Superintegrability and exactly solvable problems in classical and quantum mechanics

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Abstract

Quantum superintegrable systems are exactly solvable quantum eigenvalue problems. Their solvability is due to symmetry, but the symmetry is often "hidden". The symmetry generators of 2nd order superintegrable systems in 2 dimensions close under commutation to define quadratic algebras, a generalization of Lie algebras. The irreducible representations of these algebras yield important information about the eigenvalues and eigenspaces of the quantum systems. Distinct superintegrable systems and their quadratic algebras are related by geometric contractions, induced by generalized Inönü-Wigner Lie algebra contractions which have important physical and geometric implications, such as the Askey scheme for obtaining all hypergeometric orthogonal polynomials as limits of Racah/Wilson polynomials. We introduce the subject and and survey the theory behind the discovery and classification of these remarkable systems.

Outline

1. Introduction
2. 2nd order systems
3. Constant curvature space Helmholtz systems
4. Interbasis expansion coefficients
5. Special functions and superintegrable systems
6. Higher order superintegrable systems
7. Wrap-up

Superintegrable Systems

We call a classical or quantum Hamiltonian system on an n-dimensional manifold

$$\mathcal{H} = \sum_{i,j=0}^{n} g^{ij} p_i p_j + V(x_i), \quad \text{or} \quad H = -\Delta_n + V(x_i)$$

(maximally, Nth-order) **Superintegrable** if it admits $2n - 1$ symmetry operators, i.e.,.

$$\{\mathcal{L}_i, \mathcal{H}\} = 0, \qquad [L_i, H] = 0, \qquad \forall i = 0, \ldots, 2n - 1,$$

$L_1 = H$ such that $L_2, \cdots, L_{2n-1}$ are polynomial, degree at most N, in the momenta or as differential operators.

Superintegrable systems can be solved algebraically as well as analytically and are associated with special functions and exact solvability.

Integrability and Superintegrability

An integrable system has n algebraically independent symmetry operators in involution. A superintegrable system has $2n - 1$ algebraically independent symmetry operators (the maximum possible).

The symmetries of a merely integrable system generate an abelian algebra, those of a superintegrable system generate an algebra that is necessarily nonabelian.

Claim: Superintegrability captures what it means for a Hamiltonian system to be explicitly solvable.

Some simple but important superintegrable systems:

- Kepler-Coulomb problem, the Hohmann transfer used in celestial navigation
- hydrogen atom: periodic table of the elements
- classical and quantum harmonic oscillator

A very important but somewhat misleading example

Newton used Kepler's laws to demonstrate that the equation governing the motion of a planet about the sun is

$$m\,\ddot{\mathbf{r}} = -\frac{mk}{r^2}\hat{\mathbf{r}},$$

where $\mathbf{r}$ is the vector from the center of the sun to the center of the planet and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$.

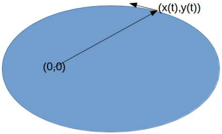
Here, M is the mass of the sun, m is the mass of the planet, $k = MG$ and G is the gravitational constant.

Newton's Gravitational force 2

1. We know today that Newton's equation for planetary motion, the 2-body problem, can be solved explicitly, not just numerically, because it is of maximal symmetry. It admits 3 independent symmetries and this is the maximum possible in two dimensions. It is a very important example of a superintegrable system.
2. It also helps explain how Kepler found the trajectories of the planets without knowing Newton's equations or calculus.
3. A basic principle here is that symmetries of a physical system lead to conservation laws obeyed by the system: quantities that do not change as the system evolves in time.

The orbital plane

A planet orbiting the sun moves in a plane. Choose coordinates (x, y) in this plane such that the center of the sun is at the origin $(0, 0)$ and at time t the center of the planet is at the point $(x(t), y(t))$ and is moving with velocity $(\dot{x}(t), \dot{y}(t))$. The speed of the planet in its orbit is $s(t) = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}$.



Hamilton's equations 1

Let q_1, q_2 be the position coordinates of a Hamiltonian system and let p_1, p_2 be the momenta. The Hamiltonian function $H(q_1, q_2, p_1, p_2)$ represents the energy of the system. Hamilton's equations give the time evolution of the system. They are

$$\dot{q}_k(t) = \frac{\partial H}{\partial_{p_k}}, \quad \dot{p}_k(t) = -\frac{\partial H}{\partial_{q_k}}, \quad k = 1, 2.$$

In our case, $r = \sqrt{q_1^2 + q_2^2}$ and

$$q_1 = x,\ q_2 = y,\ p_1 = \dot{x},\ p_2 = \dot{y}, \quad H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{k}{r}.$$

Hamilton's equations are

$$\dot{q}_1 = \dot{x},\ \dot{q}_2 = \dot{y}, \qquad \dot{p}_1 = \ddot{x} = -\frac{kx}{r^3}, \quad \dot{p}_2 = \ddot{y} = -\frac{ky}{r^3},$$

which are just Newton's equations for the 2-body problem.

