

Laplace equations, superintegrability and Bôcher contractions.

Willard Miller, [Joint with E.G. Kalnins (Waikato), Sarah Post (Hawaii)
Adrian Escobar (UNAM), and Eyal Subag (Penn State)]

University of Minnesota

Abstract

Quantum superintegrable systems are solvable eigenvalue problems. Their solvability is due to symmetry, but the symmetry is often “hidden”. The symmetry generators of 2nd order superintegrable systems in 2 dimensions close under commutation to define quadratic algebras, a generalization of Lie algebras. Distinct systems and their algebras are related by geometric limits, induced by generalized Inönü-Wigner Lie algebra contractions of the symmetry algebras of the underlying spaces. These have physical/geometric implications, such as the Askey scheme for hypergeometric orthogonal polynomials. The systems can be best understood by transforming them to Laplace conformally superintegrable systems and using ideas introduced in the 1894 thesis of Bôcher to study separable solutions of the wave equation. The contractions can be subsumed into contractions of the conformal algebra $so(4, \mathbb{C})$ to itself.

Outline

- 1 Introduction
- 2 Constant curvature space Helmholtz systems
- 3 Lie algebra contractions
- 4 Contractions of superintegrable systems
- 5 The Bôcher method
- 6 Discussion and conclusions

Superintegrable Systems: $H\psi = E\psi$

- A quantum superintegrable system is an integrable Hamiltonian system on an n -dimensional Riemannian/pseudo-Riemannian manifold with potential:

$$H = \Delta_n + V$$

that admits $2n - 1$ algebraically independent partial differential operators commuting with H , the maximum possible.

$$[H, L_j] = 0, \quad j = 1, 2, \dots, 2n - 1.$$

- Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\psi = E\psi$ to be solved exactly, analytically and algebraically.
- A system is of order K if the maximum order of the symmetry operators, other than H , is K . For $n = 2$, $K = 1, 2$ all systems are known.

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2nd order systems with potential, $K = 2$

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of H and their multiplicity
- 2nd order superintegrable systems are multiseparable.
- Smorodinsky, Winternitz and collaborators inaugurated this study in 1965 by pointing out the multiseparability of systems such as the Smorodinsky-Winternitz system

$$H = \partial_{xx} + \partial_{yy} + \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}.$$

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Nondegenerate systems ($2n - 1 = 3$ generators)

In this talk I will consider only the **nondegenerate superintegrable systems**, those with 4-parameter potentials (the maximum possible):

$$V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2 V_{(2)}(\mathbf{x}) + a_3 V_{(3)}(\mathbf{x}) + a_4$$

For these the symmetry algebra generated by H, L_1, L_2 always closes under commutation and gives the following quadratic algebra structure: Define 3rd order commutator R by $R = [L_1, L_2]$. Then

$$[L_j, R] = A_1^{(j)} L_1^2 + A_2^{(j)} L_2^2 + A_3^{(j)} H^2 + A_4^{(j)} \{L_1, L_2\} + A_5^{(j)} H L_1 + A_6^{(j)} H L_2$$

$$+ A_7^{(j)} L_1 + A_8^{(j)} L_2 + A_9^{(j)} H + A_{10}^{(j)}, \quad \{L_1, L_2\} = L_1 L_2 + L_2 L_1,$$

$$R^2 = b_1 L_1^3 + b_2 L_2^3 + b_3 H^3 + b_4 \{L_1^2, L_2\} + b_5 \{L_1, L_2^2\} + b_6 L_1 L_2 L_1 + b_7 L_2 L_1 L_2$$

$$+ b_8 H \{L_1, L_2\} + b_9 H L_1^2 + b_{10} H L_2^2 + b_{11} H^2 L_1 + b_{12} H^2 L_2 + b_{13} L_1^2 + b_{14} L_2^2 + b_{15} \{L_1, L_2\}$$

$$+ b_{16} H L_1 + b_{17} H L_2 + b_{18} H^2 + b_{19} L_1 + b_{20} L_2 + b_{21} H + b_{22},$$

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$$+ A_7^{(j)} L_1 + A_8^{(j)} L_2 + A_9^{(j)} H + A_{10}^{(j)}, \quad \{L_1, L_2\} = L_1 L_2 + L_2 L_1,$$

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Example: S9

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

where $J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

Basis symmetries: ($s_1^2 + s_2^2 + s_3^2 = 1$)

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \quad L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

Structure equations:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 +$$

$$\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3$$

$$+ \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1 a_2 + a_2 a_3 + a_3 a_1) + 64a_1 a_2 a_3, \quad R = [L_1, L_2].$$

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$$\begin{aligned} R^2 = & \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 + \\ & \frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 \\ & + \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1 a_2 + a_2 a_3 + a_3 a_1) + 64a_1 a_2 a_3, \quad R = [L_1, L_2]. \end{aligned}$$

Stäckel Equivalence Classes and Contractions

All 2nd order 2d superintegrable systems with potential are known. There are 44 nondegenerate systems, on a variety of manifolds, but they are related.

- 1 Under the Stäckel transform, an invertible structure preserving mapping, they divide into 6 equivalence classes with representatives on flat space and the 2-sphere.
- 2 Under geometric coordinate limits, non-invertible mappings, one superintegrable system can contract into another.
- 3 Every 2nd order superintegrable system can be obtained from S_9 by a sequence of Stäckel transforms and contractions.
- 4 Contractions of the superintegrable systems can be induced by ordinary Inönü-Wigner Lie algebra contractions of the symmetry algebras of the underlying Riemannian manifolds.

