$ representations

The group $O(2, 2)$ consists of the real linear transformations which leave the real form $\mathbf{z}\cdot\mathbf{z}=z_1^2+z_2^2-z_3^2-z_4^2$ invariant. The corresponding Lie algebra $o(2, 2)$

is six-dimensional with basis

$$\begin{aligned} M_{12} &= z_1\partial_2 - z_2\partial_1 = -M_{21}, & M_{34} &= z_3\partial_4 - z_4\partial_3 = -M_{43} \\ M_{13} &= z_1\partial_3 + z_3\partial_1 = M_{31}, & M_{14} &= z_1\partial_4 + z_4\partial_1 = M_{41} \\ M_{23} &= z_2\partial_3 + z_3\partial_2 = M_{32}, & M_{24} &= z_2\partial_4 + z_4\partial_2 = M_{42} \end{aligned} \tag{1.1}$$

where $\partial_j = \partial/\partial z_j$. The isomorphism $o(2, 2) \cong s1(2) \oplus s1(2)$ becomes apparent when one chooses the alternate basis

$$\begin{aligned} A_1 &= M_{21} + M_{43}, & A_2 &= M_{23} + M_{14}, & A_3 &= -M_{13} + M_{24} \\ B_1 &= M_{21} - M_{43}, & B_2 &= M_{23} - M_{14}, & B_3 &= -M_{13} - M_{24} \end{aligned} \tag{1.2}$$

with commutation relations

$$[C_1, C_2] = 2C_3, \quad [C_2, C_3] = -2C_1, \quad [C_3, C_1] = 2C_2, \quad C = A, B, \quad [A_i, B_j] = 0. \tag{1.3}$$

In this paper we are concerned with the eigenvalue equation

$$\Delta\Psi(\mathbf{z}) = \sigma(\sigma + 2)\Psi(\mathbf{z}), \quad \sigma \in R, \tag{1.4}$$

where

$$\begin{aligned} \Delta &= M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{24}^2 - M_{12}^2 - M_{34}^2 \\ &= -A_1^2 + A_2^2 + A_3^2 = -B_1^2 + B_2^2 + B_3^2 \end{aligned} \tag{1.5}$$

is the Laplace-Beltrami operator on the hyperboloid $\mathbf{z} \cdot \mathbf{z} = 1$. We shall classify the possible coordinate systems on this hyperboloid in which (1.4) admits solutions via separation of variables and study the corresponding separated solutions. To find these coordinates we make use of **[2]** which lists the complex separable coordinate systems for the Laplace-Beltrami operator on the complex sphere $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1$ and classify the distinct real forms of these coordinates which parametrize the real hyperboloid $\mathbf{z} \cdot \mathbf{z} = 1$. We will show explicitly that to each separable coordinate system $\{\alpha, \beta, \gamma\}$ there corresponds a commuting pair L_1, L_2 of second-order symmetry operators in the enveloping algebra of $o(2, 2)$ such that the corresponding separable solutions $\Psi_{\lambda_1\lambda_2}(\mathbf{z}) = A_{\lambda_1\lambda_2}(\alpha)B_{\lambda_1\lambda_2}(\beta)C_{\lambda_1\lambda_2}(\gamma)$ of (1.4) are characterized by the eigenvalue equations

$$L_1\Psi_{\lambda_j} = \lambda_1\Psi_{\lambda_j}, \quad L_2\Psi_{\lambda_j} = \lambda_2\Psi_{\lambda_j}. \tag{1.6}$$

Here λ_1, λ_2 are the separation constants. Utilizing techniques from **[2**, section 1**]** one can give a general derivation of the operators L_j. However, we shall accomplish the same end by listing these operators for each separable system.

More precisely, we shall divide the separable systems into equivalence classes of coordinates and list a representative from each equivalence class. We consider two coordinate systems as equivalent if one can be obtained from the other by an action of $O(2, 2)$. Furthermore, we do not distinguish between systems $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$ such that $\alpha = \alpha(\alpha')$, $\beta = \beta(\beta')$, $\gamma = \gamma(\gamma')$. An equivalence class of separable coordinates corresponds to an orbit of two-dimensional commuting subspaces of second order symmetries under the adjoint action of $O(2, 2)$ **[2]**.

The separation of variables problem for (1.4) is intimately related to several problems in harmonic analysis and mathematical physics. We examine four of these problems in detail.

1. Separable Solutions of $\square_2\Phi(\mathbf{x})=0$

Let

$$\square_2=\partial_{x_1x_1}+\partial_{x_2x_2}-\partial_{x_3x_3}-\partial_{x_4x_4} \tag{1.7}$$

where the coordinates $\{x_j\}$ are real. In the region $\mathbf{x}\cdot\mathbf{x}>0$ we can introduce new coordinates $\rho>0$, $\mathbf{z}$ with $\mathbf{z}\cdot\mathbf{z}=1$ such that $\mathbf{x}=\rho\mathbf{z}$. In terms of these coordinates we find

$$\square_2=-\partial_{\rho\rho}-\frac{3}{\rho}\partial_\rho+\frac{1}{\rho^2}\Delta \tag{1.8}$$

where Δ is given by (1.5). Thus the function $\Phi(\mathbf{x})=\rho^\sigma\Psi(\mathbf{z})$ is a solution of $\square_2\Phi=0$ provided $\Psi(\mathbf{z})$ is a solution of (1.4). Solution of the separation of variables problem for (1.4) is an essential step in the solution of the corresponding problem for $\square_2\Phi=0$.

2. Separable Solutions of the Wave Equation

Let

$$\square=\partial_{x_1x_1}-\partial_{x_2x_2}-\partial_{x_3x_3}-\partial_{x_4x_4} \tag{1.9}$$

be the wave operator and consider solutions $\Phi(\mathbf{x})$ of the wave equation $\square\Phi=0$ which have the form $\Phi=r^{-1}e^{im\varphi}\Psi(x_1,x_2,r)$, m an integer, where

$$x_4=r\cos\varphi,\quad x_3=r\sin\varphi,\quad r\geqq 0,\quad 0\leqq\varphi<2\pi.$$

Then Ψ satisfies the reduced equation

$$(\partial_{11}-\partial_{22}-\partial_{rr}+r\partial_r+(m^2-1)/r^2)\Psi=0. \tag{1.10}$$

If Ψ is independent of x_2 then (1.10) reduces to the Euler-Poisson-Darboux (EPD) equation [**4**]. The symmetry algebra of (1.10) is isomorphic to $o(2,2)$.

A basis of symmetry operators is

$$\begin{aligned}
M_{12}&=\tfrac12\{(1+x_1^2+x_2^2+r^2)\partial_1+2x_1x_2\partial_2+2x_1r\partial_r\}\\
M_{34}&=-\tfrac12\{(1+x_1^2+x_2^2-r^2)\partial_2+2x_2x_1\partial_1+2x_2r\partial_r\}\\
M_{13}&=\tfrac12\{(1-x_1^2-x_2^2+r^2)\partial_2-2x_2x_1\partial_1-2x_2r\partial_r\}\\
M_{14}&=-(x_1\partial_1+x_2\partial_2+r\partial_r),\quad M_{23}=x_1\partial_2+x_2\partial_1\\
M_{24}&=\tfrac12\{(1-x_1^2-x_2^2-r^2)\partial_1-2x_1x_2\partial_2-2x_1r\partial_r\}
\end{aligned} \tag{1.11}$$

with the same commutation relations as the corresponding basis (1.1). Equation (1.10) can be rewritten as

$$(M_{13}^2+M_{14}^2+M_{23}^2+M_{24}^2+M_{24}^2-M_{12}^2-M_{34}^2)\Psi=(m-1)(m+1)\Psi. \tag{1.12}$$

Moreover, equations (1.4 and 1.12) with $\sigma=m-1$ can be identified through the correspondence

$$x_1=\frac{z_2}{z_1+z_4},\quad x_2=\frac{z_3}{z_1+z_4},\quad r=\frac{1}{z_1+z_4} \tag{1.13}$$

and the operators (1.1 and 1.11) can be shown to coincide. Note that this parametrization of the hyperboloid $H=\{\mathbf{z}:\mathbf{z}\cdot\mathbf{z}=1\}$ covers only the surface $H'\subset H$ where $H'=\{\mathbf{z}\in H: z_1+z_4>0\}$. Thus the problems of determining the separable coordinate systems for equations (1.4 and 1.10) are virtually equivalent: Every separable system for (1.10) yields via (1.13) a separable system for (1.4) defined on a neighbourhood in the surface H' while every separable system for (1.4) defined on a neighbourhood in H' yields a separable system for (1.10). (Separable systems need only be defined locally and may not cover all of H.) We see from this that solution of the separation of variables problem for (1.4) is an essential step in solution of the corresponding problem for the wave equation.

As shown in [**3**], the solution space of (1.10) admits a natural Hilbert space structure inherited from the Hilbert space of positive energy solutions of the wave equation. Furthermore, the induced action of the group $SL(2,R)\times SL(2,R)$, locally isomorphic to $O(2,2)$, is unitary and irreducible.

Indeed, consider the Hilbert space $\mathcal{H}_m$ consisting of functions $f(l,k)$, ($l\geqq 0$, $-\infty<k<\infty$), Lebesgue square integrable with respect to the measure $d\rho(l,k)=l(l^2+k^2)^{-1/2}\,dl\,dk$. The inner product is

$$(f_1,f_2)=\int_0^\infty\int_{-\infty}^\infty f_1\bar{f}_2\,d\rho(l,k),\quad f_1,f_2\in\mathcal{H}_m.$$

The algebra $o(2,2)$ acts on $\mathcal{H}_m$ via

$$\begin{aligned}
M_{12}&=\frac{-i}{2}(k^2+l^2)^{1/2}(\partial_{ll}+l^{-1}\partial_l-m^2l^{-2}+\partial_{kk}-1)\\
M_{34}&=\frac{ik}{2}(\partial_{ll}+l^{-1}\partial_l-m^2l^{-2}-\partial_{kk}+1)-il\partial_{lk}-i\partial_k\\
M_{13}&=\frac{ik}{2}(\partial_{ll}+l^{-1}\partial_l-m^2l^{-2}-\partial_{kk}-1)-il\partial_{lk}-i\partial_k\\
M_{14}&=1+l\partial_l+k\partial_k,\quad M_{23}=(k^2+l^2)^{1/2}\partial_k\\
M_{24}&=\frac{i}{2}(k^2+l^2)^{1/2}(\partial_{ll}+l^{-1}\partial_l-m^2l^{-2}+\partial_{kk}+1).
\end{aligned} \tag{1.14}$$

As shown in [**3**] these operators determine the unitary irreducible representation $D^-_{(|m|-1)/2}\otimes D^-_{(|m|-1)/2}$ of $SL(2,R)\times SL(2,R)$ on $\mathcal{H}_m$. Here D^-_k is a unitary irreducible representation of $SL(2,R)$ belonging to the negative discrete series. The eigenvalues of A_1 and B_1 on $\mathcal{H}_m$ are $i(|m|+2\alpha+1)$, $i(|m|+2\beta+1)$, respectively, where $\alpha,\ \beta=0,1,2,\ldots$.

Given any $f\in\mathcal{H}_m$ there is a corresponding (weak) solution Ψ of (1.10) given by the integral transform

$$\Psi(x_1, x_2, r) = I(f) = \frac{r}{2\pi}\iint \exp\left[i(x_1\sqrt{k^2+l^2} - x_2 k)\right] J_m(lr) \times f(l, k)\, d\rho(l, k). \tag{1.15}$$

Here, $J_m(x)$ is a Bessel function [5]. If $\Psi_\alpha = I(f_\alpha)$, $\alpha = 1, 2$, then

$$\begin{aligned}(\Psi_1, \Psi_2) &\equiv (f_1, f_2)\\ &= i\int_0^\infty \frac{dr}{r}\int_{-\infty}^\infty dx_2 \Psi_1 \partial_1 \bar{\Psi}_2 \\ &= -i\int_0^\infty \frac{dr}{r}\int_{-\infty}^\infty dx_2 (\partial_1\Psi_1)\overline{\Psi_2}\end{aligned} \tag{1.16}$$

where the integrals are independent of the variable $x_1 = t$. Integrating by parts in (1.15) one can check that the operators (1.11) acting on the solution space of (1.10) correspond under the unitary transformation I to the operators (1.14).

There is another Hilbert space structure we can impose on these results. Let $\hat{\Psi}(\mathbf{z}) = \Psi(x_1, x_2, r) = I(f)$, $f \in \mathcal{H}_m$, where $\mathbf{z}$, and (x_1, x_2, r) are related by (1.13). Then $\hat{\Psi}$ is a solution of (1.4) for $\sigma = m - 1$ and $\mathbf{z} \in H'$. Moreover, the right-hand side of (1.15) is defined for $r<0$ and we find $\Psi(x_1, x_2, -r) = (-1)^{m+1}\Psi(x_1, x_2, r)$. Thus we can extend $\hat{\Psi}$ to all of H, (with the exception of the lower-dimensional manifold $z_1 + z_4 = 0$) by imposing the symmetry relation $\hat{\Psi}(-\mathbf{z}) = (-1)^{m+1}\hat{\Psi}(\mathbf{z})$. As is well-known there is an essentially unique $O(2, 2)$ invariant measure $d\mu$ on H. In the region where $z_1 > 0$ this measure is given by $d\mu(\mathbf{z}) = dz_2\, dz_3\, dz_4/z_1$. More generally, in terms of coordinates $(\alpha, \theta, \varphi)$ on H,

$$\begin{aligned} &z_1 = \cosh\alpha\cos\varphi, \quad z_2 = \cosh\alpha\sin\varphi, \quad \alpha \geqq 0,\\ &z_3 = \sinh\alpha\sin\theta, \quad z_4 = \sinh\alpha\cos\theta, \quad 0 \le \varphi, \theta < 2\pi,\end{aligned} \tag{1.17}$$

we have $d\mu(\mathbf{z}) = \cosh\alpha\sinh\alpha\, d\varphi\, d\alpha\, d\theta$. Now suppose $\Psi_j = I(f_j)$, $j = 1, 2$, where $f_j \in \mathcal{H}_m$. A straightforward computation using (1.15) and the inversion theorems for the Fourier and Hankel transforms yields the

THEOREM. *If $m = \pm 1, \pm 2, \ldots$ then*

$$\frac{1}{2}\int_H \hat{\Psi}_1\bar{\hat{\Psi}}_2\, d\mu = \int_0^\infty \frac{dr}{r^3}\int_{-\infty}^\infty dx_1\int_{-\infty}^\infty dx_2 \Psi_1\bar{\Psi}_2 = |m|^{-1}(f_1, f_2). \tag{1.18}$$

Thus we can induce a Hilbert space structure on the functions $\hat{\Psi}$ on H by means of the definition

$$(\hat{\Psi}_1, \hat{\Psi}_2) = |m/2|\int_H \hat{\Psi}_1\bar{\hat{\Psi}}_2\, d\mu = (f_1, f_2). \tag{1.19}$$

The unitary irreducible action of $O(2, 2)$ on the Hilbert space $\mathcal{H}_m$ so obtained is induced by the usual action of $O(2, 2)$ on H.