Hamilton's equations 2

A function $F(q_1, q_2, p_1, p_2)$ is a symmetry or constant of the motion if $F(q_1(t), q_2(t), p_1(t), p_2(t))$ remains constant as the system evolves in time. Thus F is a constant of the motion if and only if

$$\frac{d}{dt}F(q_1(t), q_2(t), p_1(t), p_2(t)) = 0.$$

From the chain rule,

$$\frac{d}{dt}F(t) = \frac{\partial F}{\partial_{q_1}}q_1' + \frac{\partial F}{\partial_{q_2}}q_2' + \frac{\partial F}{\partial_{p_1}}p_1' + \frac{\partial F}{\partial_{p_2}}p_2' =$$

$$\frac{\partial F}{\partial_{q_1}}\frac{\partial H}{\partial_{p_1}} + \frac{\partial F}{\partial_{q_2}}\frac{\partial H}{\partial_{p_2}} - \frac{\partial F}{\partial_{p_1}}\frac{\partial H}{\partial_{q_1}} - \frac{\partial F}{\partial_{p_2}}\frac{\partial H}{\partial_{q_2}} \equiv \{H, F\},$$

where $\{H, F\}$ is the *Poisson bracket* of F and H.

Hamilton's equations 3

Thus F is a constant of the motion provided the Poisson bracket $\{H, F\} = 0$. Note that H itself (the energy) is always a constant of the motion. In our case, in addition to the energy, we have the following constants of the motion:

1. Angular momentum $L = q_1 p_2 - p_1 q_2$.

   Proof:

$$\{H, L\} = p_2 p_1 - p_1 p_2 - \frac{k q_2 q_1}{r^3} + \frac{k q_2 q_1}{r^3} = 0.$$

Hamilton's equations 4

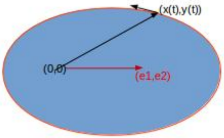
1. The first component of the Laplace vector

$$e_1 = p_2(q_1 p_2 - q_2 p_1) - \frac{kq_1}{r}.$$

2. The second component of the Laplace vector

$$e_2 = -p_1(q_1 p_2 - q_2 p_1) - \frac{kq_2}{r}.$$

are also constants of the motion.



Constants of the motion

Energy

$$\frac{1}{2}\left((\dot{x})^2 + (\dot{y})^2\right) - \frac{k}{\sqrt{x^2+y^2}} = E$$

Angular momentum

$$x\dot{y} - y\dot{x} = L$$

Laplace vector $\mathbf{e} = (e_1, e_2)$ where

$$\dot{y}(x\dot{y} - y\dot{x}) - \frac{kx}{\sqrt{x^2+y^2}} = e_1, \ -\dot{x}(x\dot{y} - y\dot{x}) - \frac{ky}{\sqrt{x^2+y^2}} = e_2.$$

Structure equations for symmetries:

$\{L, e_1\} = e_2, \ \{L, e_2\} = -e_1, \ \{e_1, e_2\} = -2LH$

Casimir: $e_1^2 + e_2^2 - 2L^2H = k^2$

This is $so(3)$ if H is constant.

The trajectories 1

By lining up the x, y coordinate system so that the x-axis is in the direction of the Laplace vector, we can assume $e_2 = 0$. (This means that the x-axis goes through the perihelion of the planet. It is called the apse axis in astronomy.) Then

1. $$e_2 = 0 \text{ and } L \text{ constant} \longrightarrow \dot{x} = -\frac{ky}{L\sqrt{x^2+y^2}}$$

2. $$e_1 \text{ and } L \text{ constant} \longrightarrow \dot{y} = \frac{e_1}{L} + \frac{kx}{L\sqrt{x^2+y^2}}$$

The trajectories 2

1. Substitute these expressions into the equation for L, and simplify to get the equation:
$$k\sqrt{x^2+y^2} = L^2 - e_1 x$$
2. Square and simplify to get the trajectory
$$(1-\frac{e_1^2}{k^2})x^2 + \frac{2L^2 e_1}{k^2}x + y^2 = \frac{L^4}{k^2}$$

The paths are conic sections!

Set $e_1 = \epsilon k \geq 0$ where ϵ is called the eccentricity.