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Nondegenerate flat space systems: $H\Psi = (\partial_x^2 + \partial_y^2 + V)\Psi = E\Psi$.

$$\textcircled{1} E1: V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}, \quad \textcircled{2} E2: V = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2},$$

$$\textcircled{3} E3': V = \alpha(x^2 + y^2) + \beta x + \gamma y,$$

$$\textcircled{4} E7: V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2 - b}} + \frac{\beta(x-iy)}{\sqrt{(x+iy)^2 - b} \left(x+iy + \sqrt{(x+iy)^2 - b}\right)^2} + \gamma(x^2 + y^2),$$

$$\textcircled{5} E8: V = \frac{\alpha(x-iy)}{(x+iy)^3} + \frac{\beta}{(x+iy)^2} + \gamma(x^2 + y^2),$$

$$\textcircled{6} E9: V = \frac{\alpha}{\sqrt{x+iy}} + \beta y + \frac{\gamma(x+2iy)}{\sqrt{x+iy}},$$

$$\textcircled{7} E10: V = \alpha(x - iy) + \beta(x + iy - \frac{3}{2}(x - iy)^2) + \gamma(x^2 + y^2 - \frac{1}{2}(x - iy)^3),$$

$$\textcircled{8} E11: V = \alpha(x - iy) + \frac{\beta(x-iy)}{\sqrt{x+iy}} + \frac{\gamma}{\sqrt{x+iy}},$$

$$\textcircled{9} E15: V = f(x - iy),$$

$$\textcircled{10} E16: V = \frac{1}{\sqrt{x^2 + y^2}} \left(\alpha + \frac{\beta}{y + \sqrt{x^2 + y^2}} + \frac{\gamma}{y - \sqrt{x^2 + y^2}} \right),$$

$$\textcircled{11} E17: V = \frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{\beta}{(x+iy)^2} + \frac{\gamma}{(x+iy)\sqrt{x^2 + y^2}},$$

$$\textcircled{12} E19: V = \frac{\alpha(x+iy)}{\sqrt{(x+iy)^2 - 4}} + \frac{\beta}{\sqrt{(x-iy)(x+iy+2)}} + \frac{\gamma}{\sqrt{(x-iy)(x+iy-2)}}.$$

$$\textcircled{13} E20: V = \frac{1}{\sqrt{x^2 + y^2}} \left(\alpha + \beta \sqrt{x + \sqrt{x^2 + y^2}} + \gamma \sqrt{x - \sqrt{x^2 + y^2}} \right),$$

Nondegenerate systems on the complex 2-sphere:

$$H\Psi = (J_{23}^2 + J_{13}^2 + J_{12}^2 + V)\Psi = E\Psi, \quad J_{k\ell} = s_k \partial_{s_\ell} - s_\ell \partial_{s_k}, \quad s_1^2 + s_2^2 + s_3^2 = 1.$$

Here,

$$\textcircled{1} \text{ S1: } V = \frac{\alpha}{(s_1 + is_2)^2} + \frac{\beta s_3}{(s_1 + is_2)^2} + \frac{\gamma(1 - 4s_3^2)}{(s_1 + is_2)^4},$$

$$\textcircled{2} \text{ S2: } V = \frac{\alpha}{s_3^2} + \frac{\beta}{(s_1 + is_2)^2} + \frac{\gamma(s_1 - is_2)}{(s_1 + is_2)^3},$$

$$\textcircled{3} \text{ S4: } V = \frac{\alpha}{(s_1 + is_2)^2} + \frac{\beta s_3}{\sqrt{s_1^2 + s_2^2}} + \frac{\gamma}{(s_1 + is_2)\sqrt{s_1^2 + s_2^2}},$$

$$\textcircled{4} \text{ S7: } V = \frac{\alpha s_3}{\sqrt{s_1^2 + s_2^2}} + \frac{\beta s_1}{s_2^2 \sqrt{s_1^2 + s_2^2}} + \frac{\gamma}{s_2^2},$$

$$\textcircled{5} \text{ S8: } V = \frac{\alpha s_2}{\sqrt{s_1^2 + s_3^2}} + \frac{\beta(s_2 + is_1 + s_3)}{\sqrt{(s_2 + is_1)(s_3 + is_1)}} + \frac{\gamma(s_2 + is_1 - s_3)}{\sqrt{(s_2 + is_1)(s_3 - is_1)}},$$

$$\textcircled{6} \text{ S9: } V = \frac{\alpha}{s_1^2} + \frac{\beta}{s_2^2} + \frac{\gamma}{s_3^2},$$

Darboux 1 systems: $H\Psi = \left(\frac{1}{4x}(\partial_x^2 + \partial_y^2) + V\right) \Psi = E\Psi$. (Winternitz et. al., 2002)

$$\textcircled{1} D1A: V = \frac{b_1(2x-2b+iy)}{x\sqrt{x-b+iy}} + \frac{b_2}{x\sqrt{x-b+iy}} + \frac{b_3}{x} + b_4,$$

$$\textcircled{2} D1B: V = \frac{b_1(4x^2+y^2)}{x} + \frac{b_2}{x} + \frac{b_3}{xy^2} + b_4,$$

$$\textcircled{3} D1C: V = \frac{b_1(x^2+y^2)}{x} + \frac{b_2}{x} + \frac{b_3y}{x} + b_4.$$

Darboux 2 systems: $H\Psi = \left(\frac{x^2}{x^2+1}(\partial_x^2 + \partial_y^2) + V\right) \Psi = E\Psi$.