3. Harmonic Analysis on H

Let $L^2(H)$ be the Hilbert space of functions on H, Lebesgue square integrable with respect to the measure $d\mu$. The inner product is

$$\langle F, G\rangle = \int_H F(\mathbf{z})\bar{G}(\mathbf{z})\, d\mu(\mathbf{z}). \tag{1.20}$$

The usual action of $O(2,2)$ on H induces a unitary representation of $O(2,2)$ on $L^2(H)$ and the decomposition of this representation into a direct integral of irreducible representations is well-known [**6, 7**]. We recall the results using the notation of [**7**]. Since the spaces of even and odd functions on H are separately invariant under $O(2,2)$, we start by performing the decomposition $L^2(H) \cong L^2_+(H) \oplus L^2_-(H)$ where

$$L^2_\pm(H) = \{F \in L^2(H) : F(-\mathbf{z}) = \pm F(\mathbf{z})\}.$$

then

$$\begin{aligned} &\text{i)}\quad L^2_+(H) \cong \left(\sum_{j=0}^{\infty} \oplus \mathcal{H}'_{2j+1}\right) \oplus \int_0^\infty \mathcal{H}'_{\rho e} a(\rho)\, d\rho \\ &\text{ii)}\quad L^2_-(H) \cong \left(\sum_{j=1}^{\infty} \oplus \mathcal{H}'_{2j}\right) \oplus \int_0^\infty \mathcal{H}'_{\rho 0} b(\rho)\, d\rho \end{aligned} \tag{1.21}$$

$$\begin{pmatrix} a(\rho) \\ b(\rho) \end{pmatrix} = \frac{\rho}{4\pi} \begin{pmatrix} \tanh(\pi\rho/2) \\ \coth(\pi\rho/2) \end{pmatrix}.$$

(The spaces $\mathcal{H}'_m$ were defined in subsection 2 above.) More specifically, let $L^2(S^1 \times S^1)$ be the Hilbert space of functions on the torus $S^1 \times S^1 = \{(\xi, \tau) : 0 \le \xi, \tau < 2\pi, \bmod 2\pi\}$, Lebesgue square integrable with respect to the measure $d\xi\, d\tau$. The inner product is

$$[\varphi_1, \varphi_2] = \int_{S^1 \times S^1} \varphi_1(\xi, \tau)\bar{\varphi}_2(\xi, \tau)\, d\xi\, d\tau.$$

As is well-known, [**7**], there is a unitary representation T^ρ of $O(2,2)$ on $L^2(S^1 \times S^1)$ for which the action of the identity component of $O(2,2)$ is induced by the Lie algebra generators

$$\begin{aligned} M_{12} &= \partial_\xi \\ M_{34} &= \partial_\tau \\ M_{13} &= -(1+i\rho)\cos\tau\cos\xi - \cos\tau\sin\xi\,\partial_\xi - \sin\tau\cos\xi\,\partial_\tau \\ M_{14} &= -(1+i\rho)\sin\tau\cos\xi - \sin\tau\sin\xi\,\partial_\xi + \cos\tau\cos\xi\,\partial_\tau \\ M_{23} &= -(1+i\rho)\cos\tau\sin\xi + \cos\tau\cos\xi\,\partial_\xi - \sin\tau\sin\xi\,\partial_\tau \\ M_{24} &= -(1+i\rho)\sin\tau\sin\xi + \sin\tau\cos\xi\,\partial_\xi + \cos\tau\sin\xi\,\partial_\tau. \end{aligned} \tag{1.22}$$

Let $L^2_\pm(S^1 \times S^1)$ be the subspaces of $L^2(S^1 \times S^1)$ consisting of even and odd functions with respect to the reflection $(\xi, \tau) \to (\xi + \pi, \tau + \pi)$:

$$L^2_\pm(S^1 \times S^1) = \{\varphi \in L^2(S^1 \times S^1) : \varphi(\xi + \pi, \tau + \pi) = \pm\varphi(\xi, \tau)\}.$$

Then $L^2(S^1\times S^1)\cong L^2_+(S^1\times S^1)\oplus L^2_-(S^1\times S^1)$ and it is easy to show that $L^2_\pm$ are invariant under $T^\rho: T^\rho\cong T^\rho_+\oplus T^\rho_-$. Moreover, it can be shown that $T^\rho_\pm$ are irreducible representations of $O(2,2)$, [**7**].

Given $\varphi\in L^2_+(S^1\times S^1)$ we define a function

$$\begin{gathered} F_+(\mathbf{z},\rho)=I^\rho_+(\varphi) \quad \text{where} \quad I^\rho_+(\varphi)=a(\rho)[\varphi, K^{\rho+}_{\mathbf{z}}],\\ K^{\rho+}_{\mathbf{z}}(\xi,\tau)=|z_1\cos\xi+z_2\sin\xi-z_3\cos\tau-z_4\sin\tau|^{-1-i\rho}. \end{gathered}\tag{1.23}$$

(Since $K^\rho_{\mathbf{z}}\notin L^2_+(S^1\times S^1)$ for some values of $\mathbf{z}$ the integral must be interpreted as a regularization [**8**, page 46].) It is easy to see that $F_+(\mathbf{z},\rho)=F_+(-\mathbf{z},\rho)$ is a (weak) solution of the wave equation in the variables $\mathbf{z}$ which is homogeneous of degree $\sigma=-1-i\rho$. Thus, for $\mathbf{z}\in H$ it follows that F_+ is a solution of (1.4) with this value of σ. It can also be shown that for $\mathbf{z}=(\cos\alpha,\sin\alpha,\cos\tau,\sin\tau)$, $F_+(\mathbf{z},\rho)\equiv F_+(\alpha,\tau,\rho)$ belongs to $L^2_+(S^1\times S^1)$ and I^ρ_+ becomes a unitary transformation on this space [**7**]. Finally, the operators (1.22) acting on φ induce the operators (1.1) acting on F_+. (Also, the action of reflections of the form $z_j\to -z_j$ on F_+ correspond via (1.23) to group operators on φ.)

Restricting $\mathbf{z}$ to H one can show that $F_+(\mathbf{z},\rho)$ is not identically zero on H if $\varphi\neq 0$ in L^2_+. Thus we can define a Hilbert space structure on $\mathcal{H}'_{\rho e}=I^\rho_+(L^2_+(S^1\times S^1))$ such that $I^\rho_+: L^2_+(S^1\times S^1)\to\mathcal{H}'_{\rho e}$ is unitary.

The specific decomposition (1.21i) is given as follows: Let $\{F_{m,k}: k=0,1,2,\dots\}$ be an orthonormal (ON) basis for $\mathcal{H}'_m$. For $F\in L^2_+(H)$ we define the transform functions

$$a_{m,k}=\int_H F(\mathbf{z})\bar{F}_{m,k}(\mathbf{z})\,d\mu(\mathbf{z}),\quad \varphi(\xi,\tau,\rho)=\int_H F(\mathbf{z})K^{\rho+}_{\mathbf{z}}(\xi,\tau)\,d\mu(\mathbf{z}),\quad 0<\rho<\infty. \tag{1.24}$$

Then we have the inversion formula

$$F(\mathbf{z})=\sum_{\substack{m\\ \text{odd}}}\sum_{k=0}^{\infty}a_{m,k}F_{m,k}(\mathbf{z})+\frac{1}{2\pi}\int_0^\infty a(\rho)[\varphi,K^{\rho+}_{\mathbf{z}}]\,d\rho. \tag{1.25}$$

Similarly, given $\varphi\in L^2_-(S^1\times S^1)$ we define a function $F_-(\mathbf{z},\rho)=I^\rho_-(\varphi)$ where

$$\begin{gathered} I^\rho_-(\varphi)=b(\rho)[\varphi,K^{\rho-}_{\mathbf{z}}]\\ K^{\rho-}_{\mathbf{z}}(\xi,\tau)=\operatorname{sgn}(z_1\cos\xi+z_2\sin\xi-z_3\cos\tau-z_4\sin\tau)K^{\rho+}_{\mathbf{z}}(\xi,\tau) \end{gathered}\tag{1.26}$$

Here $F_-(\mathbf{z},\rho)=-F_-(-\mathbf{z},\rho)$ is a solution of the wave equation which is homogeneous of degree $\sigma=-1-i\rho$ in $\mathbf{z}$, so F_- is a solution of (1.4). The operators (1.22) acting on φ induce the operators (1.1) on F_-. Restricting $\mathbf{z}$ to H we can define a Hilbert space structure on $\mathcal{H}'_{\rho o}=I^\rho_-(L^2_-(S^1\times S^1))$ such that I^ρ is unitary. For $F\in L^2_-(H)$ we define the transforms

$$a_{m,k}=\int_H F(\mathbf{z})\bar{F}_{m,k}(\mathbf{z})d\mu(\mathbf{z}),\quad \varphi(\xi,\tau,\rho)=\int_H F(\mathbf{z})K^{\rho-}_{\mathbf{z}}(\xi,\tau)d\mu(\mathbf{z}),\quad 0<\rho<\infty, \tag{1.27}$$

and obtain the inversion formula

$$F(\mathbf{z})=\sum_{\substack{m\\ \text{odd}}}\sum_{k=0}^{\infty} a_{m,k}F_{m,k}(\mathbf{z})+\frac{1}{2\pi}\int_0^{\infty} b(\rho)[\varphi, K_{\mathbf{Z}}^{\rho-}]\,d\rho.$$

4. Harmonic Analysis on $SL(2, R)$

Let $\mathscr{L}_2=L_2(SL(2,R))$ be the Hilbert space of functions f on $SL(2,R)$, Lebesgue square integrable with respect to the Haar measure **[9, 10]**. The operators $T(\mathscr{B},\mathscr{C})$,

$$T(\mathscr{B},\mathscr{C})f(\mathscr{A})=f(\mathscr{B}^{-1}\mathscr{A}\mathscr{C}),\quad f\in\mathscr{L}_2,\quad \mathscr{A},\mathscr{B},\mathscr{C}\in SL(2,R),\tag{1.29}$$

define a unitary representation of $SL(2,R)\times SL(2,R)$ on $\mathscr{L}_2$. The kernel of the homomorphism $(\mathscr{B},\mathscr{C})\to T(\mathscr{B},\mathscr{C})$ consists of the two elements $I=\{(\pm\mathscr{I},\pm\mathscr{I})\}$ where $\mathscr{I}$ is the identity matrix, so it follows that (1.29) defines a unitary representation of $O_o(2,2)\cong SL(2,R)\times SL(2,R)/I$ on $\mathscr{L}_2$ where $O_o(2,2)$ is the identity component of $O(2,2)$.

As a basis for the Lie algebra $s1(2)$ of $SL(2,R)$ we choose the matrices

$$C_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad C_2=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad C_3=\begin{pmatrix}0&1\\0&0\end{pmatrix}\tag{1.30}$$

with commutation relations (1.3) Now each element

$$\mathscr{A}=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix},\ \alpha\delta-\beta\gamma=1\tag{1.31}$$

of $SL(2,R)$ can be parametrized by the pair of complex numbers a, b

$$2a=(\delta+\alpha)+i(\gamma-\beta),\quad 2b=(\delta-\alpha)+i(\gamma+\beta)\tag{1.32}$$

subject to the constraint $\mathbf{z}\cdot\mathbf{z}=1$ where

$$a=z_1+iz_2,\quad b=z_3+iz_4,\quad z_j\text{ real.}\tag{1.33}$$

Thus topologically $SL(2,R)$ can be identified with the hyperboloid H. Furthermore, with this identification one can easily extend the representation (1.29) to obtain a representation of $O(2,2)$ on $\mathscr{L}_2$.

A basis for the $s1(2)\times s1(2)$ algebra of infinitesimal operators induced by T is $\{C_j^L, C_j^R, j=1,2,3\}$ where

$$\begin{aligned}C_j^Lf(\mathscr{A})&=\frac{d}{d\tau}T(\exp\tau\, C_j,\mathscr{I})f(\mathscr{A})|_{\tau=0},\\ C_j^Rf(\mathscr{A})&=\frac{d}{d\tau}T(\mathscr{I},\exp\tau\,\mathscr{C}_j)f(\mathscr{A})|_{\tau=0}.\end{aligned}\tag{1.34}$$

A direct computation shows that the local action of $O(2,2)$ on $SL(2,R)$-functions agrees with the action of $O(2,2)$ on H-functions provided we make the identifications

$$\begin{aligned}C_1^L&=B_1,\quad C_2^L=-B_3,\quad C_3^L=B_2\\ C_1^R&=-A_1,\quad C_2^R=A_3,\quad C_3^R=A_2\end{aligned}\tag{1.35}$$

where A_j and B_j are given by (1.1 and 1.2). It follows that the Casimir operator $(C_1^R)^2-(C_2^R)^2-(C_3^R)^2$ on $SL(2, R)$ can be identified with the Laplace operator (1.5) on H. Thus the results of this paper will determine the possible coordinate systems in which the eigenvalue equation for the Laplace operator will permit solutions via separation of variables.

In the following sections we shall list the possible separable systems along with the associated operators L_1, L_2. We shall also work out the separated solutions for representative cases and compute the spectral resolutions of the self-adjoint operators L_j corresponding to the above constructed unitary irreducible representations of $O(2, 2)$. It is well known that these spectral resolutions are vital for the derivation of special function identities and for harmonic analysis related to the separated solutions **[11, 12]**.

Section 2. Split systems

We first list the *split* orthogonal coordinate systems for (1.4). These are orthogonal systems for which the corresponding symmetry operators can be chosen as $L_1=A^2$, $L_2=B^2$, $[A, B]=0$, $A, B\in o(2,2)$. Such systems are the simplest to study since one can characterize the separated solutions as eigenfunctions of the first-order operators A and B.

The possible split systems $\{u_1, u_2, u_3\}$ are:

I1) $\mathbf{z}=(\cosh u_1 \cos u_2, \cosh u_1 \sin u_2, \sinh u_1 \cos u_3, \sinh u_1 \sin u_3)$
$L_1=M_{12}^2$, $L_2=M_{34}^2$, $u_1\geqq 0$.

I2) $\mathbf{z}=(\cosh u_1 \cosh u_2, \sinh u_1 \sinh u_3, \sinh u_1 \cosh u_3, \cosh u_1 \sinh u_2)$
$\mathbf{z}=(\cos u_1 \cosh u_2, \sin u_1 \cosh u_3, \sin u_1 \sinh u_3, \cos u_1 \sinh u_2)$
$L_1=M_{14}^2$, $L_2=M_{23}^2$,

I3) $\mathbf{z}=(\tfrac{1}{2}[e^{-u_1}+(1+u_3^2-u_2^2)e^{u_1}], u_2e^{u_1}, u_3e^{u_1},$
$$\tfrac{1}{2}[-e^{-u_1}+(1+u_2^2-u_3^2)e^{u_1}])$$
$L_1=(M_{12}+M_{24})^2$, $L_2=(M_{13}+M_{43})^2$.