$\epsilon = 0$, Circle:

$$x^2 + y^2 = (\frac{L^2}{k})^2$$

$0 < \epsilon < 1$, Ellipse:

$$(1-\epsilon^2)x^2 + \frac{2\epsilon L^2}{k}x + y^2 = (\frac{L^2}{k})^2$$

$\epsilon > 1$, Hyperbola:

$$(1-\epsilon^2)x^2 + \frac{2\epsilon L^2}{k}x + y^2 = (\frac{L^2}{k})^2$$

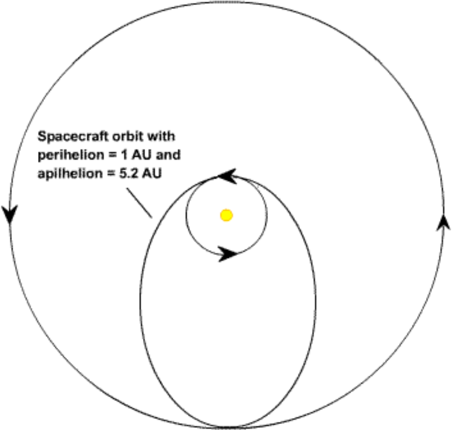
$\epsilon = 1$, Parabola: $\frac{2\epsilon L^2}{k}x + y^2 = (\frac{L^2}{k})^2$

