

Superintegrability and Quasi-Exactly Solvable Eigenvalue Problems in Quantum Mechanics

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Abstract

Quasi-exactly solvable (QES) problems in quantum mechanics are eigenvalue problems for the Schrödinger operator where it is possible to compute exactly a certain finite number of eigenvalues and eigenfunctions, even though exact algebraic expressions for the full set of eigenvalues do not exist. In the past, mathematical physicists have used hidden symmetry methods to attack these problems. We give a brief review of these methods and then show the increased insight into the structure of such problems provided by superintegrability theory and separation of variables.

Schrödinger 1D eigenvalue problem:

$$H\Psi = E\Psi$$

where

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x),$$

acting on the space of square integrable functions $L^2(\mathbb{R})$.

ES and QES problems

We say that the eigenvalue equation is **exactly-solvable, (ES)**, if there is a basis of eigenfunctions $H\Psi_n = E_n\Psi_n$ and each function $\Psi_n(x)$ can be expressed in terms of hypergeometric functions (2-term recurrence relations for the coefficients in the power series formula). (An example is the harmonic oscillator

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \omega^2 x^2.$$

It is **quasi-exactly solvable (QES)** if there is a finite-dimensional subspace W of $L^2(\mathbb{R})$ that is invariant under H , i.e., $HW \subseteq W$.

Significance of QES

Let ϕ_1, \dots, ϕ_k be a basis for W and let \mathcal{H}_W be the matrix of the restriction H_W of H to W , with respect to the basis $\{\phi_j\}$. Then H must have an eigenvector Ψ_0 in W with eigenvalue E_0 obtained by solving the characteristic equation

$$\det(\mathcal{H}_W - E_0) = 0.$$

Thus a QES problem always has at least one eigenvalue and eigenvector that can be found by algebraic means. By means of a gauge transformation we can transform a QES problem to one where W is a finite-dimensional space of polynomials in x with basis $1, x, \dots, x^{k-1}$.

Motivating example

Anharmonic oscillator with 6th order potential term:

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \left[\frac{k_1^2}{8\omega^2} - \left(2n + \frac{3}{2}\right)\omega \right] x^2 + \frac{k_1}{2} x^4 + \frac{\omega^2}{2} x^6.$$

For n a fixed positive integer, there are $n + 1$ eigenfunctions

$$\Psi_i = P_n^{(i)}(x) e^{-\frac{k_1}{4\omega} x^2 - \frac{\omega}{2} x^6},$$

$i = 0, 1, \dots, n$ where P is a polynomial of order at most n in x .

Explanation: Hidden symmetry algebra

(Turbiner, Schiffman, Ushveridze, Gonzales-Lopez, Olver, ...)

Realization of the Lie algebra gl_2 by differential operators:

$$J^+ = x^2 \frac{d}{dx} - nx, \quad J_- = \frac{d}{dx}, \quad J^0 = x \frac{d}{dx} - \frac{n}{2}, \quad E^0 = 1.$$

Here n is a non-negative integer.

Commutation relations:

$$[J^+, J^-] \equiv J^+ J^- - J^- J^+ = -2J^0, \quad [J^0, J^\pm] = \pm J^\pm,$$

$$[E^0, J^\pm] = [E^0, J^0] = 0.$$

Casimir operator:

$$C = J^+ J^- - J^0 J^0 + J^0 \equiv -\frac{n}{2} \left(\frac{n}{2} + 1 \right).$$

Strategy

Build the Hamiltonian \tilde{H} as linear combinations of products of the Lie algebra generators. Require that the subspace W_n of polynomials of order $\leq n$ be invariant under \tilde{H} . Note that

$$J^- W_n \subseteq W_n, \quad J^0 W_n \subseteq W_n, \quad J^+ W_n \subseteq W_n.$$

Easy to verify that the most general operator in the enveloping algebra of gl_2 with these properties takes the form

$$\tilde{H} = c_1 (J^+)^2 + c_2 J^+ J^0 + c_3 J^+ J^- + c_4 J^0 J^- + c_5 (J^-)^2 + c_6 J^+ + c_7 J^0 + c_8$$

where the c_j are constants.