Note that there are two parametrizations I2) corresponding to the same operators. The first set of coordinates covers the region $z_1>\sqrt{1+z_4^2}$ in H while the second set covers the region $0<z_1^2-z_4^2<1$. The parametrization I3) covers the region $z_1+z_4>0$.

Although there are only three split orthogonal systems the total number of split systems is actually six, as can be seen by utilizing the basis (1.2). The coordinates listed above can be characterized by the commuting pairs (A_1, B_1), (A_2, B_2), and (A_1-A_3, B_1+B_3). (This last commuting pair is equivalent to (A_1+A_2, B_1+B_2).)

Using elementary Lie theory one can obtain three additional split systems corresponding to the pairs (A_1, B_1+B_2), (A_2, B_1+B_2), and (A_1, B_2) and show that all remaining split systems are conjugate to one of these six under the adjoint action of $O(2, 2)$. For example, corresponding to the pair (A_1, B_1+B_2) we have the coordinates

I4) $\mathbf{z}=(u_1e^{-u_3}\sin u_2+\cosh u_3 \cos u_2, u_1e^{-u_3}\cos u_2-\cosh u_3 \sin u_2,$
$$u_1e^{-u_3}\sin u_2+\sinh u_3\cos u_2, u_1e^{-u_3}\cos u_2-\sinh u_3 \sin u_2).$$

However,

$$ds^2 = d\mathbf{z} \cdot d\mathbf{z} = -2e^{-2u_3}\, du_1\, du_2 + du_2^2 - du_3^2 \tag{2.1}$$

so these coordinates are non-orthogonal. Similarly, the other two new systems are non-orthogonal:

I5) $\mathbf{z} = (-u_1 e^{-u_3} \sinh u_2 + \cosh u_3 \cosh u_2,\ u_1 e^{-u_3} \cosh u_2 - \sinh u_3 \sinh u_2,$
$-u_1 e^{-u_3} \sinh u_2 + \sinh u_3 \cosh u_2,\ u_1 e^{-u_3} \cosh u_2 - \cosh u_3 \sinh u_2)$
$ds^2 = d\mathbf{z} \cdot d\mathbf{z} = 2e^{-2u_3}\, du_1 du_2 - du_2^2 - du_3^2$
$A_2 = \partial_{u_2},\ B_1 + B_2 = -\partial_{u_1}.$

I6) $\mathbf{z} = (\cosh u_1 \cos u_2 \cosh u_3 - \sinh u_1 \sin u_2 \sinh u_3, -\cosh u_1 \sin u_2 \cosh u_3$
$-\sinh u_1 \cos u_2 \sinh u_3, -\cosh u_1 \sin u_2 \sinh u_3 - \sinh u_1 \cos u_2 \cosh u_3,$
$-\cosh u_1 \cos u_2 \sinh u_3 + \sinh u_1 \sin u_2 \cosh u_3)$
$ds^2 = d\mathbf{z} \cdot d\mathbf{z} = -du_1^2 + du_2^2 - du_3^2 - 2 \sinh 2u_1\, du_2 du_3$
$A_1 = \partial_{u_2},\quad B_2 = \partial_{u_3}.$

We now describe how to obtain separated solutions of (1.4) corresponding to the coordinates I1) and I2), in particular how to obtain ON bases of such solutions for the Hilbert spaces $\mathcal{H}'_m$, $\mathcal{K}'_{\rho e}$ and $\mathcal{K}'_{\rho o}$.

We start with the model (1.14) of the discrete series ($m \geq 0$ an integer) and look for an ON basis $\{f_{\mu\nu}\}$ for $\mathcal{H}_m$ such that

$$A_1 f_{\mu\nu} = i\mu f_{\mu\nu} \quad B_1 f_{\mu\nu} = i\nu f_{\mu\nu}. \tag{2.2}$$

Introducing coordinates ξ, η where

$$k = (\xi^2 - \eta^2)/2, \quad l = \xi\eta, \quad 0 \leq \xi, \eta$$

we find

$$f_{\mu\nu}(\xi, \eta) = \left[\frac{2\alpha!\, \beta!}{(m+\beta)!\,(m+\alpha)!}\right]^{\frac{1}{2}} e^{-(\xi^2+\eta^2)/2} (\xi\eta)^m L_\alpha^{(m)}(\xi^2) L_\beta^{(m)}(\eta^2)$$

$$\mu = m + 2\beta + 1, \quad \nu = m + 2\alpha + 1, \quad \alpha, \beta = 0, 1, 2, \ldots, \tag{2.3}$$

[see **3**]. Here $L_\alpha^{(m)}(x)$ is a generalized Laguerre polynomial. Applying the integral transform I, (1.15), to obtain the associated ON basis $\{\Psi_{\mu\nu}\}$ in $\mathcal{H}'_m$ we find, [**3**],

$$\Psi_{\mu\nu}(u_j) = (2\pi)^{-\frac{1}{2}}(-i)^{m+\beta-\alpha}(-1)^{\alpha+1}\left[\frac{(m+\beta)!\,(m+\alpha)!}{\pi\alpha!\,\beta!}\right]^{\frac{1}{2}} \exp[i(\alpha-\beta)u_3$$
$$-i(m+\alpha+\beta+1)u_2]\,\frac{(\cosh u_1)^{-\alpha-\beta-m-1}(\sinh u_1)^{\alpha+\beta}}{m!}$$
$$\times {}_2F_1(-\alpha, -\beta; m+1; -\sinh^{-2} u_1) \tag{2.4}$$

where ${}_2F_1$ is a hypergeometric function. The coordinates are given by (1.13) and I1).

Next we find an ON basis $\{\varphi_{\mu\nu}\}$ for $L^2(S^1 \times S^1)$ such that

$$A_1 \varphi_{\mu\nu} = i\mu\varphi_{\mu\nu}, \quad B_1 \varphi_{\mu\nu} = i\nu\varphi_{\mu\nu}. \tag{2.5}$$

From (1.2 and 1.22) it follows easily that

$$\varphi_{\mu\nu}(\xi,\tau)=(2\pi)^{-1}\exp[i(\alpha\xi+\beta\tau)]$$
$$\mu=-\beta-\alpha,\quad \nu=\beta-\alpha,\quad \alpha,\beta=0,\pm1,\pm2,\ldots \tag{2.6}$$

The functions for which $\alpha+\beta$ is even form a basis for $L^2_+(S^1\times S^1)$ while those functions with $\alpha+\beta$ odd form a basis for $L^2_-(S^1\times S^1)$. Applying the transform I^ρ_+ to obtain the corresponding ON basis in $\mathcal{H}'_{\rho e}$ we find

$$\begin{aligned}F^e_{\mu\nu}(u_j)=&\frac{\Gamma[\frac12(\beta+\alpha+i\rho+1)]\Gamma[\frac12(\beta-\alpha+i\rho+1)]}{2\pi\Gamma(i\rho)\Gamma(1+\beta)}c^e(\alpha,\beta,\rho)\\ &\times(\sinh u_1)^\beta(\cosh u_1)^\alpha\exp[i(u_2\alpha+u_3\beta)]\\ &\times{}_2F_1((\beta+\alpha+i\rho+1)/2,(\beta+\alpha-i\rho+1)/2;1+\beta;-\sinh^2 u_1)\end{aligned} \tag{2.7}$$

where c is a complex number of modulus one given by

$$c^e(\alpha,\beta,\rho)=a(\rho)\int_0^{2\pi}\int_0^{2\pi}|\cos\xi-\cos\tau|^{-1+i\rho}\exp(i\alpha\xi+i\beta\tau)\,d\xi\,d\tau. \tag{2.8}$$

Similarly, an ON basis $\{F^0_{\mu\nu}\}$ in $\mathcal{H}'_{\rho 0}$ is defined by (2.7) except that the complex number c, $|c|=1$, is now given by

$$c^0(\alpha,\beta,\rho)=b(\rho)\int_0^{2\pi}\int_0^{2\pi}|\cos\xi-\cos\tau|^{-1+i\rho}\operatorname{sgn}(\cos\xi-\cos\tau) \tag{2.9}$$
$$\times\exp(i\alpha\xi+i\beta\tau)\,d\xi d\tau.$$

(Expressions (2.8, 2.9) are consequences of the remarks following (1.23).)

One can check that the above basis functions are proportional to the matrix elements of irreducible unitary representations of $SL(2,R)$ from the discrete and principal series with respect to a basis of eigenvectors of the operator C_1, (1.30), [**10**, **13**].

Before continuing with the computation of separable solutions of (1.4) it is worthwhile pointing out the relative ease in computing integrals of the form $F_{\lambda_j}=I^\rho_\pm(f_{\lambda_j})$ where $f_{\lambda_j}\in L^2(S^1\times S^1)$ satisfies the eigenvalue equations

$$L_1f_{\lambda_j}=\lambda_1 f_{\lambda_j},\quad L_2f_{\lambda_j}=\lambda_2 f_{\lambda_j} \tag{2.10}$$

and the L_j are second order commuting operators in the enveloping algebra of (1.22) which correspond to a separable coordinate system $\{u_j\}$. Indeed, $F_{\lambda_j}(\mathbf{z})$ is necessarily a solution of (1.4) which is separable in the coordinates $\{u_j\}$. Then, explicitly separating variables in (1.4) we can solve the three second-order ordinary differential equations which emerge and determine the general solutions of these equations. Thus we can express F_{λ_j} as a linear combination of at most six basic separated solutions. The expansion constants can be determined by explicit evaluation at certain special values of the parameters u_j for which the integral is most tractable. (In the above examples we used $u_1=0$ and the asymptotic value as $u_1\to+\infty$.) Thus our methods provide a powerful tool for the evaluation of integrals. We know in advance that a particular integral can be expressed simply in an appropriate coordinate system $\{u_j\}$ and we need only compute the expansion constants. Similar remarks hold for generalized eigenfunctions in $L^2_\pm(S^1\times S^1)$ and for basis functions in $\mathcal{H}'_m$.

Another useful property of the transforms $I^\rho_\pm$ which can be used for the evaluation of integrals is the fact that $I^\rho_\pm$ is unitary on $L^2_\pm(S^1\times S^1)$ and intertwines the representations $T^\rho_\pm$ and $T^{-\rho}_\pm$. Thus if $M(\rho)$ is one of the operators (1.22) we have $M(-\rho)I^\rho_\pm(f)(\mathbf{z})=I^\rho_\pm(M(\rho)f)(\mathbf{z})$ valid for all $f\in C^\infty(S^1\times S^1)$ and $z\in S^1\times S^1$. If $\{f^\pm_{\lambda_j}(\rho)\}$ is a (generalized) normalized eigenbasis for $L^2_\pm(S^1\times S^1)$ satisfying equations (2.10) where the second-order operators $L_j(\rho)$ are constructed from operators $M(\rho)$ then $I_+(f^\pm_{\lambda_j}(\rho))=c^\pm(\lambda_j,\rho)f^\pm_{\lambda_j}(-\rho)$ where $|c^\pm|=1$, as follows from the unitarity of the transform.

Note that in each case we treat, the operators L_1, L_2 are initially defined only formally on $L^2(S^1\times S^1)$. In each example we define the L_j precisely on $C^\infty(S^1\times S^1)$ in the obvious way so that they become commuting symmetric operators. Then we determine the possible self-adjoint extensions of the symmetric operators and compute the spectral resolutions. In most cases there is a single possible self-adjoint pair which extend the original operators. Only when there is a multiplicity of self-adjoint extensions will we comment on the extension process.

Next we compute basis solutions of (1.4) which separate in the second of coordinates I2). We start by constructing a generalized eigenbasis $\{f_{\alpha\beta}\}$ for $\mathcal{H}_m$ such that

$$M_{14}f_{\alpha\beta}=i\alpha f_{\alpha\beta},\quad M_{23}f_{\alpha\beta}=i\beta f_{\alpha\beta}. \tag{2.11}$$

Introducing coordinates ξ, η where

$$k=\xi\tanh\eta,\quad l=\xi/\cosh\eta,\quad 0<\xi, -\infty<\eta<\infty$$

we find

$$\begin{gathered} f_{\alpha\beta}(\xi,\eta)=(2\pi)^{-1}(\xi/\cosh\eta)^{i\alpha-1}e^{i\beta\eta},\\ (f_{\alpha\beta},f_{\alpha'\beta'})=\delta(\alpha-\alpha')\delta(\beta-\beta'),\\ -\infty<\alpha,\alpha',\beta,\beta'<\infty. \end{gathered} \tag{2.12}$$

Applying the integral transform I to find the associated eigenbasis in $\mathcal{H}'_m$ we obtain

$$\begin{aligned} F_{\alpha\beta}(u_j)=&\frac{2^{i\alpha}i^{m+1}e^{-\alpha\pi/2}}{(2\pi)^2m!}e^{i\alpha u_2}e^{i\beta u_3}\\ &\times(\sin u_1)^{-i\alpha-m-1}(\cos u_1)^{i\alpha}{}_2F_1((i\alpha+i\beta+m+1)/2,(i\alpha-i\beta+m+1)/2;\\ &\qquad 1+m;\sin^{-2}u_1),\ 0<u_1<\frac{\pi}{2}. \end{aligned} \tag{2.13}$$

(The ${}_2F_1$ is evaluated by taking the limit as $\sin^{-2}u_1$ approaches the cut on the real axis from below.)

Now we find an eigenbasis $\{f_{\alpha\beta}\}$ for $L^2(S^1\times S^1)$ such that (2.11) holds. Let the functions $g^{(\epsilon)}_\nu(\omega)$, $\epsilon=\pm1$, be defined in the interval $-\pi/2<\omega<3\pi/2$ by

$$g^{(1)}_\nu(\omega)=\begin{cases}\left(\dfrac{1+\sin\omega}{\cos\omega}\right)^{i\nu/2}(\cos\omega)^{-(1+i\rho)/2}, & -\dfrac{\pi}{2}<\omega<\dfrac{\pi}{2}\\[2ex] 0, & \dfrac{\pi}{2}\le\omega\le\dfrac{3\pi}{2}\end{cases} \tag{2.14}$$

$$g^{(-1)}_\nu(\omega)=g^{(1)}_\nu(\pi-\omega).$$

Then $\{f_{\alpha\beta}^{\pm(\epsilon,\epsilon')}\}$ is an eigenbasis for $L^2_\pm(S^1\times S^1)$ where

$$\begin{gathered} f_{\alpha\beta}^{\pm(\epsilon,\epsilon')}(u,v)=\frac{2^{-\frac{1}{2}}}{2\pi}g_{\beta+\alpha}^{(\epsilon)}(u)g_{\beta-\alpha}^{(\epsilon')}(v),-\frac{\pi}{2}<u,\quad v\le\frac{3\pi}{2} \\ f^\pm(u+2\pi,v)=\pm f^\pm(u,v) \\ u=\xi+\tau,\quad v=\xi-\tau \\ [f_{\alpha_1\beta_1}^{\alpha_1(\epsilon_1\epsilon_1')},f_{\alpha_2\beta_2}^{\alpha_2(\epsilon_2,\epsilon_2')}]=\delta_{\alpha_1\alpha_2}\delta_{\epsilon_1\epsilon_2}\delta_{\epsilon_1'\epsilon_2'}\delta(\alpha_1-\alpha_2)\delta(\beta_1-\beta_2) \\ \alpha_j=\pm,\quad \epsilon,\epsilon'=\pm1,\quad -\infty<\alpha,\beta<\infty. \end{gathered}\tag{2.15}$$

Thus each point (α,β) in the continuous spectrum has multiplicity four. Applying the transforms $I_\pm^\rho$ to $f_{\alpha\beta}^{\pm(\epsilon,\epsilon')}$ we can obtain eigenbases for $\mathcal{K}'_{\rho e}$ and $\mathcal{K}'_{\rho o}$. However, the expressions for these bases are rather complicated due to the multiplicity of the spectra and the fact that the functional form of the basis elements differs in each u_1 quadrant. These functions are linear combinations of matrix elements of irreducible unitary representations of $SL(2,R)$ from the principal series with respect to a basis of eigenvectors of the operator C_2, (1.30), **[10, 13]**.