$$\textcircled{1} D2A: V = \frac{x^2}{x^2+1} \left(b_1(x^2 + 4y^2) + \frac{b_2}{x^2} + b_3y \right) + b_4.$$

$$\textcircled{2} D2B: V = \frac{x^2}{x^2+1} \left(b_1(x^2 + y^2) + \frac{b_2}{x^2} + \frac{b_3}{y^2} \right) + b_4,$$

$$\textcircled{3} D2C: V = \frac{x^2}{\sqrt{x^2+y^2}(x^2+1)} \left(b_1 + \frac{b_2}{y+\sqrt{x^2+y^2}} + \frac{b_3}{y-\sqrt{x^2+y^2}} \right) + b_4,$$

Darboux 3 systems: $H\Psi = \left(\frac{1}{2} \frac{e^{2x}}{e^x+1}(\partial_x^2 + \partial_y^2) + V\right) \Psi = E\Psi$.

$$\textcircled{1} D3A: V = \frac{b_1}{1+e^x} + \frac{b_2 e^x}{\sqrt{1+2e^{x+iy}}(1+e^x)} + \frac{b_3 e^{x+iy}}{\sqrt{1+2e^{x+iy}}(1+e^x)} + b_4,$$

$$\textcircled{2} D3B: V = \frac{e^x}{e^x+1} \left(b_1 + e^{-\frac{x}{2}} \left(b_2 \cos \frac{y}{2} + b_3 \sin \frac{y}{2} \right) \right) + b_4,$$

$$\textcircled{3} D3C: V = \frac{e^x}{e^x+1} \left(b_1 + e^x \left(\frac{b_2}{\cos^2 \frac{y}{2}} + \frac{b_3}{\sin^2 \frac{y}{2}} \right) \right) + b_4,$$

$$\textcircled{4} D3D: V = \frac{e^{2x}}{1+e^x} (b_1 e^{-iy} + b_2 e^{-2iy}) + \frac{b_3}{1+e^x} + b_4.$$

Darboux 4 systems: $H\Psi = \left(-\frac{\sin^2 2x}{2 \cos 2x+b}(\partial_x^2 + \partial_y^2) + V\right) \Psi = E\Psi.$

$$\textcircled{1} D4(b)A: V = \frac{\sin^2 2x}{2 \cos 2x+b} \left(\frac{b_1}{\sinh^2 y} + \frac{b_2}{\sinh^2 2y} \right) + \frac{b_3}{2 \cos 2x+b} + b_4,$$

$$\textcircled{2} D4(b)B: V = \frac{\sin^2 2x}{2 \cos 2x+b} \left(\frac{b_1}{\sin^2 2x} + b_2 e^{4y} + b_3 e^{2y} \right) + b_4.$$

$$\textcircled{3} D4(b)C: V = \frac{e^{2y}}{\frac{b+2}{\sin^2 x} + \frac{b-2}{\cos^2 x}} \left(\frac{b_1}{Z+(1-e^{2y})\sqrt{Z}} + \frac{b_2}{Z+(1+e^{2y})\sqrt{Z}} + \frac{b_3 e^{-2y}}{\cos^2 x} \right) + b_4.$$

Generic Koenigs spaces:

$$\textcircled{1} K[1, 1, 1, 1]: H\Psi = \frac{1}{V(b_1, b_2, b_3, b_4)} (\partial_x^2 + \partial_y^2 + V(a_1, a_2, a_3, a_4)) \Psi = E\Psi,$$

$$V(a_1, a_2, a_3, a_4) = \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2+y^2-1)^2} - \frac{4a_4}{(x^2+y^2+1)^2},$$

$$\textcircled{2} K[2, 1, 1]: H\Psi = \frac{1}{V(b_1, b_2, b_3, b_4)} (\partial_x^2 + \partial_y^2 + V(a_1, a_2, a_3, a_4)) \Psi = E\Psi,$$

$$V(a_1, a_2, a_3, a_4) = \frac{a_1}{x^2} + \frac{a_2}{y^2} - a_3(x^2 + y^2) + a_4,$$

$$\textcircled{3} K[2, 2]: H\Psi = \frac{1}{V(b_1, b_2, b_3, b_4)} (\partial_x^2 + \partial_y^2 + V(a_1, a_2, a_3, a_4)) \Psi = E\Psi,$$

$$V(a_1, a_2, a_3, a_4) = \frac{a_1}{(x+iy)^2} + \frac{a_2(x-iy)}{(x+iy)^3} + a_3 - a_4(x^2 + y^2),$$

Generic Koenigs spaces:

$$\textcircled{1} \quad K[3, 1]: H\Psi = \frac{1}{V(b_1, b_2, b_3, b_4)} (\partial_x^2 + \partial_y^2 + V(a_1, a_2, a_3, a_4)) \Psi = E\Psi,$$

$$V(a_1, a_2, a_3, a_4) = a_1 - a_2x + a_3(4x^2 + y^2) + \frac{a_4}{y^2},$$

$$\textcircled{2} \quad K[4]: H\Psi = \frac{1}{V(b_1, b_2, b_3, b_4)} (\partial_x^2 + \partial_y^2 + V(a_1, a_2, a_3, a_4)) \Psi = E\Psi,$$

$$V(a_1, a_2, a_3, a_4) = a_1 - a_2(x + iy) + a_3(3(x + iy)^2 + 2(x - iy)) - a_4(4(x^2 + y^2) + 2(x + iy)^3),$$

$$\textcircled{3} \quad K[0]: H\Psi = \frac{1}{V(b_1, b_2, b_3, b_4)} (\partial_x^2 + \partial_y^2 + V(a_1, a_2, a_3, a_4)) \Psi = E\Psi,$$

$$V(a_1, a_2, a_3, a_4) = a_1 - (a_2x + a_3y) + a_4(x^2 + y^2),$$

Special functions and superintegrable systems

Special functions arise in two distinct ways:

- As separable eigenfunctions of the quantum Hamiltonian. Second order superintegrable systems are multiseparable. (The separated solutions are characterized as eigenfunctions of a 2nd order symmetry operator.)
- As interbasis expansion coefficients relating distinct separable coordinate eigenbases for a fixed energy eigenspace. Often these are solutions of difference equations.
- The properties of all these functions are closely related to the representation theory of the symmetry algebras.