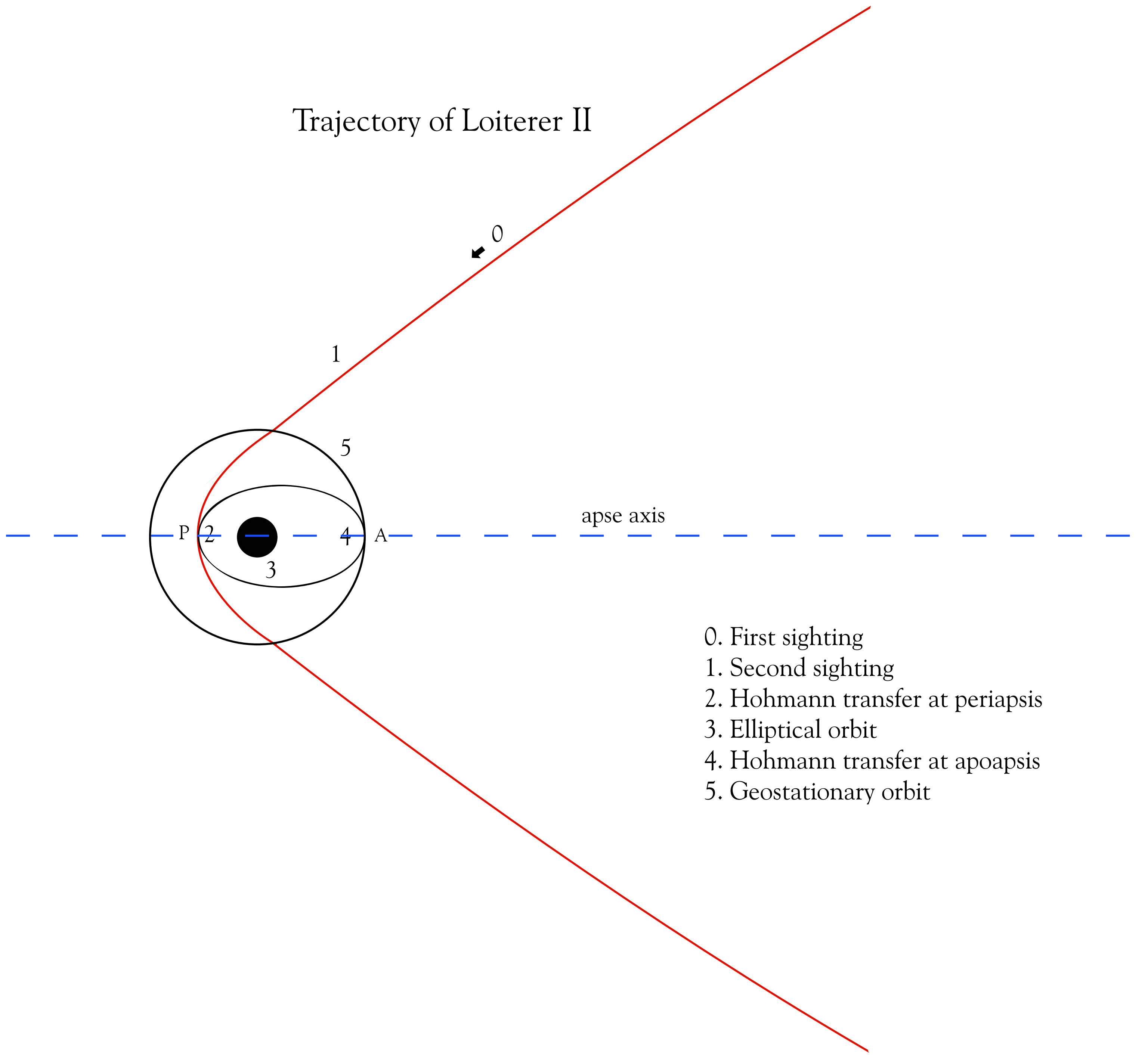
Impulse maneuvers

1. Position and velocity (x_0, y_0, x_0', y_0') at a single instant determines the trajectory: Just compute the constants of the motion E, L, e_1, e_2 at the instant and they in term uniquely define the trajectory.
2. This is the basis for impulse maneuvers in rocket science. The Hohmann transfer.

The Hohmann transfer

1. Space ship (with engines turned off) on trajectory with constants of the motion E, L, e_1, e_2.
2. At time t_0 ship has position and velocity (x_0, y_0, x_0', y_0').
3. Turn on the engine, for an instant, at time t_0: impulse propulsion. This pulse changes the velocity of the ship instantaneously, but not the position.
4. Immediately after the impulse the ship has position and velocity $(x_0, y_0, \tilde{x}_0', \tilde{y}_0')$.
5. This gives us the new trajectory with constants of the motion $\tilde{E}, \tilde{L}, \tilde{e}_1, \tilde{e}_2$.
6. The change of trajectories is determined by simple algebra.


Spacecraft orbit with
perihelion = 1 AU and
apilhelion = 5.2 AU



Trajectory of Loiterer II

The quantum Coulomb problem 1

The Hamiltonian and the the constants of the motion are replaced by differential operators: We make the formal replacement $p_1 \to \partial_{q_1}$ and handle the ambiguity of replacements $q_i p_i \to q_i \partial_{q_i},\ \partial_{q_i} q_i$ by symmetrizing: $q_i p_i \to \frac{1}{2}(q_i \partial_{q_i} + \partial_{q_i} q_i)$.

$$H = \frac{1}{2}(\partial_{q_1}^2 + \partial_{q_2}^2) - \frac{k}{\sqrt{x^2 + y^2}}$$

$$L = q_1 \partial_{q_2} - q_2 \partial_{q_1},\ e_1 = \frac{1}{2}(L\partial_{q_2} + \partial_{q_2} L) - \frac{kx}{\sqrt{x^2 + y^2}}$$

$$e_2 = -\frac{1}{2}(L\partial_{q_1} + \partial_{q_1} L) - \frac{ky}{\sqrt{x^2 + y^2}}$$

The quantum Coulomb problem 2

The Poisson bracket $\{F, G\}$ is replaced by the operator commutator $[F, G] = FG - GF$ and the constants of the motion are differential operators that commute with H.

$$[H, L] = [H, e_1] = [H, e_2] = [H, H] = 0$$

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1. Structure equations for symmetries: $[L, e_1] = e_2, \ [L, e_2] = -e_1, \ [e_1, e_2] = -2LH$
2. Casimir: $e_1^2 + e_2^2 - 2L^2H + \frac{1}{2}H = k^2$
3. Note that this is NOT a Lie algebra, unless H is a multiple of the identity operator.

Transition to Quantum Mechanics 1

Bound states are eigenfunctions of H that are square integrable:

$$H\Psi = E\Psi, \quad <\Psi, \Psi>= 1.$$

The symmetries commute with H, so they map the eigenspace into itself. We look for irreducible representations of the algebra generated by the symmetries. Necessarily, the Hamiltonian is the constant E in these cases, so we can consider the structure equations as defining $so(3)$.

1. the possible states of energy E and angular momentum L have values

$$E_n = \frac{4k^2}{(n+1)^2}, \quad L_m = im/2, \quad m = n, n-1, n-2, \cdots, -n$$

so the possible multiplicities of the eigenvalues are $2n+1$ where n is an integer. These results can be derived entirely from the representation theory of $so(3)$ (highest weight, lowest weight, etc.).