Similarly, basis functions corresponding to the three non-orthogonal split systems are proportional to matrix elements and mixed basis matrix elements of irreducible $SL(2,R)$ representations **[14]**. System I3) will be treated in the following section. (In a future paper we will establish that the three non-orthogonal split systems are the only possible non-orthogonal separable systems for (1.4).)

Section 3. $E(1,1)$ subgroup coordinates

Next we list the orthogonal coordinates corresponding to the reduction $o(2,2)\supset\mathscr{E}(1,1)$ where $\mathscr{E}(1,1)$ is the pseudo-Euclidean subalgebra of $o(2,2)$ with basis

$$P_1=M_{12}+M_{24},\quad P_2=-M_{13}+M_{34},\quad M=M_{23}\tag{3.1}$$

and commutation relations

$$[M,P_1]=P_2,\quad [M,P_2]=P_1,\quad [P_1,P_2]=0.\tag{3.2}$$

The simplest such system is I3) corresponding to the operators P_1^2, P_2^2. (Note that $P_1=\partial_{u_2}$, $P_2=-\partial_{u_3}$ and $M=u_2\partial_{u_3}+u_3\partial_{u_2}$). We look for an eigenbasis $\{f_{\alpha\beta}\}$ for $\mathcal{H}_m$ such that

$$P_1f_{\alpha\beta}=i\alpha\cosh\beta f_{\alpha\beta},\quad P_2f_{\alpha\beta}=i\alpha\sinh\beta f_{\alpha\beta}.\tag{3.3}$$

From the expressions (1.14) we find $P_1=i(k^2+l^2)^{\frac{1}{2}}$, $P_2=ik$, $M=(k^2+l^2)^{\frac{1}{2}}\partial_k$. Thus in terms of new coordinates (ξ,η) where $l=\xi$, $k=\xi\sinh\eta$ we have

$$\begin{gathered} f_{\alpha\beta}(\xi,\eta)=\delta(\xi-\alpha)\delta(\eta-\beta),\quad 0\le\alpha,\quad -\infty<\beta<\infty \\ \langle f_{\alpha\beta},f_{\alpha'\beta'}\rangle=\alpha\delta(\alpha-\alpha')\delta(\beta-\beta'). \end{gathered}\tag{3.4}$$

Furthermore,

$$\Psi_{\alpha\beta}(\mathbf{z})=I(f_{\alpha\beta})=\frac{\alpha}{2\pi}e^{-u_1}J_m(\alpha e^{-u_1})\exp[i\alpha(u_2\cosh\beta-u_3\sinh\beta)].\tag{3.5}$$

Note that

$$P_1=i\xi\cosh\eta,\quad P_2=i\xi\sinh\eta,\quad M=\partial_\eta\tag{3.6}$$

on $\mathcal{H}_m$ and for fixed $\xi=\alpha$ these operators determine an irreducible unitary representation $T^{(\alpha)}$ of the pseudo-Euclidean group $E(1,1)$ on the Hilbert space $L^2(R)$ [**15**, **16**]. Indeed, $P_1^2-P_2^2=-\xi^2=-\alpha^2$ on the space. Thus the irreducible representation of $O(2,2)$ on $\mathcal{H}_m$ decomposes to the direct integral $\int_0^\infty \oplus T^{(\alpha)}\, d\alpha$ on restriction to $E(1,1)$. We can use the commuting operators $L_1=P_1^2-P_2^2$ and $L_2=P_2^2$ to label the basis functions $f_{\alpha\beta}$.

In addition to I3) there are ten other systems related to the subgroup reduction $O(2,2)\supset E(1,1)$. These systems correspond to the operator $L_1=P_1^2-P_2^2$ and to the operators $L_2=M^2$, $\{M,P_2\}$, $\{M,P_1\}$, $\{M,P_1-P_2\}+(P_1+P_2)^2$, $M^2-P_1P_2$, $M^2\pm(P_1+P_2)^2$, $M^2\pm P_2^2$, respectively [**17**]. (Here, $\{A,B\}=AB+BA$). For five of these systems the spectral resolutions and separated solutions can be found in [**16**]. In each case the eigenfunctions can be written in the form

$$\Psi_{\alpha\gamma}(\mathbf{z})=\frac{\alpha}{2\pi}\,e^{-u_1}J_m(\alpha e^{-u_1})G_{\alpha\gamma}(u_2,u_3) \tag{3.7}$$

where $(P_1^2-P_2^2)G_{\alpha\gamma}=-\alpha^2G_{\alpha\gamma}$ the Klein-Gordon equation, and $G_{\alpha\gamma}$ is an eigenfunction of one of the operators L_2 listed above. Returning to the system I3) we look for an eigenbasis of $L^2(S^1\times S^1)$ such that equations (3.3) hold. We introduce new variables s, t such that

$$\begin{aligned}
&\cos\tau=\pm 2tX^{-\frac12},\ \sin\tau=\pm(1+s^2-t^2)X^{-\frac12}\\
&\cos\xi=\pm(1-s^2+t^2)X^{-\frac12},\ \sin\xi=\pm 2sX^{-\frac12}\\
&X=(1+s^2-t^2)^2+4t^2\\
&s=(\sin\tau+\cos\xi)^{-1}\sin\xi,\ t=(\sin\tau+\cos\xi)^{-1}\cos\tau
\end{aligned} \tag{3.8}$$

where the plus sign is adopted in the region R_1: $\sin\tau+\cos\xi>0$, and the minus sign in the region R_2: $\sin\tau+\cos\xi<0$. For the computation of eigenfunctions $f_{\alpha\beta}(\xi,\tau)$ we can restrict our attention to R_1 since $f_{\alpha\beta}(\xi+\pi,\tau+\pi)=\pm f_{\alpha\beta}(\xi,\tau)$ for $f_{\alpha\beta}\in L^2_\pm(S^1\times S^1)$. It is straightforward to verify that each $f\in L^2(R_1)$ can be expressed uniquely in the form

$$f(\xi,\tau)=X^{(1+i\rho)/2}g(s,t),\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g|^2\,ds\,dt<\infty$$

and the action of the operators P_1, P_2, M on the functions g is

$$P_1=\partial_s,\quad P_2=-\partial_t,\quad M=t\partial_s+s\partial_t. \tag{3.9}$$

(Here, $d\xi\,d\tau=4X^{-1}\,ds\,dt$.) It follows that this action is equivalent to the regular representation of $E(1,1)$ as a transformation group on the pseudo-Euclidean plane. As is well known, [**15**, ch. 5] the regular representation can be decomposed into a direct integral of irreducible representations $\langle\alpha\rangle$ where $\alpha>0$, each $\langle\alpha\rangle$ occurring with multiplicity four. The spectra of P_1, P_2 are $\pm i\alpha\cosh\beta$, $i\alpha\sinh\beta$ and $i\alpha\sinh\beta$, $\pm i\alpha\cosh\beta$, respectively where $-\infty<\beta<\infty$. In the region R_1 the eigenbasis corresponding to (3.3) is

$$\begin{gathered}
g_{\alpha\beta}(s,t)=\pi^{-1}2^{-\frac32}\exp\left[i\alpha(s\cosh\beta-t\sinh\beta)\right],\\
\alpha>0,\ -\infty<\beta<\infty,\\
[f_{\alpha\beta},f_{\alpha'\beta'}]=\alpha\,\delta(\alpha-\alpha')\delta(\beta-\beta').
\end{gathered} \tag{3.10}$$

Applying the transforms $I^\rho_\pm$ to $f^\pm_{\alpha\beta}$ we obtain the eigenbasis

$$\begin{aligned}F^\pm_{\alpha\beta}(\mathbf{z}) &= I^\rho_\pm(f^+_{\alpha\beta})\\ &= 2^{\frac{3}{2}}\binom{a(\rho)}{b(\rho)}\Gamma(i\rho)(\alpha/2)^{-i\rho}e^{-u_1}[J_{i\rho}(\alpha e^{-u_1})\\ &\quad \pm J_{-i\rho}(\alpha e^{-u_1})]\exp[i\alpha(u_2\cosh\beta - u_3\sinh\beta)].\end{aligned}$$

The other three choices for spectra of P_1, P_2 give similar results. For the bases corresponding to the nine other operators L_2 the basis functions in R_1 take the form $g_{\alpha\beta}(s,t)$ where $g_{\alpha\beta}$ is a solution of the Klein–Gordon equation in the variables s, t and can be found in [**16** and **18**]. The corresponding basis functions in $\mathcal{K}'_{\rho e}$ and $\mathcal{K}'_{\rho o}$ are

$$\begin{aligned}G^\pm_{\alpha\beta}(\mathbf{z}) &= I^\rho_\pm(g^\pm_{\alpha\beta})\\ &= 8\pi\binom{a(\rho)}{b(\rho)}\Gamma(i\rho)(\alpha/2)^{-i\rho}g^\pm_{\alpha\beta}(u_2,u_3)\\ &\quad \times[J_{i\rho}(\alpha e^{-u_1})\pm J_{-i\rho}(\alpha e^{-u_1})].\end{aligned} \tag{3.12}$$

Section 4. $O(2,1)$ subgroup coordinates

There are two types of orthogonal coordinates which correspond to a reduction $O(2,2)\supset O(2,1)$:

IIIi) $\qquad \mathbf{z} = (\xi_3\cosh x_1, \xi_2\cosh x_1, \xi_1\cosh x_1, \sinh x_1)$

$$L_1 = M_{13}^2 + M_{23}^2 - M_{12}^2, \quad \xi_1^2 - \xi_2^2 - \xi_3^2 = -1.$$

Here, (ξ_1, ξ_2, ξ_3) correspond to one of the nine classes of coordinates on the single sheet hyperboloid as listed in Appendix B. The operator L_2 can be read off from Appendix B where $N_3 = M_{23}$, $N_2 = M_{13}$ and $M_1 = M_{12}$. Thus there are nine coordinate systems of this type.

IIIii) $\qquad \mathbf{z} = (\cosh x_1, \xi_1\sinh x_1, \xi_2\sinh x_1, \xi_3\sinh x_1)$

$$L_1 = M_{23}^2 + M_{24}^2 - M_{34}^2, \quad \xi_1^2 - \xi_2^2 - \xi_3^2 = -1.$$

We have the same comments as for *i*) but with $N_3 = M_{23}$, $N_2 = M_{24}$ and $M_1 = M_{34}$. Note that these nine coordinates are valid only for $z_1 > 1$. For $|z_1| < 1$ we have

$$\mathbf{z} = (\cos x_1, \xi_1\sin x_1, \xi_2\sin x_1, \xi_3\sin x_1)$$

where now (ξ_1, ξ_2, ξ_3) corresponds to one of the nine coordinates on the double-sheet hyperboloid $\xi_1^2 - \xi_2^2 - \xi_3^2 = 1$. The operator characterization of these coordinates is identical with that given above.

We confine our attention to coordinates IIIi) since the results for cases IIIii) are so similar. This system corresponds to the reduction of $o(2,2)$ to the subalgebra $o(2,1)$ with basis

$$N_3 = M_{23}, \quad N_2 = M_{13}, \quad M_1 = M_{12} \tag{4.1}$$

and commutation relations (A.3). The model (1.14) for discrete series representations is not very convenient. However, from (1.2) and the well-known Clebsch-Gordan expansion

$$D^-_{(m-1)/2}\otimes D^-_{(m-1)/2}\big|_{so(2,1)} \cong \sum_{s=0}^{\infty}\oplus D^-_{m+s} \tag{4.2}$$

we see that on restriction to $so(2,1)$ our irreducible representation of $so(2,2)$ on $\mathcal{H}'_m$ splits into a direct sum of irreducible representations D^-_{m+s}. Corresponding to the representation D^-_{m+s} the Casimir operator $L_1 = N_3^2 + N_2^2 - M_1^2$ has the eigenvalue $(m+s+1)(m+s)$ and M_1 has eigenvalues $i(m+s+n+1)$, $n = 0, 1, 2, \ldots$. From these facts the basis functions and spectral resolutions can easily be obtained. Indeed, one needs to know the spectral resolution of the operator L_2 acting on the eigenspace $\mathcal{L}_{m+s}$ of eigenfunctions of the Laplace operator L_1 on the hyperboloid $\xi_1^2 - \xi_2^2 - \xi_3^2 = -1$ with eigenvalue $(m+s+1)(m+s)$, a realization of D^-_{m+s}. These results can be read off from [4] on the EPD equation and Appendix A. If $G_{s,\alpha}(\xi)$ is an eigenbasis for $\mathcal{L}_{m+s}$

$$L_1 G_{s,\alpha} = (m+s+1)(m+s)G_{s,\alpha}, \quad L_2 G_{s\alpha} = \alpha G_{s,\alpha} \tag{4.3}$$

then $\Psi_{s,\alpha}$ is the corresponding eigenbasis for $\mathcal{H}'_m$ where

$$\Psi_{s,\alpha}(\mathbf{z}) = \left[\frac{m(s!)}{\Gamma(2m+s+1)}\right]^{\frac{1}{2}} \frac{\Gamma(2m+1)}{\Gamma(m+1)} 2^{-m-1}(\cosh x_1)^{-m-1} C_s^{m+\frac{1}{2}}(\tanh x_1)$$
$$\times G_{s,\alpha}(\boldsymbol{\xi}), \quad s = 0, 1, 2, \ldots, m > 0. \tag{4.4}$$

As an example, in the case where $L_2 = M_3^2$ we have $\boldsymbol{\xi} = (\sinh\eta, \cosh\eta \sin\theta, \cosh\eta\cos\theta)$ and $M_1 G_{s,\alpha} = i(m+s+\alpha+1)G_{s,\alpha}$, $\alpha = 0, 1, 2, \ldots$, where

$$G_{s,\alpha}(\boldsymbol{\xi}) = \left[\frac{(m+s+\frac{1}{2})\alpha!}{\pi\Gamma(2m+2s+\alpha+2)}\right]^{\frac{1}{2}} \frac{\Gamma(2m+2s+2)}{\Gamma(m+s+\frac{3}{2})} 2^{-m-s-1}$$
$$\times \exp[-i(m+s-\tfrac{1}{2})\pi/2](\cosh\eta)^{-(m+s+1)} C_\alpha^{m+s+1}(\tanh\eta)$$
$$\times \exp[-i(m+s+\alpha+1)\theta]. \tag{4.5}$$

In this case $\{\Psi_{s,\alpha}\}$ is an ON basis for $\mathcal{H}'_m$.