The Schrödinger equation

As an ordinary differential operator

$$\tilde{H} = -p_2(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_0(x),$$

where $p_j(x)$ is a polynomial in x of order at most $j + 2$. We can turn this into a Schrödinger operator through the standard change of variable $\tau = \pm \int \frac{dx}{\sqrt{p_2(x)}}$ and the gauge transformation

$$H \equiv e^{-\rho(\tau)} \tilde{H} e^{\rho(\tau)} = -\frac{1}{2} \frac{d^2}{d\tau^2} + (\rho')^2 - \rho'' + p_0(x(\tau)),$$

where

$$\rho(\tau) = - \int \frac{p_1}{p_2} dx + \ln \tau'.$$

The Schrödinger equation 2

Then the polynomial eigenfunctions of \tilde{H} form an orthonormal set of eigenfunctions of H with respect to the weight function $\rho^2(\tau)$. Note: we have to check that the eigenfunctions are square integrable with respect to the weight function.

Example: The τ^6 oscillator

Take

$$\tilde{H} = \frac{2g}{\omega^2} J^+ - 2J^0 J^- + 2J^0 - J^- - nE^0.$$

Then

$$H = -\frac{1}{2} \frac{d^2}{d\tau^2} + \left[\frac{k_1^2}{8\omega^2} - \left(2n + \frac{3}{2}\right)\omega \right] \tau^2 + \frac{k_1}{2} \tau^4 + \frac{\omega^2}{2} \tau^6,$$

and the equation $H\Psi = E\Psi$ has $n + 1$ exact eigenfunctions

$$\Psi_i = P_n^{(i)}(x(\tau)) e^{-\frac{k_1}{4\omega} \tau^2 - \frac{\omega}{2} \tau^6},$$

$i = 0, 1, \dots, n$ where $P^{(i)}$ is a polynomial of order at most n in the computable variable $x(\tau)$.

The hidden symmetry algebra method

Advantages:

1. Can compute and classify the possibilities
2. Extends to other Lie algebras and to d -dimensional spaces
3. Extends to difference and q -difference operators.

Disadvantages:

1. Lack of insight into why the parameters must take certain special values in order that computable eigenvalues exist
2. There are examples of QES systems not explainable via the Lie algebra approach.

Another approach

The singular anisotropic oscillator potential (2D):

$$(1) \quad V_1(x, y) = \frac{1}{2}\omega^2(4x^2 + y^2) + k_1x + \frac{k_2^2 - \frac{1}{4}}{2y^2}$$

The Schrödinger equation has the form

$$(2) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi + \left[2E - \omega^2(4x^2 + y^2) - 2k_1x - \frac{k_2^2 - \frac{1}{4}}{y^2} \right] \Psi = 0.$$

Another approach 2

For $k_2 > 1/2$ the singular term at $y = 0$ is repulsive and the motion takes place only on one of the half planes ($-\infty < x < \infty, y > 0$) or ($-\infty < x < \infty, y < 0$), whereas for $0 < k_2 < 1/2$ in whole plane (x, y) . The Schrödinger equation separates in two systems: *Cartesian and parabolic coordinates*. This is an example of a superintegrable system.

Cartesian separation

Separation of variables in Cartesian coordinates leads to the two *independent* one-dimensional Schrödinger equations

$$(3) \quad \frac{d^2\psi_1}{dx^2} + (2\lambda_1 - 4\omega^2x^2 - 2k_1x)\psi_1 = 0.$$

$$(4) \quad \frac{d^2\psi_2}{dy^2} + \left(2\lambda_2 - \omega^2y^2 - \frac{k_2^2 - \frac{1}{4}}{y^2} \right) \psi_2 = 0.$$

where

$$(5) \quad \Psi(x, y; k_1, k_2) = \psi_1(x; k_1)\psi_2(y; k_2)$$

and λ_1, λ_2 are *Cartesian separation constants* with $\lambda_1 + \lambda_2 = E$.

1st Cartesian separation equation

The first equation represents the well-known linear singular oscillator system. The complete set of orthonormalized eigenfunctions, (on $1/2$) in the interval $0 < y < \infty$ can be expressed in terms of finite confluent hypergeometric series or Laguerre polynomials

$$\psi_{n_2}(y; k_2) = \sqrt{\frac{2\omega^{(1+k_2)} n_2!}{\Gamma(n_2 + k_2 + 1)}} y^{\frac{1}{2}+k_2} e^{-\frac{1}{2}\omega y^2} L_{n_2}^{k_2}(\omega y^2)$$

where $\lambda_2 = \omega(2n_2 + 1 + k_2)$.

2nd Cartesian separation equation

The second equation easily transforms to the ordinary one-dimensional oscillator problem. In terms of Hermite polynomials the orthonormal solutions are

$$\psi_{n_1}(x; k_1) = \left(\frac{2\omega}{\pi}\right)^{1/4} \frac{e^{-\omega z^2}}{\sqrt{2^{n_1} n_1!}} H_{n_1}(\sqrt{2\omega}z),$$

where $z = x + \frac{k_1}{4\omega^2}$, and $\lambda_1 = \omega(2n_1 + 1) - \frac{k_1^2}{8\omega^2}$.