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The big picture: Contractions and special functions

- Taking contractions starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and, again, limit relations between the associated special functions.
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Lie algebra contractions

Let $(A; [\cdot, \cdot]_A)$, $(B; [\cdot, \cdot]_B)$ be two complex Lie algebras. We say B is a *contraction* of A if for every $\epsilon \in (0; 1]$ there exists a linear invertible map $t_\epsilon : B \rightarrow A$ such that for every $X, Y \in B$,

$$\lim_{\epsilon \rightarrow 0} t_\epsilon^{-1} [t_\epsilon X, t_\epsilon Y]_A = [X, Y]_B.$$

Thus, as $\epsilon \rightarrow 0$ the 1-parameter family of basis transformations can become singular but the structure constants go to a finite limit.

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Contractions of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$

These are the symmetry algebras of free systems on constant curvature spaces. Their contractions have long since been classified. There are 6 nontrivial contractions of $e(2, \mathbb{C})$ and 4 of $o(3, \mathbb{C})$. They are each induced by coordinate limits.

Example: An Inönü-Wigner-contraction of $o(3, \mathbb{C})$. We use the classical realization for $o(3, \mathbb{C})$ acting on the 2-sphere, with basis

$J_1 = s_2 \partial_3 - s_3 \partial_2$, $J_2 = s_3 \partial_1 - s_1 \partial_3$, $J_3 = s_1 \partial_2 - s_2 \partial_1$, commutation relations

$$[J_2, J_1] = J_3, \quad [J_3, J_2] = J_1, \quad [J_1, J_3] = J_2,$$

and Hamiltonian $H = J_1^2 + J_2^2 + J_3^2$. Here $s_1^2 + s_2^2 + s_3^2 = 1$.

Basis change : $\{J'_1, J'_2, J'_3\} = \{\epsilon J_1, \epsilon J_2, J_3\}$, $0 < \epsilon \leq 1$

New structure relations : $[J'_2, J'_1] = \epsilon^2 J'_3$, $[J'_3, J'_2] = J'_1$, $[J'_1, J'_3] = J'_2$,

Let $\epsilon \rightarrow 0$: $[J'_2, J'_1] = 0$, $[J'_3, J'_2] = J'_1$, $[J'_1, J'_3] = J'_2$, get $e(2, \mathbb{C})$

coordinate implementation $x = \frac{s_1}{\epsilon}$, $y = \frac{s_2}{\epsilon}$, $s_3 \approx 1$, $J = J_3$

Contractions of quadratic algebras.

Just as for Lie algebras we can define a contraction of a quadratic algebra in terms of 1-parameter families of basis changes in the algebra: **As $\epsilon \rightarrow 0$ the 1-parameter family of basis transformations becomes singular but the structure constants go to a finite limit.**

Lie algebra contractions \Rightarrow quadratic algebra contractions

Constant curvature spaces:

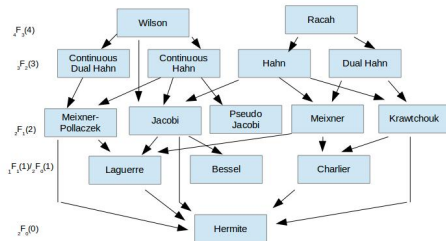
Theorem

(Kalnins-Miller, 2014) *Every Lie algebra contraction of $A = \mathfrak{e}(2, \mathbb{C})$ or $A = \mathfrak{o}(3, \mathbb{C})$ induces uniquely a contraction of a free quadratic algebra \tilde{Q} based on A , which in turn induces uniquely a contraction of the quadratic algebra Q with potential. This is true for both classical and quantum algebras.*

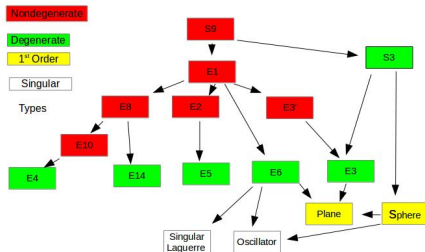
Complications:

Darboux spaces: They support superintegrable systems, but their Lie symmetry algebras are only 1-dimensional so Wigner contractions don't apply.

Konigs spaces: They support superintegrable systems, but have no Lie symmetry algebra.