For the principal series representations our results are more complicated. We introduce new variables $\mathbf{Y} = (Y_1, Y_2, Y_3)$ in $S^1 \times S^1$ such that

$$Y_1 = \cot\tau,\ Y_2 = \sin\xi \operatorname{cosec}\tau,\ Y_3 = \cos\xi \operatorname{cosec}\tau,\ Y_2^2 + Y_3^2 - Y_1^2 = 1, \tag{4.6}$$

and

$$\begin{aligned} \sin\tau &= \pm(Y_1^2+1)^{-\frac{1}{2}}, \cos\tau = Y_1(Y_1^2+1)^{-\frac{1}{2}}, \\ \cos\xi &= Y_3(Y_1^2+1)^{\frac{1}{2}}, \quad \sin\xi = Y_2(Y_1^2+1)^{-\frac{1}{2}}. \end{aligned} \tag{4.7}$$

Here the plus sign is adopted in the region R_1': $0 < \tau < \pi$, $0 \le \xi < 2\pi$, and the minus sign in the region R_2': $\pi < \tau < 2\pi$, $0 \le \xi < 2\pi$. For the computation of eigenfunctions $f_{\alpha\beta}(\xi, \tau)$ we restrict our attention to R_1' since $f_{\alpha\beta}(\xi+\pi, \tau+\pi) = \pm f_{\alpha\beta}(\xi, \tau)$ for $f_{\alpha\beta} \in L^2_\pm(S^1 \times S^1)$. Now for fixed $\rho > 0$ each $f \in L^2(R_1')$ can be expressed uniquely in the form

$$f(\xi, \tau) = (1+Y_1^2)^{(1+i\rho)/2} g(\mathbf{Y}), \quad g \in L^2(H').$$

Here, $L^2(H')$ is the Hilbert space of measurable functions g such that

$$\iint_{H'} |g|^2 \, d\nu(\mathbf{Y}) < \infty, \quad d\nu(\mathbf{Y}) = \frac{dY_2 dY_3}{|Y_1|}$$

where the integration surface H' is the single-sheet hyperboloid $Y_2^2+Y_3^2-Y_1^2=1$. Furthermore, the action of $o(2,1)$ on g induced by (1.22 and 4.1) is

$$N_3=Y_1\partial_{Y_2}+Y_2\partial_{Y_1},\quad N_2=Y_1\partial_{Y_3}+Y_3\partial_{Y_1},\quad M_1=Y_3\partial_{Y_2}-Y_2\partial_{Y_3}. \tag{4.8}$$

Thus, the induced action of $o(2,1)$ on $L_2(R_1')$ is unitary equivalent to the regular representation of $o(2,1)$ on H'. The decomposition of this regular representation into a direct integral of irreducible representations is well known [7, 15]. We find that

$$L^2(H')\cong L_+^2(H')\oplus L_-^2(H')$$
$$L_\pm^2(H')=\{g\in L^2(H'): g(-\mathbf{Y})=\pm g(\mathbf{Y})\}$$

where $L_\pm^2(H')$ are invariant under the natural action of $O(2,1)$. Let $\{g_{m,k}: k=0,1,2,\ldots\}$ be an ON basis for the space $\mathscr{L}_m$ defined in Appendix A. For $g\in L_+^2(H')$ we define the transform functions

$$b_{m,k}=\int_{H'} g(\mathbf{Y})\bar{g}_{m,k}(\mathbf{Y})\,d\nu(\mathbf{Y}),$$

$$\psi(\zeta,\epsilon,\mu)=\int_{H'} g(\mathbf{Y})|Y_2\cos\zeta+Y_3\sin\zeta+Y_1\epsilon|^{-\frac12-i\mu}\,d\nu(\mathbf{Y}) \tag{4.9}$$
$$0<\mu<\infty,\quad \epsilon=\pm1.$$

Then,

$$g(\mathbf{Y})=\sum_m\sum_{k=0}^{\infty} b_{m,k}g_{m,k}(\mathbf{Y})+\sum_{\epsilon=\pm1}\int_0^\infty \frac{1}{2\pi}J_{+,\epsilon}^{\mu}(\psi)\,d\mu,$$

$$J_{+,\epsilon}^{\mu}(\psi)=c(\mu)\int_0^{2\pi}|Y_2\cos\zeta+Y_3\sin\zeta+Y_1\epsilon|^{-\frac12+i\mu}\psi(\zeta,\epsilon,\mu)\,d\xi, \tag{4.10}$$

$$c(\mu)=\frac{2}{\sqrt{2\pi}}\frac{|\Gamma(\frac12+i\mu)|}{|\Gamma(i\mu)|}$$

Similarly, for $g\in L_-^2(H')$ we define transforms

$$b_{m,k}=\int_{H'} g(\mathbf{Y})\bar{g}_{m,k}(\mathbf{Y})\,d\nu(\mathbf{Y}),$$

$$\psi(\zeta,\epsilon,\mu)=\int_{H'} g(\mathbf{Y})\,\mathrm{sgn}\,(Y_2\cos\zeta+Y_3\sin\zeta+Y_1\epsilon)|Y_2\cos\zeta+Y_3\sin\zeta+\epsilon Y_1|^{-\frac12-i\mu}\,d\nu(\mathbf{Y}). \tag{4.11}$$

Then

$$g(\mathbf{Y})=\sum_m\sum_{k=0}^{\infty} b_{m,k}g_{m,k}(\mathbf{Y})+\frac{1}{2\pi}\sum_{\epsilon=\pm1}\int_0^\infty J_{-,\epsilon}^{\mu}(\psi)\,d\mu,$$

$$J_{-,\epsilon}^{\mu}(\psi)=c(\mu)\int_0^{2\pi}|Y_2\cos\zeta+Y_3\sin\zeta+Y_1\epsilon|^{-\frac12+i\mu}\,\mathrm{sgn}\,(Y_2\cos\zeta+Y_3\sin\zeta+Y_1\epsilon)\psi(\zeta,\epsilon,\mu)\,d\zeta. \tag{4.12}$$

The action (4.8) of $0(2,1)$ induces via (4.10 and 4.12) an action on the functions $\psi(\zeta)$ given by

$$N_3=-\epsilon\left(\sin\zeta\frac{d}{d\zeta}+(\tfrac{1}{2}+i\mu)\cos\zeta\right),\quad N_2=\epsilon\left(\cos\zeta\frac{d}{d\zeta}-(\tfrac{1}{2}+i\mu)\sin\zeta\right),$$
$$M_1=-\frac{d}{d\zeta}. \tag{4.13}$$

The basis eigenfunctions $\Psi_{\alpha\beta}(\varphi)$ on S^1 corresponding to each of the nine separable systems can be obtained directly from **[19]**, together with the spectral decomposition. The corresponding solutions $I^{\rho}_{\pm}(J^{\mu}_{+,\epsilon}(\psi))$ take the form

$$\begin{aligned}&\left({}^{a(\rho)}_{b(\rho)}\right)a'(\mu)\Gamma(i\rho)(\cosh\pi\rho\pm1)2^{i\rho}e^{-\pi\rho/4}\Gamma(\tfrac{1}{2}+i\mu-i\rho)\Gamma(\tfrac{1}{2}-i\mu-i\rho)\\&\cdot(1-\tanh^2x_1)^{i\rho/2}[P^{i\rho/2}_{-\frac{1}{2}-i\mu}(\tanh x_1)\pm P^{i\rho/2}_{-\frac{1}{2}-i\mu}(-\tanh x_1)]\\&\cdot\int_0^{2\pi}|\xi_2\cos\zeta+\xi_3\sin\zeta+\xi_1\epsilon|^{-\frac{1}{2}+i\mu}\,\psi(\zeta,\epsilon,\mu)\,d\varphi\end{aligned} \tag{4.14}$$

where $P^{\mu}_{\nu}(x)$ is an associated Legendre function. Similarly the solutions $I^{\rho}_{\pm}(J^{\mu}_{-,\epsilon}(\psi))$ take the same form except that the quantity in brackets is now $[P^{i\rho/2}_{-\frac{1}{2}-i\mu}(\tanh x_1)\mp P^{i\rho/2}_{-\frac{1}{2}-i\mu}(-\tanh x_1)]$ and the integral reads

$$\int_0^{2\pi}|\xi_2\cos\zeta+\xi_3\sin\zeta+\xi_1\epsilon|^{-\frac{1}{2}+i\mu}\,\mathrm{sgn}\,(\xi_2\cos\zeta+\xi_3\sin\zeta+\xi_1\epsilon)\cdot\psi(\zeta,\epsilon,\mu)\,d\zeta. \tag{4.15}$$

The integrals (4.14, 4.15) considered as functions of $\boldsymbol{\xi}$ are eigenfunctions of the Laplace operator on the single sheet hyperboloid and can be evaluated by exactly the same methods as for the double sheet hyperboloid carried out in **[19]**. However, it seems that these rather straightforward computations have not yet appeared in the literature.

The solutions of (1.4) corresponding to the discrete series in (4.10 and 4.12) take the form $G_{m-\frac{1}{2},\alpha}(\boldsymbol{\xi})$ times a linear combination of $(\cosh x_1)^{m-\frac{1}{2}}C^{m+\frac{1}{2}}_{-i\rho-m-\frac{1}{2}}(i\sinh x_1)$ and $(\cosh x_1)^{+i\rho-1}C^{-i\rho+\frac{1}{2}}_{m+i\rho-\frac{1}{2}}(\tanh x_1)$ where $G_{m-\frac{1}{2},\alpha}(\boldsymbol{\xi})$ is an eigenbasis for $\mathscr{L}_{m-\frac{1}{2}}$ with $L_2G_{m-\frac{1}{2},\alpha}=\alpha G_{m-\frac{1}{2},\alpha}$ and $C^{\nu}_{\mu}(z)$ is a Gegenbauer function.

Section 5. Semi-split systems

Semi-split coordinate systems are those for which the corresponding symmetry operators can be chosen as $L_1=A^2$, L_2, $[A,L_2]=0$, $A\in o(2,2)$. We have discussed several semi-split systems above, but these systems were also either split or associated with the reduction of $o(2,2)$ to a three parameter subalgebra. We now examine the remaining semi-split systems. As we shall see, these systems are all related to coordinates on the single and double sheet hyperboloids but are not obtained by a reduction from $o(2,2)$ to $o(2,1)$, the symmetry algebra of the hyperboloid. The coordinate types on the hyperboloids are those listed in Appendix B.

Class IV. Rotational systems associated with coordinates of type 1 on the single and double sheeted hyperboloids.

IV1) $\mathbf{z}=(\xi_1^{(1)}\cos x_3, \xi_1^{(1)}\sin x_3, \xi_2^{(1)}, \xi_3^{(1)}), \quad \boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=1,$
$L_1=M_{12}^2, \quad L_2=M_{13}^2+M_{23}^2+a(M_{14}^2+M_{24}^2)$

IV2) $\mathbf{z}=(\xi_2^{(1)}\sinh x_3, \xi_1^{(1)}, \xi_3^{(1)}, \xi_2^{(1)}\cosh x_3), \quad \boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=1,$
$L_1=M_{14}^2, \quad L_2=M_{24}^2-M_{12}^2+aM_{23}^2$

IV3) $\mathbf{z}=(\xi_3^{(1)}\sinh x_3, \xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}\cosh x_3), \quad \boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=1,$
$L_1=M_{14}^2, \quad L_2=M_{23}^2+a(M_{24}^2-M_{12}^2)$

IV4) $\mathbf{z}=(\xi_2^{(1)}, \xi_3^{(1)}, \xi_1^{(1)}\cos x_3, \xi_1^{(1)}\sin x_3), \quad \boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=-1,$
$L_1=M_{34}^2, \quad L_2=M_{13}^2+M_{14}^2+a(M_{23}^2+M_{24}^2)$

IV5) $\mathbf{z}=(\xi_2^{(1)}\cosh x_3, \xi_3^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}\sinh x_3), \quad \boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=-1,$
$L_1=M_{14}^2, \quad L_2=M_{13}^2-M_{34}^2+aM_{23}^2$

IV6) $\mathbf{z}=(\xi_3^{(1)}\cosh x_3, \xi_2^{(1)}, \xi_1^{(1)}, \xi_3^{(1)}\sinh x_3), \quad \boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=-1,$
$L_1=M_{14}^2, \quad L_2=M_{23}^2+a(M_{13}^2-M_{34}^2).$

Class V. Rotational systems associated with coordinates of type 2 on the single and double sheeted hyperboloids.

V1) $\mathbf{z}=(\xi_1^{(2)}\cos x_3, \xi_1^{(2)}\sin x_3, \xi_2^{(2)}, \xi_3^{(2)}), \quad \boldsymbol{\xi}^{(2)}\cdot\boldsymbol{\xi}^{(2)}=1,$
$L_1=M_{12}^2, \quad L_2=M_{13}^2+M_{23}^2-aM_{34}^2$

V2) $\mathbf{z}=(\xi_2^{(2)}\sinh x_3, \xi_1^{(2)}, \xi_3^{(2)}, \xi_2^{(2)}\cosh x_3), \quad \boldsymbol{\xi}^{(2)}\cdot\boldsymbol{\xi}^{(2)}=1,$
$L_1=M_{14}^2, \quad L_2=M_{24}^2-M_{12}^2-a(M_{34}^2-M_{13}^2)$

V3) $\mathbf{z}=(\xi_2^{(2)}, \xi_3^{(2)}, \xi_1^{(2)}\cos x_3, \xi_1^{(2)}\sin x_3), \quad \boldsymbol{\xi}^{(2)}\cdot\boldsymbol{\xi}^{(2)}=-1,$
$L_1=M_{34}^2, \quad L_2=M_{13}^2+M_{14}^2-aM_{12}^2$

V4) $\mathbf{z}=(\xi_2^{(2)}\cosh x_3, \xi_3^{(2)}, \xi_1^{(2)}, \xi_2^{(2)}\sinh x_3), \quad \boldsymbol{\xi}^{(2)}\cdot\boldsymbol{\xi}^{(2)}=-1,$
$L_1=M_{14}^2, \quad L_2=M_{13}^2-M_{34}^2+a(M_{24}^2-M_{12}^2).$

Class VI. Rotational systems associated with coordinates of type 3 on the hyperboloids.