Energy spectrum

$$E = \lambda_1 + \lambda_2 = \omega[2n + 2 + k_2] - \frac{k_1^2}{8\omega^2}, \quad n = n_1 + n_2 = 0, 1, 2, \dots$$

The degree of degeneracy for fixed principal quantum number n is $(n + 1)$. The separation of variables in Cartesian coordinates leads to two **exactly solvable** one-dimensional Schrödinger equations.

Parabolic separation

Parabolic coordinates ξ and η are connected with the Cartesian x and y by

$$x = \frac{1}{2}(\xi^2 - \eta^2) , \quad y = \xi\eta , \quad \xi \in \mathbf{R}, \eta > 0.$$

The Schrödinger equation in parabolic coordinates is

$$\frac{1}{\xi^2 + \eta^2} \left(\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} \right) + \left[2E - \omega^2(\xi^4 - \xi^2\eta^2 + \eta^4) - k_1(\xi^2 - \eta^2) - \frac{k_2^2 - \frac{1}{4}}{\xi^2\eta^2} \right] \Psi = 0.$$

Parabolic separation equations

Upon substituting

$$\Psi(\xi, \eta) = X(\xi)Y(\eta)$$

and introducing the parabolic separation constant λ , we find the two separation equations:

$$\frac{d^2 X}{d\xi^2} + \left(2E\xi^2 - \omega^2\xi^6 - k_1\xi^4 - \frac{k_2^2 - \frac{1}{4}}{\xi^2} \right) X = -\lambda X,$$

$$\frac{d^2 Y}{d\eta^2} + \left(2E\eta^2 - \omega^2\eta^6 + k_1\eta^4 - \frac{k_2^2 - \frac{1}{4}}{\eta^2} \right) Y = +\lambda Y.$$

New eigenvalue problem

The separation equations are transformed into one another by the change $\xi \longleftrightarrow i\eta$. We have

$$\Psi(\xi, \eta; E, \lambda) = C(E, \lambda) Z(\xi; E, \lambda) Z(i\eta; E, \lambda)$$

where $C(E, \lambda)$ is the normalization constant determined by the condition

$$(6) \quad \int_0^\infty d\eta \int_{-\infty}^\infty d\xi (\xi^2 + \eta^2) |\Psi(\xi, \eta; E, \lambda)|^2 = 1$$

and the function $Z(\mu; E, \lambda)$ is a solution of the equation

$$\left[-\frac{d^2}{d\mu^2} + \left(\omega^2 \mu^6 + k_1 \mu^4 - 2E\mu^2 + \frac{k_2^2 - \frac{1}{4}}{\mu^2} \right) \right] Z = \lambda Z.$$

Parabolic spectrum

To solve the unified separation equation we make the substitution

$$Z(\mu; E, \lambda) = \exp\left(-\frac{\omega}{4}\mu^4 - \frac{k_1}{4\omega}\mu^2\right) \mu^{\frac{1}{2}-k_2} \psi(\mu; E, \lambda).$$

The λ -spectrum corresponds to the polynomial solutions $\psi(z)$. **Note: In the special case $k_2 = \frac{1}{2}$, and E replaced by its spectral values this is exactly the 1D anharmonic oscillator QES problem, where now the energy eigenvalue is $\lambda/2$.**

Ushveridze's order 10 polynomial

Ushveridze took two copies of an ordinary differential QES problem (polynomial potential of order 10) and combined them to form a single 2D partial differential equation from which the original ordinary differential equations can be obtained by separation of variables. However, the partial differential equation that he obtains is merely separable, not multiseparable or superintegrable. Here we show the increased insight and greater simplicity obtained by using three copies of the QES problem to form a 3d superintegrable system.