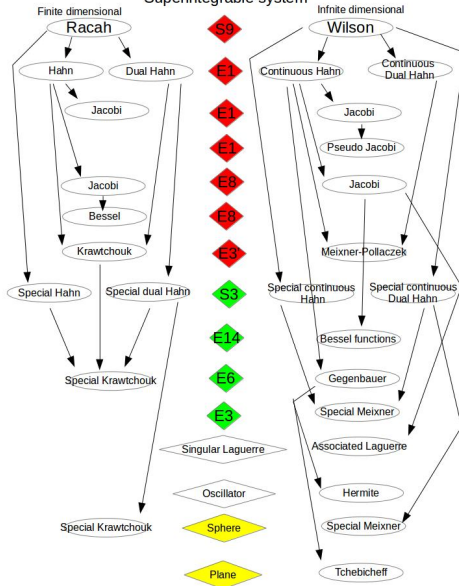


Askey Scheme of Hypergeometric Orthogonal Polynomials



Partial list of contractions of superintegrable systems

Superintegrable system



Unification of the theory: The Laplace equation

Every 2D Riemannian manifold is conformally flat, so we can always find a Cartesian-like coordinate system with coordinates $(x, y) \equiv (x_1, x_2)$ such that the Helmholtz eigenvalue equation takes the form

$$(*) \quad \tilde{H}\Psi = \left(\frac{1}{\lambda(x, y)} (\partial_x^2 + \partial_y^2) + \tilde{V}(\mathbf{x}) \right) \Psi = E\Psi.$$

However, this equation is equivalent to the flat space Laplace equation

$$(**) \quad H\Psi \equiv (\partial_x^2 + \partial_y^2 + V(\mathbf{x})) \Psi = 0, \quad V(\mathbf{x}) = \lambda(\mathbf{x})(\tilde{V}(\mathbf{x}) - E).$$

In particular, the symmetries of $(*)$ correspond to the conformal symmetries of $(**)$. Thus without loss of generality we can assume the manifold is flat space with $\lambda \equiv 1$.

2D conformal superintegrability of the 2nd order

Systems of Laplace type are of the form

$$H\psi \equiv \Delta_n \psi + V\psi = 0.$$

Here Δ_n is the Laplace-Beltrami operator on a conformally flat n D Riemannian or pseudo-Riemannian manifold.

A conformal symmetry of this equation is a partial differential operator S in the variables $\mathbf{x} = (x_1, \dots, x_n)$ such that $[S, H] \equiv SH - HS = R_S H$ for some differential operator R_S .

The system is *conformally superintegrable* for $n > 2$ if there are $2n - 1$ functionally independent conformal symmetries, S_1, \dots, S_{2n-1} with $S_1 = H$. It is second order conformally superintegrable if each symmetry S_i can be chosen to be a differential operator of at most second order.

The conformal Stäckel transform

Suppose we have a second order conformal superintegrable system

$$H = \partial_{xx} + \partial_{yy} + V(x, y) = 0, \quad H = H_0 + V$$

where $V(x, y) = W(x, y) - E U(x, y)$ for arbitrary parameter E .

Theorem

The transformed (Helmholtz) system

$$\tilde{H}\Psi = E\Psi, \quad \tilde{H} = \frac{1}{U}(\partial_{xx} + \partial_{yy}) + \tilde{V}$$

is truly superintegrable, where $\tilde{V} = \frac{W}{U}$.

There is a similar definition of Stäckel transforms of Helmholtz superintegrable systems $H\Psi = E\Psi$ which takes superintegrable systems to superintegrable systems, essentially preserving the quadratic algebra structure.

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Thus any second order conformal Laplace superintegrable system admitting a nonconstant potential U can be Stäckel transformed to a Helmholtz superintegrable system.

By choosing all possible special potentials U associated with the fixed Laplace system we generate the equivalence class of all Helmholtz superintegrable systems obtainable through this process.

Theorem

There is a one-to-one relationship between flat space conformally superintegrable Laplace systems with nondegenerate potential and Stäckel equivalence classes of superintegrable Helmholtz systems with nondegenerate potential.

The conformal symmetry algebra of the underlying flat space is $so(4, \mathbb{C})$, much bigger than $e(2, \mathbb{C})$ and $so(3, \mathbb{C})$. We can use contractions of $so(4, \mathbb{C})$ to compute our limits.

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The equivalence classes for nondegenerate systems

System	Non-degenerate potentials in flat space $(\partial_{xx} + \partial_{yy} + V(x, y))\Psi = 0$
[1, 1, 1, 1]	$\frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2+y^2-1)^2} - \frac{4a_4}{(x^2+y^2+1)^2}$
[2, 1, 1]	$\frac{a_1}{x^2} + \frac{a_2}{y^2} - a_3(x^2 + y^2) + a_4$
[2, 2]	$\frac{a_1}{(x+iy)^2} + \frac{a_2(x-iy)}{(x+iy)^3} + a_3 - a_4(x^2 + y^2)$
[3, 1]	$a_1 - a_2x + a_3(4x^2 + y^2) + \frac{a_4}{y^2}$
[4]	$a_1 - a_2(x + iy) + a_3(3(x + iy)^2 + 2(x - iy)) - a_4(4(x^2 + y^2) + 2(x + iy)^3)$
[0]	$a_1 - (a_2x + a_3y) + a_4(x^2 + y^2)$
(1)	$\frac{a_1}{(x+iy)^2} + a_2 - \frac{a_3}{(x+iy)^3} + \frac{a_4}{(x+iy)^4}$
(2)	$a_1 + a_2(x + iy) + a_3(x + iy)^2 + a_4(x + iy)^3$

A Stäckel transform example

$$[1, 1, 1, 1]: \quad (\partial_x^2 + \partial_y^2 + \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2 + y^2 - 1)^2} + \frac{4a_4}{(x^2 + y^2 + 1)^2})\psi = 0.$$

Multiplying on the left by x^2 we obtain

$$\left(x^2(\partial_x^2 + \partial_y^2) + a_1 + a_2 \frac{x^2}{y^2} + 4a_3 \frac{x^2}{(x^2 + y^2 - 1)^2} - 4a_4 \frac{x^2}{(x^2 + y^2 + 1)^2} \right) \psi = 0.$$