VI1) $\mathbf{z}=(\xi_3^{(3)}\sinh x_3, \xi_1^{(3)}, \xi_2^{(3)}, \xi_3^{(3)}\cosh x_3), \quad \boldsymbol{\xi}^{(3)}\cdot\boldsymbol{\xi}^{(3)}=1,$
$\mathbf{z}=(\xi_3^{(3)}\cosh x_3, \xi_2^{(3)}, \xi_1^{(3)}, \xi_3^{(3)}\sinh x_3), \quad \boldsymbol{\xi}^{(3)}\cdot\boldsymbol{\xi}^{(3)}=-1,$
$L_1=M_{14}^2, \quad L_2=\alpha(M_{34}^2-M_{13}^2-M_{24}^2+M_{12}^2)+\beta(\{M_{43}, M_{24}\}-\{M_{31}, M_{12}\}).$

Here, $\{A, B\}=AB+BA$. Note that there are two parametrizations VI1) corresponding to the same operators. These two coordinate systems cover disjoint regions in H.

The eleven systems listed above are distinct real forms of the complex system 13) listed in [**2**].

To explain the relationship between these coordinates and the hyperboloids $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=\pm 1$ we consider the system IV1) in some detail. A straightforward computation shows that when applied to a function $f(\boldsymbol{\xi})e^{i\mu x_3}$ the identities

$$\begin{aligned} M_{14}^2+M_{24}^2 &= N_2^2+\xi_3\partial_{\xi_3}-\mu^2\xi_3^2/\xi_1^2 \\ M_{13}^2+M_{23}^2 &= N_3^2+\xi_2\partial_{\xi_2}-\mu^2\xi_2^2/\xi_1^2 \\ M_{34} &= M_1 \end{aligned} \tag{5.1}$$

hold where N_3, N_2, M_1 are defined by (B1). Thus the differential equation (1.4) becomes

$$(N_3^2+N_2^2-M_1^2+\xi_2\partial_{\xi_2}+\xi_3\partial_{\xi_3}+\mu^2/\xi_1^2)f(\boldsymbol{\xi})=\sigma(\sigma+2)f(\boldsymbol{\xi}). \tag{5.2}$$

Now the second order terms in (5.2) agree with the second order terms in the Laplace–Beltrami operator $N_3^2+N_2^2-M_1^2$ on the hyperboloid $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=1$. In order that variables separate in the reduced equation (5.2) it is necessary that they separate in the second order terms alone, hence that they separate in the eigenvalue equation for $N_3^2+N_2^2-M_1^2$. Among the nine coordinate systems in which this last equation separates only two still yield separation when the perturbing terms in (5.2) are added. These are coordinates IV1) and V1). In the case IV1) we see from Appendix B that coordinates of type 1 are characterized by the operator $N_3^2+aN_2^2$, $a>0$. From (5.1) it follows that L_2 agrees with this operator in the second order terms. Similar remarks hold for all of the other systems listed in this section.

The eleven coordinate systems IV–VI are real forms of the complex system (13) listed in **[2]**, as are the two elliptic cylindrical coordinate systems on S^3, **[20]** and the five systems 17)–21) on the hyperboloids $z_1^2-z_2^2-z_3^2-z_4^2=\pm 1$ **[1]**. In each case the separated solutions of (4.1) can be expressed as an exponential function times a product of associated Lamé functions. We omit the tedious derivations of the spectral resolutions since they are so similar to those worked out in detail in [**1** and **20**]. Due to their relative intractability the associated Lamé functions have not yet proved to be of great practical importance.

Class VII. Rotational systems associated with coordinates of type 4 on the hyperboloids $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=\pm 1$.

VII1) $\mathbf{z}=(\xi_3^{(4)}\sinh x_3, \xi_1^{(4)}, \xi_2^{(4)}, \xi_3^{(4)}\cosh x_3)$, $\boldsymbol{\xi}^{(4)}\cdot\boldsymbol{\xi}^{(4)}=1$,
$L_1=M_{14}^2$, $L_2=M_{23}^2+(M_{21}+M_{31})^2-(M_{24}+M_{34})^2$

VII2) $\mathbf{z}=(\xi_3^{(4)}\cosh x_3, \xi_2^{(4)}, \xi_1^{(4)}, \xi_3^{(4)}\sinh x_3)$, $\boldsymbol{\xi}^{(4)}\cdot\boldsymbol{\xi}^{(4)}=-1$,
$L_1=M_{14}^2$, $L_2=M_{23}^2+(M_{24}+M_{43})^2-(M_{12}+M_{31})^2$.

Class VII′. Rotational systems associated with coordinates of type 5 on the hyperboloids $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=\pm 1$.

VII1)′ $\mathbf{z}=(\xi_3^{(5)}\cosh x_3, \xi_2^{(5)}, \xi_1^{(5)}, \xi_3^{(5)}\sinh x_3)$, $\boldsymbol{\xi}^{(5)}\cdot\boldsymbol{\xi}^{(5)}=-1$,
$L_1=M_{14}^2$, $L_2=M_{23}^2+(M_{21}+M_{31})^2-(M_{24}+M_{34})^2$

VII2)′ $\mathbf{z}=(\xi_3^{(5)}\sinh x_3, -\xi_1^{(5)}, \xi_2^{(5)}, -\xi_3^{(5)}\cosh x_3)$, $\boldsymbol{\xi}^{(5)}\cdot\boldsymbol{\xi}^{(5)}=1$,
$L_1=M_{14}^2$, $L_2=M_{23}^2+(M_{24}+M_{43})^2-(M_{12}+M_{31})^2$.

There are only two distinct systems here since VII1) and VII1)′, respectively VII2) and VII2)′, correspond to the same operators L_1, L_2 and cover disjoint regions of $\mathbf{z}\cdot\mathbf{z}=1$. These systems are real forms of the complex system (14) in **[2]** as are 25) and 26) in **[1]**. We compute the spectral resolution only for VII1) since the other case is similar.

The generalized eigenbasis $\{f_{\mu\kappa}\}$ for $\mathcal{H}_m$ such that

$$M_{14}f_{\mu\kappa}=i\mu f_{\mu\kappa},\quad L_2 f_{\mu\kappa}=-\kappa^2 f_{\mu\kappa} \tag{5.3}$$

where L_2 is given by VII1), takes the form

$$\begin{aligned} f_{\mu\kappa}(\kappa, l) &= \pi^{\frac{3}{2}}\frac{\Gamma\left(\dfrac{m+1+i\kappa+i\mu}{2}\right)\Gamma\left(\dfrac{m+1+i\kappa-i\mu}{2}\right)}{m!\Gamma(1-i\kappa)}\zeta^{i\mu-1} \\ &\times(\tanh x)^m(\cosh x)^{1-i(\kappa+\mu)}{}_2F_1((m+1+i\kappa+i\mu)/2, (m+1+i\kappa-i\mu)/2; \\ &\qquad m+1; \tanh^2 x),\quad 0<\kappa<\infty,\quad -\infty<\mu<\infty, \end{aligned} \tag{5.4}$$

$$\langle f_{\mu\kappa}, f_{\mu'\kappa'}\rangle=\delta(\mu-\mu')\delta(\kappa-\kappa').$$

Here, $l=2\zeta\tanh x/\cosh x$, $k=\zeta(1-\sinh^2 x)/\cosh^2 x$ with $x, \zeta\geqq 0$. The corresponding eigenbasis for $\mathcal{H}'_m$ can be expressed as

$$e^{i\mu u_3}(\cosh u_2)^{i\kappa}(\sinh u_2)^{i\mu}{}_2F_1((i\kappa+i\mu+m+1)/2,$$
$$(i\kappa+i\mu-m+1)/2; i\mu+1; -\sinh^2 u_2)$$

times a linear combination of the functions

$$(\cosh u_1)^{i\mu}(\sinh u_1)^{i\kappa}{}_2F_1((i\kappa+i\mu+m+1)/2, (i\kappa+i\mu-m+1)/2; i\mu+1; -\sinh^2 u_1)$$
$$(\cosh u_1)^{i\mu}(\sinh u_1)^{-i\kappa}{}_2F_1((-i\kappa+i\mu+m+1)/2, (-i\kappa+i\mu-m+1)/2;$$
$$(-i\kappa+1; -\sinh^2 u_1)$$

Here, $\xi_1^{(4)}+\xi_2^{(4)}=\sinh u_1\cosh u_2$, $\xi_1^{(4)}-\xi_2^{(4)}=\cosh u_2/\sinh u_1-\sinh u_1/\cosh u_2+\sinh u_1\cosh u_2$, $\xi_3^{(4)}=\cosh u_1\sinh u_2$.

The corresponding basis for $L_2(S^1\times S^1)$ is much more complicated. Appropriate coördinates for $S^1\times S^1$ are of the form

$$\sin\xi=R^{-1},\quad \cos\xi=2\tan\beta\cosh\alpha/R\cos\beta$$
$$\sin\tau=2\tan\beta\sinh\alpha/R\cos\beta,\quad \cos\tau=(2-\cos^2\beta)/R\cos^2\beta$$
$$d\xi\, d\tau=\frac{4\tan\beta}{\cos^2\beta}d\alpha\, d\beta,\quad R=(1+4\tan^2\beta\cosh^2\alpha/\cos^2\beta)^{\frac{1}{2}},$$
$$-\infty<\alpha<\infty,\quad 0\leqq\beta<\pi/2$$

with seven similar coordinate patches needed to cover $S^1\times S^1$. In any one of these regions, eigenfunctions satisfying equations (5.3) take the form

$$f_{\mu\kappa}(\alpha, \beta)=R^{1+i\rho}e^{i\mu\alpha}g(\beta),\quad -\infty<\mu<\infty, \tag{5.5}$$

where g is a linear combination of the functions

$$(\tan\beta)^{\pm i\rho}(\cos\beta)^{1-i(\kappa+\mu)}{}_2F_1((i\kappa+i\mu\pm i\rho+1)/2,$$
$$(i\kappa-i\mu+i\rho\pm 1)/2; 1\pm i\rho; -\tan^2\beta). \tag{5.6}$$

However, the deficiency indices of L_2 are (2, 2) in this case so there is a two-parameter family of possible self-adjoint extensions, for each of which the spectrum is discrete. The corresponding solutions of (1.4) are obtained in the usual fashion.

Class VIII. Euclidean systems associated with coordinates of type 4 on the hyperboloids $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=\pm 1$.

VIII1) $\mathbf{z}=(\xi_1^{(4)}-\frac{1}{2}x_3^2(\xi_1^{(4)}+\xi_2^{(4)}), (\xi_1^{(4)}+\xi_2^{(4)})x_3, \xi_3^{(4)},$
$\xi_2^{(4)}+\frac{1}{2}x_3^2(\xi_1^{(4)}+\xi_2^{(4)})), \quad \boldsymbol{\xi}^{(4)}\cdot\boldsymbol{\xi}^{(4)}=1$

$$L_1=(M_{12}+M_{42})^2,$$
$$L_2=-2M_{24}^2+M_{34}^2+M_{13}^2-M_{14}^2-\{M_{12}, M_{42}\}-\{M_{13}, M_{34}\}.$$

VIII2) $\mathbf{z}=(\xi_2^{(4)}+\frac{1}{2}x_3^2(\xi_1^{(4)}+\xi_2^{(4)}), \xi_3^{(4)}, (\xi_1^{(4)}+\xi_2^{(4)})x_3,$
$\xi_1^{(4)}-\frac{1}{2}x_3^2(\xi_1^{(4)}+\xi_2^{(4)})), \quad \boldsymbol{\xi}^{(4)}\cdot\boldsymbol{\xi}^{(4)}=-1$

$$L_1=(M_{43}+M_{13})^2,$$
$$L_2=-2M_{13}^2+M_{12}^2+M_{42}^2-M_{14}^2-\{M_{43}, M_{13}\}+\{M_{42}, M_{12}\}.$$

Class IX. Euclidean systems associated with coordinates of type 5 on the hyperboloids $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=\pm 1$.

IX1) As in case VIII1) with $\boldsymbol{\xi}^{(5)}\cdot\boldsymbol{\xi}^{(5)}=1$.

$$L_1=(M_{12}+M_{42})^2,$$
$$L_2=-2M_{12}^2+M_{13}^2+M_{14}^2+M_{34}^2+\{M_{21}, M_{24}\}-\{M_{13}, M_{34}\},$$

IX2) As in case VIII2) with $\boldsymbol{\xi}^{(5)}\cdot\boldsymbol{\xi}^{(5)}=-1$.

$$L_1=(M_{43}+M_{13})^2,$$
$$L_2=-2M_{34}^2+M_{42}^2+M_{14}^2+M_{12}^2+\{M_{34}, M_{31}\}-\{M_{42}, M_{21}\}.$$

The four systems VIII–IX are real forms of the complex system (15) in **[2]**, as are the two systems 23), 24) studied in **[1]**. In each case the separated solutions of (4.1) can be expressed as an exponential function times a product of spheroidal wave functions. We omit derivations of the spectral resolutions which are similar to those carried out in **[1** and **20]**.

Class X. Euclidean systems associated with coordinates of type 6 on the hyperboloids $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=\pm 1$.