The hydrogen atom on the 2-sphere 1

These striking results led to mathematical physicists to concentrate on finding systems with symmetries that generated Lie algebras. However, consider this analog analog of the hydrogen atom on the 2-sphere.

$$H = \sum_{i=1}^{3} J_i^2 + \frac{\alpha s_3}{\sqrt{s_1^2 + s_2^2}},$$

where $s_1^2 + s - 2^2 + s_3^2 = 1$, $J_1 = s_2\partial_3 - s_3\partial_2$ and J_2, J_3 by cyclic permutation. A basis for the symmetry operators is

$$L_1 = J_1J_3 + J_3J_1 - \frac{\alpha s_1}{\sqrt{s_1^2 + s_2^2}}, \quad L_1 = J_1J_3 + J_3J_1 - \frac{\alpha s_1}{\sqrt{s_1^2 + s_2^2}}, \quad X = J_3,$$

along with H.

The hydrogen atom on the 2-sphere 2

The symmetry operators satisfy the structure relations

$$[X, L_1] = -L_2, \; [X, L_2] = L_1, \; [L_1, L_2] = 4HX - 8X^3 + X,$$

$$L_1^2 + L_2^2 + 4X^4 - 4HX^2 + H - 5X^2 - \alpha^2 = 0.$$

clearly not a Lie algebra. The irreducible representations of this algebra yield the energy eigenvalues

$$E_n = -\frac{1}{4}(n+1)^2 + \frac{1}{4} + \frac{\alpha^2}{(n+1)^2}.$$

with multiplicity $2n = 1$.

(Studied by Schrödinger.) As the radius of the sphere goes to ∞ this system converges to the hydrogen atom system in 2D Euclidean space.

Example: Smorodinski-Winternitz $\sim$ 1967

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + \frac{b_1}{x^2} + \frac{b_2}{y^2}$$

Generators:

$$L_1 = \partial_x^2 - \omega^2 x^2 + \frac{b_1}{x^2},\ L_2 = \partial_y^2 - \omega^2 y^2 + \frac{b_2}{y^2},\ L_3 = (x\partial_y - y\partial_x)^2 + y^2\frac{b_1}{x^2} + x^2\frac{b_2}{y^2}$$

Structure relations:

$$[R, L_1] = 8L_1^2 - 8HL_1 - 16\omega^2 L_3 + 8\omega^2,$$

$$[R, L_3] = 8HL_3 - 8\{L_1, L_3\} + (16b_1 + 8)H - 16(b_1 + b_2 + 1)L_1,$$

$$R^2 + \frac{8}{3}\{L_1, L_1, L_3\} - 8H\{L_1, L_3\} + (16b_1 + 16b_2 + \frac{176}{3})L_1^2 - 16\omega^2 L_3^2 - (32b_1 + \frac{176}{3})HL_1$$

$$+(16b_1 + 12)H^2 + \frac{176}{3}\omega^2 L_3 + 16\omega^2(3b_1 + 3b_2 + 4b_1b_2 + \frac{2}{3}) = 0$$

The quantum Tremblay, Turbiner, Winternitz system (in polar coordinates in the plane) is Here, $R \equiv [L_1, L_2]$. This is called a nondegenerate quadratic algebra.

$\{A, B\} \equiv AB + BA$, and $\{A, B, C\}$ are symmetrizers.

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2nd order superintegrable systems

2nd order systems are the easiest to construct and classify, due to their connection with separation of variables: every orthogonal separable coordinate system is characterized by n 2nd order symmetry operators, mutually commuting. All such superintegrable systems for 2nd order systems for 2D have been classified.

There are 44 nondegenerate (3 parameter potential) systems, on a variety of manifolds, but under the Stäckel transform, an invertible structure preserving mapping, they divide into 6 equivalence classes with representatives on flat space and the 2-sphere.

There are also a similar number of degenerate (1 parameter potential) systems that divide into 6 equivalence classes.

All of these systems are multiseparable

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Nondegenerate flat space systems: $H\Psi = (\partial_x^2 + \partial_y^2 + V)\Psi = E\Psi$.