Introduce variables $x = e^{-a}$, $y = r$. Then

$$\left(\partial_a^2 + \partial_a + e^{-2a} \partial_r^2 + a_1 + a_2 \frac{e^{-2a}}{r^2} + a_3 \frac{4}{(e^{-a} + e^a(r^2 - 1))^2} - a_4 \frac{4}{(e^{-a} + e^a(r^2 + 1))^2} \right) \psi$$

Horospherical coordinates on the complex 2-sphere, $s_1^2 + s_2^2 + s_3^2 = 1$:

$$s_1 = \frac{i}{2}(e^{-a} + (r^2 + 1)e^a), \quad s_2 = re^a, \quad s_3 = \frac{1}{2}(e^{-a} + (r^2 - 1)e^a)$$

Now $\left(\partial_{s_1}^2 + \partial_{s_2}^2 + \partial_{s_3}^2 + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2} + \frac{a_4}{s_1^2} \right) \psi = -a_1 \psi$, explicitly the superintegrable system S_9 .

The Bôcher approach

In his 1894 thesis Bôcher developed a geometrical method for finding and classifying the R-separable orthogonal coordinate systems for the flat space Laplace equation $\Delta_n \Psi = 0$ in n dimensions. It was based on the conformal symmetry of these equations. The conformal symmetry algebra in the complex case is $so(n+2, \mathbb{C})$. We will use his ideas for $n=2$, but applied to the Laplace equation with potential

$$H\Psi \equiv (\partial_x^2 + \partial_y^2 + V)\Psi = 0.$$

The $so(4, \mathbb{C})$ conformal symmetry algebra in the case $n=2$ has the basis

$$P_1 = \partial_x, \quad P_2 = \partial_y, \quad J = x\partial_y - y\partial_x, \quad D = x\partial_x + y\partial_y,$$

$$K_1 = (x^2 - y^2)\partial_x + 2xy\partial_y, \quad K_2 = (y^2 - x^2)\partial_y + 2xy\partial_x.$$

Bôcher linearizes this action by introducing tetraspherical coordinates. These are 4 projective coordinates (x_1, x_2, x_3, x_4) confined to the nullcone $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$.

Tetraspherical coordinates

They are complex projective coordinates on the null cone

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

Relation to Cartesian coordinates (x, y) :

$$x = -\frac{x_1}{x_3 + ix_4}, \quad y = -\frac{x_2}{x_3 + ix_4},$$

$$H\Psi = \left(\partial_{xx} + \partial_{yy} + \tilde{V} \right) \Psi = (x_3 + ix_4)^2 \left(\sum_{k=1}^4 \partial_{x_k}^2 + V \right) \Psi = 0$$

where $\tilde{V} = (x_3 + ix_4)^2 V$.

Relation to flat space 1st order conformal symmetries

We define

$$L_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}, \quad 1 \leq j, k \leq 4, \quad j \neq k,$$

where $L_{jk} = -L_{kj}$. The generators for flat space conformal symmetries are related to these via

$$P_1 = \partial_x = L_{13} + iL_{14}, \quad P_2 = \partial_y = L_{23} + iL_{24}, \quad D = iL_{34},$$

$$J = L_{12}, \quad K_1 = L_{13} - iL_{14}, \quad K_2 = L_{23} - iL_{24}.$$

Here

$$D = x\partial_x + y\partial_y, \quad J = x\partial_y - y\partial_x, \quad K_1 = 2xD - (x^2 + y^2)\partial_x,$$

etc.

Relation to separation of variables and Bôcher's limits

Bôcher uses symbols of the form $[n_1, n_2, \dots, n_p]$ where $n_1 + \dots + n_p = 4$, to define coordinate surfaces as follows. Consider the quadratic forms

$$\Omega = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \quad \Phi = \frac{x_1^2}{\lambda - e_1} + \frac{x_2^2}{\lambda - e_2} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4}.$$

If the parameters e_j are pairwise distinct, the elementary divisors of these two forms are denoted by $[1, 1, 1, 1]$.

Given a point in 2D flat space with Cartesian coordinates (x^0, y^0) , there corresponds a set of tetraspherical coordinates $(x_1^0, x_2^0, x_3^0, x_4^0)$, unique up to multiplication by a nonzero constant. **If we substitute into Φ we see that there are exactly 2 roots $\lambda = \rho, \mu$ such that $\Phi = 0$. These are elliptic coordinates.** They are orthogonal with respect to the metric $ds^2 = dx^2 + dy^2$ and are R -separable for the Laplace equations $(\partial_x^2 + \partial_y^2)\Theta = 0$ or $(\sum_{j=1}^4 \partial_{x_j}^2)\Theta = 0$.

The potential $V_{[1,1,1,1]}$

Consider the potential

$$V_{[1,1,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}.$$

It is the only potential V such that equation

$$\left(\sum_{j=1}^4 \partial_{x_j}^2 + V\right)\Theta = 0$$

is R -separable in elliptic coordinates for *all* choices of the parameters e_j . The separation is characterized by 2nd order conformal symmetry operators that are linear in the parameters e_j . In particular the symmetries span a 3-dimensional subspace of symmetries, so the system $H\Theta = (\sum_{j=1}^4 \partial_{x_j}^2 + V_{[1,1,1,1]})\Theta = 0$ must be conformally superintegrable.