X1) $\mathbf{z}=(\xi_1^{(6)}-\frac{1}{2}x_3^2(\xi_1^{(6)}+\xi_2^{(6)}), x_3(\xi_1^{(6)}+\xi_2^{(6)}),$
$\xi_3^{(6)}, \xi_2^{(6)}+\frac{1}{2}x_3^2(\xi_1^{(6)}+\xi_2^{(6)})), \quad \boldsymbol{\xi}^{(6)}\cdot\boldsymbol{\xi}^{(6)}=1$

$$L_1=(M_{12}+M_{42})^2, \quad L_2=\{M_{23}, M_{12}+M_{42}\}+\{M_{14}, M_{13}+M_{43}\},$$

X2) $\mathbf{z}=(\xi_2^{(6)}+\frac{1}{2}x_3^2(\xi_1^{(6)}+\xi_2^{(6)}), \xi_3^{(6)}, x_3(\xi_1^{(6)}+\xi_2^{(6)}),$
$\xi_1^{(6)}-\frac{1}{2}x_3^2(\xi_1^{(6)}+\xi_2^{(6)})), \quad \boldsymbol{\xi}^{(6)}\cdot\boldsymbol{\xi}^{(6)}=-1$

$$L_1=(M_{43}+M_{13})^2, \quad L_2=\{M_{23}, M_{43}+M_{13}\}+\{M_{14}, M_{42}+M_{12}\}.$$

These systems are real forms of the complex system (16) in **[2]** as is 27) in **[1]**. We examine the spectral resolution for X2) alone since X1) is similar. The generalized eigenbasis $\{f_{\mu\lambda}\}$ for $\mathcal{H}_m$ such that

$$(M_{43}+M_{13})f_{\mu\lambda}=-i\mu f_{\mu\lambda},\quad L_2 f_{\mu\lambda}=-2\mu\lambda f_{\mu\lambda} \tag{5.7}$$

where L_2 is given by X2), takes the form

$$f_{\mu\lambda}(k,l)=\frac{|k|^{\frac{1}{2}}\,\delta(\mu-\eta)e^{i\lambda\xi}}{l\sqrt{2\pi}},\quad \eta=k,\quad \sinh\xi=-k/l$$
$$-\infty<\mu,\lambda<\infty,\quad \langle f_{\mu\lambda},f_{\mu'\lambda'}\rangle=\delta(\mu-\mu')\delta(\lambda-\lambda'). \tag{5.8}$$

The corresponding eigenbasis for $\mathcal{H}'_m$ is

$$\Psi_{\mu\lambda}(\mathbf{x})=I(f_{\mu\lambda})=(8\pi^3\mu)^{-\frac{1}{2}}\frac{e^{-i\mu x_2}}{m!}\Gamma\left(\frac{i+m+i\lambda}{2}\right)$$
$$\times(e^{-i\pi/2}\mu a_2)^{(1+m)/2}e^{i\mu a_2/2}{}_1F_1((1+m-i\lambda)/2;1+m;-i\mu a_2) \tag{5.9}$$

where

$$2\sqrt{a_1}=\sqrt{2(x_1-r)}+\sqrt{2(x_1+r)},\quad 2i\sqrt{a_2}=\sqrt{2(x_1-r)}-\sqrt{2(x_1+r)}.$$

(These coordinates cover only the region $x_1>r$ of (x_1, x_2, r) space. There is a similar expression for $x_1<-r$.) Also, $\Psi_{-\mu,\lambda}(x)=\bar{\Psi}_{\mu,\lambda}(x)$. Here $W_{\alpha,\beta}(z)$ is a Whittaker function [**5**].

On $L_2(S^1\times S^1)$ the appropriate coordinates are $(\cos\xi,\sin\xi,\cos\tau,\sin\tau)=(1+\nu^2-\eta^2,2\eta,2\nu,1-\nu^2+\eta^2)X^{-1}$, $X=[(\nu^2-1-\eta^2)^2+4\nu^2]^{\frac{1}{2}}$, $-\infty<\eta,\nu<\infty$, $d\xi\,d\tau=2X^{-2}\,d\eta\,d\nu$ for $\cos\xi+\sin\tau>0$. In this region eigenfunctions g satisfying $(M_{43}+M_{13})g=i\mu g$, $L_2 g=\lambda g$ take the form $g(\eta,\nu)=X^{1+i\rho}\eta^{-i\rho/2}e^{i\mu(\nu-\eta)}h(\eta)$ where h is a linear combination of the functions

$$\eta^{\pm\rho/2}{}_1F_1((1\pm\rho)/2-\lambda i/4\mu;\ 1\pm\rho;\ 2i\mu\eta).$$

The spectral resolution of L_2 is fairly complicated in this case and we shall not give the details. In particular, L_2 does not have a unique self-adjoint extension for $0<\rho<1$.

Section 6. Non-split systems

Non-split coordinate systems are separable systems which are not semi-split, i.e. those for which one cannot write either of the defining operators as the square of a first order symmetry. We have already discussed some semi-split systems in sections 3 and 4 that were related to subgroup reductions. The remaining non-split systems that we now list are not so related. Moreover, the ordinary differential equations for the separated solutions always lead to two-parameter eigenvalue problems, about which relatively little is known. Therefore, we merely list the separable systems and their operator characterizations without attempting spectral resolutions.

Class XI. These are real forms of the complex coordinates (17) in [**2**].

XI1) $z_1^2=-x_1x_2x_3/ab,\quad z_2^2=(x_1-1)(x_2-1)(x_3-1)/(a-1)(b-1)$

$z_3^2=(x_1-b)(x_2-b)(x_3-b)/(a-b)(b-1)b$

$z_4^2=-(x_1-a)(x_2-a)(x_3-a)/(a-b)(a-1)a$

for $a>x_1>b>1>x_2>0>x_3$. The operators are

$$L_1=abM_{12}^2-aM_{13}^2-bM_{14}^2,$$

$$L_2=(a+b)M_{12}^2-(a+1)M_{13}^2-(b+1)M_{14}^2-aM_{32}^2-bM_{42}^2+M_{43}^2.$$

The following four systems and their operator characterizations are obtained from XI1) via the indicated mappings.

XI2) $(z_1, z_2, z_3, z_4)_1 \to (iz_3, iz_4, iz_1, iz_2)$

$$x_1>a>x_2>b>1>x_3>0.$$

XI3) $(\mathbf{z})_1 \to (z_1, iz_4, z_3, iz_2)$

$$x_1>a>b>x_2>1>0>x_3.$$

XI4) $(\mathbf{z})_1 \to (iz_3, z_2, iz_1, z_4)$

$$x_1, x_2>a>b>x_3>1 \quad \text{or} \quad a>x_1, x_2>b>x_3>1 \quad \text{or}$$

$$b>x_1, x_2, x_3>1.$$

XI5) $(\mathbf{z})_1 \to (z_1, iz_3, iz_2, z_4)$

$$x_1, x_2>a>b>1>0>x_3 \quad \text{or} \quad a>x_1, x_2>b>1>0>x_3$$

$$\text{or} \quad b>x_1, x_2>1>0>x_3.$$

XI6) $z_1^2=-x_1x_2x_3/ab, \quad z_2^2=(x_1-1)(x_2-1)(x_3-1)/(a-1)(b-1)$

$$(z_3+iz_4)^2=2(x_1-b)(x_2-b)(x_3-b)/(a-b)(b-1)b,$$

$$x_1>1>x_2>0>x_3, \quad a=\bar{b}=\alpha+i\beta.$$

The operators are

$$L_1=(\alpha^2+\beta^2)M_{12}^2+\alpha(M_{14}^2-M_{13}^2)-\beta\{M_{13}, M_{14}\}$$

$$L_2=2\alpha M_{12}^2+(\alpha+1)(M_{14}^2-M_{13}^2)+\alpha(M_{24}^2-M_{23}^2)$$

$$-\beta\{M_{13}, M_{14}\}-\beta\{M_{23}, M_{24}\}+M_{43}^2.$$

XI7) $(\mathbf{z})_6 \to (iz_3, iz_4, iz_1, iz_2)$,

$$x_1, x_2>1>x_3>0 \quad \text{or} \quad 1>x_1, x_2, x_3>0.$$

XI8) $(z_1+iz_2)^2=2(x_1-c)(x_2-c)(x_3-c)/(c-b)(c-a)(c-d)$

$$(z_3+iz_4)^2=-2(x_1-b)(x_2-b)(x_3-b)/(b-a)(b-c)(b-d)$$

$$a=\bar{b}=\alpha+i\beta, \quad c=\bar{d}=\gamma+i\delta, \quad x_1, x_2, x_3 \quad \text{real}$$

$$L_1=(\alpha^2+\beta^2)M_{12}^2+(\gamma^2+\delta^2)M_{34}^2+\alpha\gamma(M_{13}^2+M_{42}^2-M_{23}^2-M_{14}^2)$$

$$+\beta\delta(\{M_{23}, M_{14}\}-\{M_{13}, M_{42}\})+\alpha\delta(\{M_{13}, M_{23}\}+\{M_{42}, M_{14}\})$$

$$+\gamma\beta(\{M_{13}, M_{14}\}+\{M_{42}, M_{23}\})$$

$$L_2=2\alpha M_{12}^2+2\gamma M_{34}^2+(\alpha+\gamma)(M_{13}^2+M_{42}^2-M_{23}^2-M_{14}^2)$$

$$+\delta(\{M_{13}, M_{23}\}+\{M_{42}, M_{14}\})+\beta(\{M_{13}, M_{14}\}+\{M_{42}, M_{23}\}).$$

Class XII. Real forms of the complex coordinates (18) in [**2**]

XII1) $(z_1+z_3)^2=x_1x_2x_3/a, \quad z_1^2-z_3^2=[(a+1)x_1x_2x_3-a(x_1x_2+x_1x_3+x_2x_3)]/a^2$

$$z_2^2=(x_1-a)(x_2-a)(x_3-a)/a^2(a-1)$$

$$z_4^2=(x_1-1)(x_2-1)(x_3-1)/(a-1), \quad x_1, x_2, x_3>a>1,$$

$$L_1=(M_{21}+M_{32})^2+a(M_{41}+M_{34})^2+aM_{13}^2,$$

$$L_2=-(a+1)M_{13}^2-M_{23}^2+M_{12}^2+a(M_{14}^2-M_{34}^2)+(M_{21}+M_{32})^2 + (M_{41}+M_{34})^2.$$

XII2) $(\mathbf{z})_1 \to (iz_3, iz_4, iz_1, iz_2)$

$$x_1, x_2>a>1>0>x_3 \quad \text{or} \quad a>x_1, x_2>1>0>x_3.$$

XII3) $(\mathbf{z})_1 \to (iz_3, z_2, iz_1, z_4)$

$$x_1>a>1>x_2>0>x_3.$$

XII4) $(\mathbf{z})_1 \to (z_1, iz_4, z_3, iz_2)$

$$x_1, x_2>a>1>x_3>0 \quad \text{or} \quad a>x_1, x_2>1>x_3>0.$$

XII5) $(z_1+z_3)^2=x_1x_2x_3/ab, \quad (z_2+iz_4)^2=2(x_1-b)(x_2-b)(x_3-b)/(b-a)b^2$

$$z_1^2-z_3^2=\frac{(a+b)}{a^2b^2}x_1x_2x_3-\frac{1}{ab}(x_1x_2+x_1x_3+x_2x_3)$$

$$a=\bar{b}=\alpha+i\beta, \quad x_1, x_2, x_3 \text{ real}, \quad \operatorname{sign}(x_1x_2x_3)=+,$$

$$L_1=(\alpha^2+\beta^2)M_{13}^2+\alpha(M_{21}+M_{23})^2-\alpha(M_{41}+M_{43})^2 + \beta\{M_{21}+M_{23}, M_{41}+M_{43}\},$$

$$L_2=-2\alpha M_{13}^2+\alpha(M_{21}^2+M_{43}^2-M_{41}^2-M_{23}^2)+\beta\{M_{21}, M_{41}\} - \beta\{M_{23}, M_{43}\}+(M_{21}+M_{23})^2-(M_{41}+M_{43})^2.$$

XII6) $(\mathbf{z})_5 \to (iz_3, iz_4, iz_1, iz_2), \quad \operatorname{sign}(x_1x_2x_3)=-.$

Class XIII. Real forms of the complex coordinates (19).

XIII1) $(z_4+z_2)^2=(x_1-1)(x_2-1)(x_3-1), \quad (z_3+z_1)^2=x_1x_2x_3$

$$z_2^2-z_4^2=2x_1x_2x_3-(x_1x_3+x_2x_3+x_1x_2)+1$$

where $x_1>1>x_2, x_3>0$ or $x_1>1>0>x_2, x_3$ or $x_1, x_2, x_3>1$,

$$L_1=4(M_{34}-M_{23})^2+2\{M_{34}-M_{23}, M_{21}+M_{14}\}-M_{24}^2+M_{13}^2$$

$$L_2=M_{24}^2+M_{13}^2.$$

XIII2) $(\mathbf{z})_1 \to (iz_3, z_2, iz_1, z_4), \quad x_1>1>x_2>0>x_3.$

XIII3) $(\mathbf{z})_1 \to (z_1, iz_4, z_3, iz_2)$

$$1>x_1>0>x_2, x_3 \quad \text{or} \quad 1>x_1, x_2, x_3>0.$$

XIII4) $(z_1+z_4+iz_3+iz_2)^2=4(x_1-a)(x_2-a)(x_3-a)/(a-b)^2$

$$(z_1+iz_3)^2-(z_4+iz_2)^2=2[(x_1-a)(x_2-a)+(x_1-a)(x_3-a)+(x_2-a)\times(x_3-a)]/(a-b)^2$$
$$4(x_1-a)(x_2-a)(x_3-a)/(a-b)^3,$$

$a=\bar{b}=\alpha+i\beta,\quad x_1, x_2, x_3$ real,

$$\begin{aligned}L_1=&-\tfrac{1}{4}(M_{24}+M_{34}+M_{12}-M_{13})^2-\alpha\beta\{M_{13}-M_{24}, M_{14}+M_{23}\}\\&+\tfrac{1}{2}(\alpha^2+\beta^2)[(M_{12}+M_{34})^2-(M_{24}-M_{13})^2]\\&+\tfrac{1}{2}(\alpha^2-\beta^2)[(M_{13}-M_{24})^2-(M_{14}+M_{23})^2]\\&-\tfrac{1}{2}(\alpha^2+\beta^2)[(M_{23}-M_{14})^2+(M_{13}+M_{24})^2]\\&+\tfrac{1}{2}\{M_{24}+M_{34}+M_{12}-M_{13}, \beta(M_{14}-M_{23})-\alpha(M_{13}+M_{24})\},\end{aligned}$$

$$\begin{aligned}L_2=&\alpha[(M_{12}+M_{34})^2-2M_{14}^2-2M_{23}^2-(M_{13}+M_{24})^2]\\&+2\beta\{M_{13}-M_{24}, M_{14}+M_{23}\}\\&-\tfrac{1}{2}\{M_{24}+M_{34}+M_{12}-M_{13}, M_{13}+M_{24}\}.\end{aligned}$$

Class XIV. Real forms of the complex coordinates (20).

XIV1) $(z_3-z_2)^2=x_1x_2x_3,\quad 2z_1(z_3-z_2)=x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3,$

$z_4^2=(x_1-1)(x_2-1)(x_3-1),$

$x_1>1>x_2, x_3>0$ or $x_1>1>0>x_2, x_3$ or $1>x_1, x_2>0>x_3,$

$L_1=(M_{42}+M_{43})^2+\{M_{12}+M_{13}, M_{32}\}$

$L_2=M_{34}^2+M_{41}^2-M_{24}^2-(M_{42}+M_{43})^2+\{M_{42}-M_{43}, M_{14}\}.$

XIV2) $(\mathbf{z})_1\to(iz_3, iz_4, iz_1, iz_2)$

$x_1>1>x_2>0$ or $1>x_1, x_2>0>x_3$ or $0>x_1, x_2, x_3.$

Class XV. Real forms of the complex coordinates (21).