1. $E1$: $V = \alpha(x^2+y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}$, (2) $E2$: $V = \alpha(4x^2+y^2) + \beta x + \frac{\gamma}{y^2}$,
3. $E3'$: $V = \alpha(x^2+y^2) + \beta x + \gamma y$,
4. $E7$: $V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-b}} + \frac{\beta(x-iy)}{\sqrt{(x+iy)^2-b}\left(x+iy+\sqrt{(x+iy)^2-b}\right)^2} + \gamma(x^2+y^2)$,
5. $E8$ $V = \frac{\alpha(x-iy)}{(x+iy)^3} + \frac{\beta}{(x+iy)^2} + \gamma(x^2+y^2)$,
6. $E9$: $V = \frac{\alpha}{\sqrt{x+iy}} + \beta y + \frac{\gamma(x+2iy)}{\sqrt{x+iy}}$,
7. $E10$: $V = \alpha(x-iy) + \beta(x+iy-\frac{3}{2}(x-iy)^2) + \gamma(x^2+y^2-\frac{1}{2}(x-iy)^3)$,
8. $E11$: $V = \alpha(x-iy) + \frac{\beta(x-iy)}{\sqrt{x+iy}} + \frac{\gamma}{\sqrt{x+iy}}$,
9. $E15$: $V = f(x-iy)$,
10. $E16$: $V = \frac{1}{\sqrt{x^2+y^2}}(\alpha + \frac{\beta}{y+\sqrt{x^2+y^2}} + \frac{\gamma}{y-\sqrt{x^2+y^2}})$,
11. $E17$: $V = \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{\beta}{(x+iy)^2} + \frac{\gamma}{(x+iy)\sqrt{x^2+y^2}}$,
12. $E19$: $V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2-4}} + \frac{\beta}{\sqrt{(x-iy)(x+iy+2)}} + \frac{\gamma}{\sqrt{(x-iy)(x+iy-2)}}$.
13. $E20$: $V = \frac{1}{\sqrt{x^2+y^2}}\left(\alpha + \beta\sqrt{x+\sqrt{x^2+y^2}} + \gamma\sqrt{x-\sqrt{x^2+y^2}}\right)$,

Nondegenerate systems on the complex 2-sphere:

$H\Psi = (J_{23}^2 + J_{13}^2 + J_{12}^2 + V)\Psi = E\Psi, \quad J_{k\ell} = s_k\partial_{s_\ell} - s_\ell\partial_{s_k} \quad s_1^2 + s_2^2 + s_3^2 = 1.$

Here,

1. $S1$: $V = \frac{\alpha}{(s_1+is_2)^2} + \frac{\beta s_3}{(s_1+is_2)^2} + \frac{\gamma(1-4s_3^2)}{(s_1+is_2)^4}$,
2. $S2$: $V = \frac{\alpha}{s_3^2} + \frac{\beta}{(s_1+is_2)^2} + \frac{\gamma(s_1-is_2)}{(s_1+is_2)^3}$,
3. $S4$: $V = \frac{\alpha}{(s_1+is_2)^2} + \frac{\beta s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\gamma}{(s_1+is_2)\sqrt{s_1^2+s_2^2}}$,
4. $S7$: $V = \frac{\alpha s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\beta s_1}{s_2^2\sqrt{s_1^2+s_2^2}} + \frac{\gamma}{s_2^2}$,
5. $S8$: $V = \frac{\alpha s_2}{\sqrt{s_1^2+s_3^2}} + \frac{\beta(s_2+is_1+s_3)}{\sqrt{(s_2+is_1)(s_3+is_1)}} + \frac{\gamma(s_2+is_1-s_3)}{\sqrt{(s_2+is_1)(s_3-is_1)}}$,
6. $S9$: $V = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2}$,

2nd order systems with potential, $K = 2$

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of H and their multiplicity
- All the 2nd order superintegrable systems are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere,
- The coordinate limits are induced by generalized Inönü-Wigner Lie algebra contractions of the symmetry algebras of the manifolds underlying the superintegrable systems, either flat space or the complex sphere. These contractions have been classified.

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$S9$: the generic system on the 2-sphere

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad a_j = \frac{1}{4} - k_j^2,$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

Basis symmetries: $(J_3 = s_2\partial_{s_1} - s_1\partial_{s_2}, \cdots)$

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \; L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \; L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

Structure equations:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 +$$
$$\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3$$
$$+ \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3, \quad R = [L_1, L_2].$$

$S9$: the generic system on the 2-sphere

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad a_j = \frac{1}{4} - k_j^2,$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

Basis symmetries: $(J_3 = s_2\partial_{s_1} - s_1\partial_{s_2}, \cdots)$

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \; L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \; L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

Structure equations:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 +$$

$$\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3$$

$$+ \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3, \quad R = [L_1, L_2].$$

ON basis of eigenfunctions of L_1, H:

$$\Psi_{N-n,n} = (s_1^2+s_2^2)^{\frac12(2n+k_1+k_2+1)}(1-s_1^2-s_2^2)^{\frac12(k_3+\frac12)}\left(\frac{s_2^2}{s_1^2+s_2^2}\right)^{\frac12(k_2+\frac12)}\left(\frac{s_1^2}{s_1^2+s_2^2}\right)^{\frac12(k_1+\frac12)}$$

$$\times P_n^{(k_2,k_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right)P_{N-n}^{(2n+k_1+k_2+1,k_3)}(1-2s_1^2-2s_2^2),$$

$$L_1\Psi_{N-n,n} = (k_1^2+k_2^2-\frac12-(2n+1+k_1+k_2)^2\Psi_{N-n,n}, \quad n=0,1,\cdots,N,$$

$$H\Psi_{N-n,n} = E_N\Psi_{N-n,n}, \quad E_N = -[2N+k_1+k_2+k_3+2]^2+\frac14, \quad N=0,1,\cdots.$$