Contraction $[1, 1, 1, 1] \rightarrow [2, 1, 1]$

If some of the e_i become equal then Bôcher shows that the process of making $e_1 \rightarrow e_2$ together with suitable transformations of the x'_j 's produces a conformally equivalent H :

$$e_1 = e_2 + \epsilon^2, \quad x_1 \rightarrow \frac{i(x'_1 + ix'_2)}{\sqrt{2}\epsilon}, \quad x_2 \rightarrow \frac{(x'_1 + ix'_2)}{\sqrt{2}\epsilon} + \epsilon \frac{(x'_1 - ix'_2)}{\sqrt{2}}, \quad x_j \rightarrow x'_j, j = 3, 4,$$

This coordinate limit induces a contraction of $\mathfrak{so}(4, \mathbb{C})$ to itself:

$$L'_{12} = L_{12}, \quad L'_{13} = -\frac{i}{\sqrt{2}\epsilon}(L_{13} - iL_{23}) - \frac{i\epsilon}{\sqrt{2}}L_{13}, \quad L'_{23} = -\frac{i}{\sqrt{2}\epsilon}(L_{13} - iL_{23}) - \frac{\epsilon}{\sqrt{2}}L_{13}$$

$$L'_{34} = L_{34}, \quad L'_{14} = -\frac{i}{\sqrt{2}\epsilon}(L_{14} - iL_{24}) - \frac{i\epsilon}{\sqrt{2}}L_{14}, \quad L'_{24} = -\frac{i}{\sqrt{2}\epsilon}(L_{14} - iL_{24}) - \frac{\epsilon}{\sqrt{2}}L_{14}.$$

We call this the Bôcher contraction $[1, 1, 1, 1] \rightarrow [2, 1, 1]$. There is a family of such contractions to $[2, 2]$, $[3, 1]$, $[4]$, based on the roots of quadratic forms.

What is a Bôcher contraction?

Let $\mathbf{x} = \mathbf{A}(\epsilon)\mathbf{y}$, and $\mathbf{x} = (x_1, \dots, x_4)$, $\mathbf{y} = (y_1, \dots, y_4)$ be column vectors, and $\mathbf{A} = (A_{jk}(\epsilon))$, be a 4×4 matrix with matrix elements

$$A_{kj}(\epsilon) = \sum_{\ell=-N}^N a_{kj}^{\ell} \epsilon^{\ell}$$

where N is a nonnegative integer and the a_{kj}^{ℓ} are complex constants. We say that the matrix \mathbf{A} defines a **special Bôcher contraction** of the conformal algebra $so(4, \mathbb{C})$ to itself provided $A(\epsilon) \in SO(4, \mathbb{C})$ for all $\epsilon \neq 0$.

The contraction is implemented as $\epsilon \rightarrow 0$. It takes the null cone $\mathbf{x} \cdot \mathbf{x} = 0$ to the null cone $\mathbf{y} \cdot \mathbf{y} = 0$. Bôcher contractions can be multiplied., since $A \in SO(4, \mathbb{C})$. To generate all such contractions we simply compute the possible 1-parameter subgroups of $SO(4, \mathbb{C})$ from Gantmacher (Matrix Theory, 1960).

To generate all such contractions we simply compute the possible normal forms for of $so(4, \mathbb{C})$ from Gantmacher (Matrix Theory, 1960)

$$C_1 = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 1+i & 0 & 0 \\ -1-i & 0 & -1+i & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_4 = \frac{1}{2} \begin{pmatrix} 0 & 1 & i & 2\lambda \\ -1 & 0 & 2\lambda & i \\ -i & -2\lambda & 0 & -1 \\ -2\lambda & -i & 1 & 0 \end{pmatrix}.$$

Every 1-parameter subgroup $\mathbf{A}(t)$ of $SO(4, \mathbb{C})$ (i.e., $\mathbf{A}(t_1 + t_2) = \mathbf{A}(t_1)\mathbf{A}(t_2)$), is conjugate to one of the forms $\mathbf{A}_j(t) = \exp(tC_j)$, $j = 1, 2, 3, 4$. By making an appropriate change of coordinate $t = t(\epsilon)$ we can obtain a special Böcher contraction matrix.

$$\mathbf{A}_1(t) = \frac{1}{2} \begin{pmatrix} \frac{\epsilon^2+1}{\epsilon} & -\frac{i(\epsilon^2-1)}{\epsilon} & 0 & 0 \\ \frac{i(\epsilon^2-1)}{\epsilon} & \frac{\epsilon^2+1}{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon = e^{i\lambda t},$$

$$\mathbf{A}_2(t) = \frac{1}{2} \begin{pmatrix} \frac{\epsilon_1^2+1}{\epsilon_1} & -\frac{i(\epsilon_1^2-1)}{\epsilon_1} & 0 & 0 \\ \frac{i(\epsilon_1^2-1)}{\epsilon_1} & \frac{\epsilon_1^2+1}{\epsilon_1} & 0 & 0 \\ 0 & 0 & \frac{\epsilon_2^2+1}{\epsilon_2} & -\frac{i(\epsilon_2^2-1)}{\epsilon_2} \\ 0 & 0 & \frac{i(\epsilon_2^2-1)}{\epsilon_2} & \frac{\epsilon_2^2+1}{\epsilon_2} \end{pmatrix}, \quad \epsilon_1 = e^{i\lambda t}, \epsilon_2 = e^{i\mu t}$$