XV1) $(z_4+z_2)^2=-2x_1x_2x_3,\quad (z_4+z_2)(z_3+z_1)=x_1x_2+x_2x_3+x_1x_3$

$(z_4+z_2)(z_3-z_1)-\tfrac{1}{2}(z_1+z_3)^2=x_1+x_2+x_3,\quad \operatorname{sign}(x_1x_2x_3)=-,$

$L_1=\{M_{24}, M_{41}-M_{23}+M_{34}+M_{21}\}-\tfrac{1}{2}(M_{43}+M_{21}+M_{23}-M_{41})^2$

$$\begin{aligned}L_2=&\{M_{24}, M_{23}+M_{14}+M_{43}+M_{21}\}+2\{M_{13}, M_{14}+M_{21}\}\\&+(M_{12}+M_{23})^2-(M_{43}-M_{14})^2.\end{aligned}$$

XV2) $(\mathbf{z})_1\to(iz_3, iz_4, iz_1, iz_2),\quad \operatorname{sign}(x_1x_2x_3)=+.$

Appendix A. The EPD equation.

We choose the EPD equation in the form

$$(\partial_{tt}-\partial_{rr}+r\partial_r+(m^2-1)/r^2)\Psi(t,r)=0, \tag{A.1}$$

where t is real and $r>0$. The symmetry algebra for this equation is isomorphic to $o(2,1)$ with basis

$$\begin{aligned}N_3=\tfrac{1}{2}(1-t^2-r^2)\partial_t-tr\partial_r,\quad N_2=-t\partial_t-r\partial_r\\M_1=\tfrac{1}{2}(1+t^2+r^2)\partial_t+tr\partial_r\end{aligned} \tag{A.2}$$

and commutation relations

$$[N_3, N_2] = -M_1, \quad [N_3, M_1] = -N_2, \quad [N_2, M_1] = N_3. \tag{A.3}$$

Indeed, (A.1) is equivalent to $Q\Psi = (\frac{1}{4} - m^2)\Psi$ where $Q = M_1^2 - N_2^2 - N_3^2$ is the Casimir operator for $O(2, 1)$. The EPD equation was studied in detail in [**4**] from the viewpoint of separation of variables. As pointed out there, for any C^∞ function $f(k)$ with compact support in $(0, \infty)$ the function

$$\Psi(t, r) = U(f) = r^{\frac{1}{2}} \int_0^\infty e^{itk} J_m(k, r) f(k)\, dk \tag{A.4}$$

is a solution of (A.1). It follows easily that U can be extended to a unitary mapping from the Hilbert space $L_2(0, \infty)$ onto a Hilbert space of generalized solutions of (A.1). The induced action of the operators (A.2) on $L_2(0, \infty)$ takes the form

$$\begin{gathered} N_3 = \frac{i}{2} k \left(\frac{d^2}{dk^2} + \frac{1}{k} \frac{d}{dk} - \frac{m^2}{k^2} + 1 \right), \quad N_2 = \tfrac{1}{2} + k \frac{d}{dk} \\ M_1 = \frac{i}{2} k \left(\frac{-d^2}{dk^2} - \frac{1}{k} \frac{d}{dk} + \frac{m^2}{k^2} + 1 \right), \end{gathered} \tag{A.5}$$

and for $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, iN_3, iN_2, iM_1 can be extended to self-adjoint operators on $L_2(0, \infty)$ which determine unitary irreducible representations $D^-_{m-\frac{1}{2}}$ of $SO(2, 1)$ from the discrete series.

$SO(2, 1)$ also acts on the single-sheet hyperboloid $\xi_1^2 - \xi_2^2 - \xi_3^2 = -1$ according to

$$N_3 = \xi_1 \partial_{\xi_2} + \xi_2 \partial_{\xi_1}, \quad N_2 = \xi_1 \partial_{\xi_3} + \xi_3 \partial_{\xi_1}, \quad M_1 = \xi_3 \partial_{\xi_2} - \xi_2 \partial_{\xi_3}. \tag{A.6}$$

Moreover, the eigenvalue equation $Q\Psi(\boldsymbol{\xi}) = (\frac{1}{4} - m^2)\Psi(\boldsymbol{\xi})$ for the Casimir operator Q on the single-sheet hyperboloid can be identified with the EPD equation (A.1) through the correspondence

$$t = \frac{\xi_2}{\xi_1 + \xi_3}, \quad r = \frac{1}{\xi_1 + \xi_3}, \quad \xi_1^2 - \xi_2^2 - \xi_3^2 = -1 \tag{A.7}$$

and the operators (A.2), (A.6) can be shown to coincide. However, this parametrization of the hyperboloid covers only the surface where $\xi_1 + \xi_3 = r^{-1} > 0$.

It follows that the separation of variables problems for the EPD equation and the Casimir eigenvalue equation on $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = -1$ are virtually equivalent.

In analogy with our comments on $O(2, 2)$ in section 1 we can impose a Hilbert space structure on solutions of the eigenvalue equation for the hyperboloid. Let $\hat{\Psi}(\boldsymbol{\xi}) = \Psi(t, r) = U(f)$, $f \in L_2(0, \infty)$ for fixed $m = \frac{1}{2}, \frac{3}{2}, \ldots$ where $\boldsymbol{\xi}$ and t, r are related by (A.7). Then $\hat{\Psi}$ is a solution of the eigenvalue equation $Q\hat{\Psi} = (\frac{1}{4} - m^2)\hat{\Psi}$ on the part of the hyperboloid such that $\xi_1 + \xi_3 > 0$. Moreover, the right-hand side of (A.4) makes sense for $r < 0$ and we have $\Psi(t, -r) = (-1)^{m+\frac{1}{2}} \Psi(t, r)$. Thus we can extend $\hat{\Psi}$ to the entire hyperboloid $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = -1$ by imposing the relation

$$\hat{\Psi}(-\boldsymbol{\xi}) = (-1)^{m+\frac{1}{2}} \hat{\Psi}(\boldsymbol{\xi}).$$

Let $d\nu$ be the $SO(2, 1)$-invariant measure on $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = -1$ which for $\xi_1 > 0$ takes the form $d\nu(\boldsymbol{\xi}) = d\xi_2\, d\xi_3/\xi_1$. Now suppose $\Psi_j = U(f_j)$, $j = 1, 2$, where $f_j \in L_2(0, \infty)$.

A straightforward computation making use of (A.4) yields the

THEOREM. *If* $m=\frac{1}{2},\frac{3}{2},\ldots$ *then*

$$\int_{\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1}\hat{\Psi}_1\bar{\hat{\Psi}}_2\,d\nu(\boldsymbol{\xi})=\frac{2\pi}{m}\int_0^\infty f_1(k)\overline{f_2(k)}\,dk.$$

This result permits us to introduce a Hilbert space structure on the eigenspace $\mathcal{L}_{m-\frac{1}{2}}$ of eigenfunctions of the operator $-Q$ on $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1$ with eigenvalue $(m+\frac{1}{2})(m-\frac{1}{2})$, a realization of $D^-_{m-\frac{1}{2}}$. Thus the spectral resolutions computed in [**4**] can be carried over directly to $\mathcal{L}_{m-\frac{1}{2}}$.

Appendix B. Coordinate systems on the hyperboloids $\xi_1^2-\xi_2^2-\xi_3^2=\pm1$.

The above mentioned hyperboloids are invariant under the group $SO(2,1)$ with induced Lie algebra action provided by operators

$$N_3=\xi_1\partial_{\xi_2}+\xi_2\partial_{\xi_1},\quad N_2=\xi_1\partial_{\xi_3}+\xi_3\partial_{\xi_1}$$
$$M_1=\xi_3\partial_{\xi_2}-\xi_2\partial_{\xi_3}\tag{B.1}$$

obeying commutation relations

$$[N_3,N_2]=-M_1,\quad [N_3,M_1]=-N_2,\quad [N_2,M_1]=N_3.\tag{B.2}$$

We consider the eigenvalue equation

$$(M_1^2-N_2^2-N_3^2)f(\boldsymbol{\xi})=\lambda f(\boldsymbol{\xi})\tag{B.3}$$

on each of these hyperboloids. As follows from [**4, 21, 22**], for each hyperboloid (B.3) admits separable solutions in exactly nine coordinate systems and the separated solutions $f(\xi)=X_1(x_1)X_2(x_2)$ for each system are characterized as eigenfunctions of a second-order symmetric operator L in the enveloping algebra of $o(2,1)$: $Lf=\mu f$. The eigenvalue μ is the separation constant. We now list the nine systems and their operator characterizations.

type 1

$$(\xi_1^{(1)})^2=\frac{x_1x_2}{a},\quad (\xi_2^{(1)})^2=\frac{(x_1-1)(x_2-1)}{a-1},$$

$$(\xi_3^{(1)})^2=\frac{(x_1-a)(a-x_2)}{a(a-1)},\quad 1<x_1<a<x_2,$$

$$\boldsymbol{\xi}^{(1)}\cdot\boldsymbol{\xi}^{(1)}=(\xi_1^{(1)})^2-(\xi_2^{(1)})^2-(\xi_3^{(1)})^2=1.$$

The coordinates on $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1$ are obtained by the substitution $\boldsymbol{\xi}^{(1)}\to i\boldsymbol{\xi}^{(1)}$ and $x_1<0<1<x_2<a$. Here $L_1=N_3^2+aN_2^2$.

type 2

$$(\xi_3^{(2)})^2=\frac{(x_1-1)(1-x_2)}{a-1},\quad (\xi_2^{(2)})^2=\frac{-x_1x_2}{a},$$

$$(\xi_3^{(2)})^2=\frac{(x_1-a)(a-x_2)}{a(a-1)},\quad x_1<0<1<a<x_2,\quad \boldsymbol{\xi}^{(2)}\cdot\boldsymbol{\xi}^{(2)}=1.$$

The coordinates on $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1$ are obtained via the substitution $\boldsymbol{\xi}\to i\boldsymbol{\xi}$ and $1<x_1, x_2<a$ or $x_1, x_2>a>1$. The operator is $L=N_3^2-aM_1^2$.

type 3

$$(\xi_1^{(3)}+i\xi_2^{(3)})^2=2(x_1-a)(x_2-a)/a(a-b),\quad (\xi_3^{(3)})^2=-x_1x_2/ab,$$
$$a=\bar{b}=\alpha+i\beta,\quad x_1<0<x_2,\quad \boldsymbol{\xi}^{(3)}\cdot\boldsymbol{\xi}^{(3)}=1.$$

For $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1$ we use the substitution $\boldsymbol{\xi}\to i\boldsymbol{\xi}$ and $x_1, x_2>0$. The operator is $L=\alpha(M_1^2-N_2^2)-\beta\{M_1, N_2\}$.

type 4

$$\xi_1^{(4)}+\xi_2^{(4)}=[-x_1x_2]^{\frac{1}{2}},\quad \xi_3^{(4)}=[(1-x_1)(x_2-1)]^{\frac{1}{2}},$$
$$\xi_1^{(4)}-\xi_2^{(4)}=-[(-x_1)/x_2]^{\frac{1}{2}}+[x_2/(-x_1)]^{\frac{1}{2}}+[-x_1x_2]^{\frac{1}{2}}$$
$$x_1<0<1<x_2,\quad \boldsymbol{\xi}^{(4)}\cdot\boldsymbol{\xi}^{(4)}=1.$$

The coordinates on the single sheet hyperboloid are obtained via the substitution $\boldsymbol{\xi}\to i\boldsymbol{\xi}$ with either $x_1, x_2>1$, $0<x_1, x_2<1$ or $x_1, x_2<0$. Here, $L=N_3^2-(N_2-M_1)^2$.

type 5

$$\xi_1^{(5)}+\xi_2^{(5)}=[x_1x_2]^{\frac{1}{2}},\quad \xi_3^{(5)}=[(1-x_1)(x_2-1)]^{\frac{1}{2}},$$
$$\xi_1^{(5)}-\xi_2^{(5)}=[x_1/x_2]^{\frac{1}{2}}+[x_2/x_1]^{\frac{1}{2}}-[x_1x_2]^{\frac{1}{2}},\quad 0<x_1<1<x_2,\quad \boldsymbol{\xi}^{(5)}\cdot\boldsymbol{\xi}^{(5)}=1.$$

The coordinates for $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1$ are obtained via the substitution $\boldsymbol{\xi}\to i\boldsymbol{\xi}$ with $x_1<0<x_2<1$. Here, $L=N_3^2+(N_2-M_1)^2$.

type 6

$$\xi_1^{(6)}+\xi_2^{(6)}=[-x_1x_2]^{\frac{1}{2}},\quad \xi_3^{(6)}=\tfrac{1}{2}[x_2/(-x_1)]^{\frac{1}{2}}-\tfrac{1}{2}[-x_1/x_2]^{\frac{1}{2}},$$
$$\xi_1^{(6)}-\xi_2^{(6)}=(x_1-x_2)^2/4(-x_1x_2)^{\frac{3}{2}},\quad x_1<0<x_2,\quad \boldsymbol{\xi}^{(6)}\cdot\boldsymbol{\xi}^{(6)}=1.$$

The coordinates for $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=-1$ are obtained via the substitution $\boldsymbol{\xi}\to i\boldsymbol{\xi}$ with $x_1, x_2>0$. The operator is $L=\{N_3, N_2+M_1\}$.

type 7

$$\xi_1^{(7)}+\xi_2^{(7)}=x_1^{\frac{1}{2}},\quad \xi_3^{(7)}=x_2x_1^{\frac{1}{2}},\quad \xi_1^{(7)}-\xi_2^{(7)}=x_1^{-\frac{1}{2}}+x_2^2x_1^{\frac{1}{2}},$$
$$x_1, x_2>0,\quad \boldsymbol{\xi}^{(7)}\cdot\boldsymbol{\xi}^{(7)}=1.$$

The coordinates on the single sheet hyperboloid are obtained via the substitution $\boldsymbol{\xi}\to i\boldsymbol{\xi}$ with $x_1<0<x_2$. Here, $L=(N_2-M_1)^2$.

type 8

$\boldsymbol{\xi}^{(8)}=(\cosh x_1\cosh x_2, \cosh x_1\sinh x_2, \sinh x_1)$ with $\boldsymbol{\xi}^{(8)}\cdot\boldsymbol{\xi}^{(8)}=1$ or

$$\hat{\boldsymbol{\xi}}^{(8)}=(\sinh x_1\cosh x_2, \sinh x_1\sinh x_2, \cosh x_1)$$
$$\hat{\boldsymbol{\xi}}^{(8)}=(\sin x_1\sinh x_2, \sin x_1\cosh x_2, \cos x_1)$$

with $\hat{\boldsymbol{\xi}}^{(8)}\cdot\hat{\boldsymbol{\xi}}^{(8)}=-1$. Here, $L=N_3^2$.

type 9

$$\boldsymbol{\xi}^{(9)} = (\cosh x_1, \sinh x_1 \cos x_2, \sinh x_1 \sin x_2),\ \boldsymbol{\xi}^{(9)} \cdot \boldsymbol{\xi}^{(9)} = 1, \text{ or}$$

$$\hat{\boldsymbol{\xi}}^{(9)} = (\sinh x_1, \cosh x_1 \cos x_2, \cosh x_1 \sin x_2)$$

with $\hat{\boldsymbol{\xi}}^{(9)} \cdot \hat{\boldsymbol{\xi}}^{(9)} = -1$. The operator is $L = M_1^2$.

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