Separable in spherical coordinates, orthogonal with respect to area measure on the 1st octant of the 2-sphere. Dimension of eigenspace E_N is $N+1$.

$$P_n^{(\alpha,\beta)}(y) = \binom{n+\alpha}{n}\, {}_2F_1\left(\begin{matrix}-n & \alpha+\beta+n+1\\ \alpha+1 & \end{matrix}; \frac{1-y}{2}\right), \quad \text{Jacobi polynomials}$$

These functions are defined even for n, N complex.

ON basis of eigenfunctions of L_2, H:

Get immediately by permutation $1 \leftrightarrow 3$, $n \leftrightarrow q$, of L_1 basis:

$$L_2\Lambda_{N-q,q} = (k_3^2 + k_2^2 - \frac{1}{2} - (2q + 1 + k_3 + k_2)^2\Lambda_{N-q,q}, \quad q = 0, 1, \cdots, N,$$

$$H\Lambda_{N-q,q} = E_N\Lambda_{N-q,q}, \quad E_N = -[2N + k_1 + k_2 + k_3 + 2]^2 + \frac{1}{4}, \quad N = 0, 1, \cdots.$$

Separable in a different set of spherical coordinates, orthogonal with respect to area measure on the 1st octant of the 2-sphere. Dimension of eigenspace E_N is $N + 1$.

Interbasis expansion coefficients 1

The action of L_1 on and L_2 eigenbasis follows immediately from permutation symmetry. Now we expand the L_2 eigenbasis in terms of the L_1 eigenbasis:

$$\Lambda_{N-q,q} = \sum_{n=0}^{N} R_q^n \, \Psi_{N-n,n}, \quad q = 0, \cdots, N$$

Applying L_2 to both sides of this expression one can show that the expansion coefficients R_q^n satisfy a 3-term recurrence relation in n with respect to multiplication by $t = (q + \frac{k_1+k_3+1}{2})^2$, so the expansion coefficients are polynomials in t^2 of order n. The action of the symmetry operators can be transferred to the expansion coefficients so that these polynomials form a basis for an irreducible representation of the quadratic symmetry algebra acting as difference operators on polynomials.

The Wilson and Racah polynomials

$$R_q^n \sim {}_4F_3\left(\begin{matrix} -n, & \alpha+\beta+\gamma+\delta+n-1, & \alpha-t, & \alpha+t \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta & \end{matrix} ; 1 \right)$$

$$a_j = \frac{1}{4} - k_j^2,\ k_1 = \delta+\beta-1,\ k_2 = \alpha+\gamma-1,\ k_3 = \alpha-\gamma,$$

$$N = -\alpha-\beta,\ t = q + \frac{k_1+k_3+1}{2},$$

a polynomial in t^2.

The quadratic structure algebra of $S9$ can be identified with the Askey-Wilson algebra of these orthogonal polynomial.

The big picture: Special functions

Special functions arise in two distinct ways:

- As separable eigenfunctions of the quantum Hamiltonian. Second order superintegrable systems are multiseparable.
- As eigenfunctions in the model. Often solutions of difference equations. These are interbasis expansion coefficients relating distinct separable coordinate eigenbases.

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The big picture: Contractions and special functions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $o(3, \mathbb{C})$ and $e(2, \mathbb{C})$.,
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)
- Contractions have geometrical and physical significance: $c \to \infty$, $\hbar \to 0$, radius of sphere $\to \infty$, etc.