$$\mathbf{A}_3(t) = \begin{pmatrix} 1 - \frac{1}{2\epsilon^2} & \frac{1}{\epsilon} & \frac{i}{2\epsilon^2} & 0 \\ -\frac{1}{\epsilon} & 1 & \frac{i}{\epsilon} & 0 \\ \frac{i}{2\epsilon^2} & -\frac{i}{\epsilon} & 1 + \frac{1}{2\epsilon^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \epsilon = \frac{2}{t(1+i)},$$

$$\mathbf{A}_4(t) = \frac{1}{2} \begin{pmatrix} \frac{\epsilon_1^2+1}{\epsilon_1} & \frac{1}{\epsilon_1\epsilon_2} & \frac{i}{\epsilon_1\epsilon_2} & \frac{i(\epsilon_1^2-1)}{\epsilon_1} \\ -\frac{\epsilon_1}{\epsilon_2} & \frac{\epsilon_1^2+1}{\epsilon_1} & \frac{i(\epsilon_1^2-1)}{\epsilon_1} & \frac{i\epsilon_1}{\epsilon_2} \\ -\frac{i\epsilon_1}{\epsilon_2} & \frac{i(\epsilon_1^2-1)}{\epsilon_1} & \frac{\epsilon_1^2+1}{\epsilon_1} & -\frac{\epsilon_1}{\epsilon_2} \\ \frac{i(\epsilon_1^2-1)}{\epsilon_1} & \frac{i}{\epsilon_1\epsilon_2} & \frac{1}{\epsilon_1\epsilon_2} & \frac{\epsilon_1^2+1}{\epsilon_1} \end{pmatrix}. \quad \epsilon_1 = e^{i\lambda t}, \epsilon_2 = \frac{1}{t}.$$

Partial list of Bôcher contractions of Laplace systems

① **[1, 1, 1, 1] ↓ [2, 1, 1] contraction:**

$$V_{[1,1,1,1]} \downarrow V_{[2,1,1]}, \quad V_{[3,1]} \downarrow V_{[2,1]}, \quad V_{[4]} \downarrow V_{[0]},$$

② **[1, 1, 1, 1] ↓ [2, 2] contraction:**

$$V_{[1,1,1,1]} \downarrow V_{[2,2]}, \quad V_{[2,1,1]} \downarrow V_{[2,2]}, \quad V_{[3,1]} \downarrow V_{(1)}, \quad V_{[4]} \downarrow V_{(2)},$$

③ **[2, 1, 1] ↓ [3, 1] contraction:**

$$V_{[1,1,1,1]} \downarrow V_{[3,1]}, \quad V_{[2,1,1]} \downarrow V_{[3,1]}, \quad V_{[2,2]} \downarrow V_{[0]}, \quad V_{[4]} \downarrow V_{[0]}, \quad V_{(1)} \downarrow V_{(2)},$$

④ **[1, 1, 1, 1] ↓ [4] contraction:**

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⑤ **[2, 2] ↓ [4] contraction:**

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Helmholtz contractions from Bôcher contractions

Bôcher contractions of conformal superintegrable systems induce contractions of Helmholtz superintegrable systems.

The basic idea is that the procedure of taking a conformal Stäckel transform of a conformal superintegrable system, followed by a Helmholtz contraction yields the same result as taking a Bôcher contraction followed by an ordinary Stäckel transform: The diagrams commute.

All quadratic algebra contractions are induced by Lie algebra contractions of $so(4, \mathbb{C})$, even those for Darboux and Konigs spaces.

Helmholtz contractions from Bôcher contractions

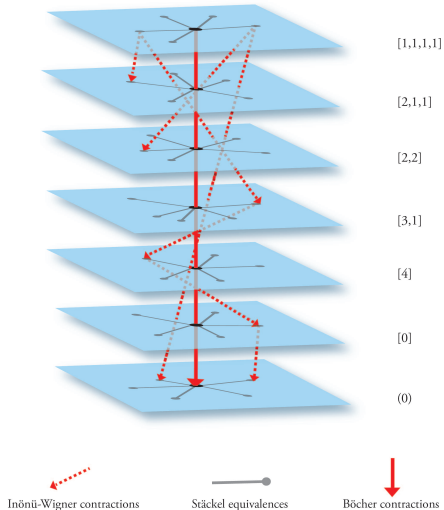
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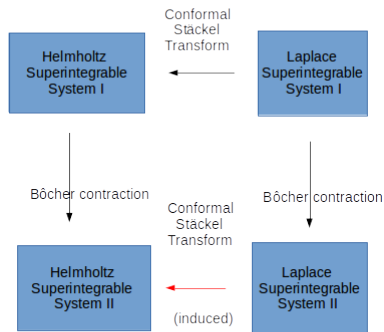
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Schematic of Laplace and Helmholtz superintegrable systems

Stackel Equivalence Classes





The diagram commutes

Figure : Relationship between conformal Stäckel transforms and Bôcher contractions

Wrap-up. 1

- There are 3 basic structures in superintegrability theory: 1) the structures of the symmetry algebras, and their representation theory, 2) Stäckel equivalence of superintegrable systems on distinct manifolds, and 3) contractions relating different systems.
- Taking contractions step-by-step from the S_9 model we can recover the Askey Scheme. However, the contraction method is more general. It applies to all special functions that arise from the quantum systems via separation of variables, not just polynomials of hypergeometric type, and it extends to higher dimensions.
- The theory can be unified and simplified by recognizing that each equivalence class of Stäckel equivalent superintegrable systems corresponds to a single flat space Laplace conformally superintegrable system. Contractions of these systems are determined by Lie algebra contractions of the conformal symmetry algebra $so(4, \mathbb{C})$.
- This work can be generalized to higher dimensions. The case $n = 3$ is under active investigation.