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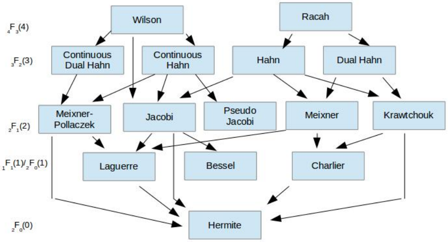
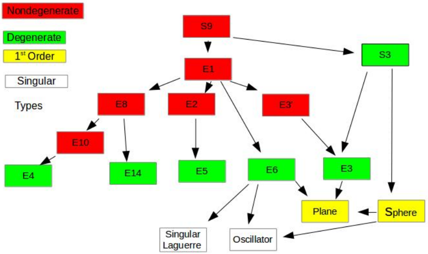
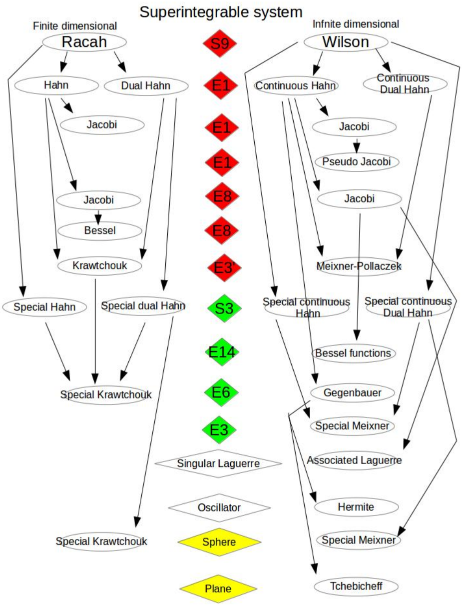


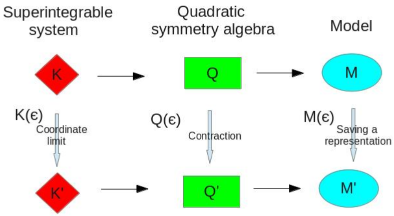
Askey Scheme of Hypergeometric Orthogonal Polynomials



Partial list of contractions of superintegrable systems


Superintegrable system
Finite dimensional
Racah
Hahn
Dual Hahn
Jacobi
Jacobi
Bessel
Krawtchouk
Special Hahn
Special dual Hahn
Special Krawtchouk
Special Krawtchouk
S9
E1
E1
E1
E8
E8
E3'
S3
E14
E6
E3
Singular Laguerre
Oscillator
Sphere
Plane
Infnite dimensional
Wilson
Continuous Hahn
Continuous Dual Hahn
Jacobi
Pseudo Jacobi
Jacobi
Meixner-Pollaczek
Special continuous Hahn
Special continuous Dual Hahn
Bessel functions
Gegenbauer
Special Meixner
Associated Laguerre
Hermite
Special Meixner
Tchebicheff

Model interplay



The Contraction of Models

The TTW System 1

Before 2009, relatively few examples of superintegrable systems of higher order than 2 were known. This changed dramatically with the introduction of the TTW system, built on the Smorodinski Winternitz potential.

The Smorodinski-Winternitz system in polar coordinates $x = r\cos\theta,\ y = r\sin\theta$ is:

$$H = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 - \omega^2 r^2 + \frac{1}{r^2}\left(\frac{\alpha}{\sin^2(\theta)} + \frac{\beta}{\cos^2(\theta)}\right)$$

The Tremblay, Turbiner, Winternitz system (TTW, 2009) is

$$H = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 - \omega^2 r^2 + \frac{1}{r^2}\left(\frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)}\right)$$

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where $k = \frac{p}{q}$ is rational.

The TTW System 2

For $k = 1$ this is the caged isotropic oscillator, for $k = 2$ it is a Calogero system on the line, etc. TTW conjectured that this system was classically and quantum superintegrable for all rational k.

Proved for the classical case by Kalnins, Pogosyan and Miller and in the quantum case by Kalnins, Kress and Miller (2010). The 3rd constant of the motion is of arbitrarily high order. The algebra generated by the symmetries closes.

Using this idea of obtaining higher order superintegrable systems from 2nd order systems, many families of higher order superintegrable systems have now been discovered and for some of them the structure of the symmetry algebras has been worked out. However, there is still no general theory of higher order systems.

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Wrap-up. 1

- Superintegrable systems and their associated symmetry algebras are, essentially, those quantum and classical mechanical systems that can be solved exactly.
- These systems are related to one another by Stäckel transforms or (coupling constant metamorphosis) which preserve the symmetry algebra structure, and by contractions.
- Special functions are identified as functions that express solutions of solvable problems. Thus there are deep connections between the special functions of mathematical physics and superintegrable systems.
- For 2nd order systems, by taking contractions step-by-step from the $S9$ model we can recover the Askey Scheme. However, the contraction method is more general. It applies to all special functions that arise from the quantum systems via separation of variables, not just polynomials of hypergeometric type, and it extends to higher dimensions.

Wrap-up. 2

- For 2nd order superintegrable systems there is a reasonably mature classification and structure theory and a large number of applications.
- For 3rd order systems there are some classification results.
- For higher order superintegrable systems there are many examples but, as yet, no classification and structure theory.