3D Schrödinger equation

Consider the Schrödinger equation $H\Psi = E\Psi$ where

$$H = \frac{1}{(u^2 - v^2)(u^2 - w^2)} \left[\frac{\partial^2}{\partial u^2} - 36k_1^2 u^{10} - 48k_1 k_2 u^8 \right. \\ \left. - 8(2k_2^2 + 3k_1 k_3) u^6 + \frac{p(1-p)}{u^2} \right] \\ + \frac{1}{(v^2 - u^2)(v^2 - w^2)} \left[\frac{\partial^2}{\partial v^2} - 36k_1^2 v^{10} - 48k_1 k_2 v^8 \right. \\ \left. - 8(2k_2^2 + 3k_1 k_3) v^6 + \frac{p(1-p)}{v^2} \right] +$$

3D Schrödinger equation 2

$$+\frac{1}{(w^2 - v^2)(w^2 - u^2)} \left[\frac{\partial^2}{\partial w^2} - 36k_1^2 w^{10} - 48k_1 k_2 w^8 \right. \\ \left. - 8(2k_2^2 + 3k_1 k_3) w^6 + \frac{p(1-p)}{w^2} \right].$$

This equation is clearly separable in the u, v, w coordinates.

Equation in x, y, z coordinates

Passing to Cartesian coordinates

$$z = iuvw, \quad x + iy = \frac{1}{2}(u^2v^2 + u^2w^2 + v^2w^2) - \frac{1}{4}(u^4 + v^4 + w^4),$$

$$x - iy = \frac{1}{2}(u^2 + v^2 + w^2),$$

we can recognize the Hamiltonian operator in the form

$$H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 36k_1^2(2(x - iy)^3 - 4(x^2 + y^2) - z^2) \\ + 48k_1k_2(3(x - iy)^2 - (x + iy)) - 16(2k_2^2 + 3k_1k_3)(x + iy) - \frac{p(p - 1)}{z^2}.$$

3D superintegrable system

This in turn can be recognized as essentially the complex Euclidean space superintegrable system with complex nondegenerate potential

$$V = \alpha (z^2 - 2(x - iy)^3 + 4(x^2 + y^2)) + \beta (2(x + iy) - 3(x - iy)^2) \\ + \gamma(x + iy) + \frac{\delta}{z^2}.$$

3D superintegrable system 2

The six basis second-order symmetry operators can be taken in the form

$$H = \partial_x^2 + \partial_y^2 + \partial_z^2 + V,$$

$$S_1 = (\partial_x - i\partial_y)^2 + f_1, \quad S_2 = \partial_z^2 + f_2, \quad S_3 = \{\partial_z, J_2 + iJ_1\} + f_3,$$

$$S_4 = \frac{1}{2}\{J_3, \partial_x - i\partial_y\} - \frac{i}{4}(\partial_x + i\partial_y)^2 + f_4, \quad S_5 = (J_2 + iJ_1)^2 + 2i\{\partial_z, J_1\}$$

where $\{A, B\} = AB + BA$ and the f_i are appropriate functions.

Separated eqns. in u, v, w coords.

The separation equations for the Schrödinger equation have the form

$$\left[\frac{\partial^2}{\partial \lambda^2} - 36k_1^2 \lambda^{10} - 48k_1 k_2 \lambda^8 - 8(2k_2^2 + 3k_1 k_3) \lambda^6 + \frac{p(1-p)}{\lambda^2} + E \lambda^4 + \ell_2 \lambda^2 \right] \Lambda(\lambda) = -\ell_3 \Lambda(\lambda),$$

for $\lambda = u, v, w$, essentially Ushveridze's 1D QES problem.

Can find the quantized values of E for QES solution by determining the energy levels of the 3D problem.

A new set of separable coordinates

We choose new separable coordinates u, v, z defined by

$$x + iy = -\frac{1}{2}(u - v)^2, \quad x - iy = u + v.$$

The Schrödinger equation has the separable form

$$H\Psi = \left[\frac{1}{u - v} \left[\left(\frac{\partial}{\partial u^2} - 144k_1^2 u^4 - 96k_1 k_2 u^3 + 16(2k_2^2 + 3k_1 k_3) u^2 \right) - \left(\frac{\partial}{\partial v^2} - 144k_1^2 v^4 - 96k_1 k_2 v^3 + 16(2k_2^2 + 3k_1 k_3) v^2 \right) \right] + \frac{\partial^2}{\partial z^2} - 36k_1^2 z^2 + \frac{p(1 - p)}{z^2} \right] \Psi = E\Psi.$$

We already know the quantized values of E from the 3D problem.

New separation equations

The separation equations for the variables $\ell = u, v$ each have the form (with E replaced by its quantized values)

$$\left(\frac{\partial^2}{\partial \ell^2} - 144k_1^2 \ell^4 - 96k_1 k_2 \ell^3 + 16(2k_2^2 + 3k_1 k_3) \ell^2 - \left\{ 16k_2 k_3 + 16 \frac{k_3^2}{k_1} + 6k_1(4n - 4r + 2p - 3) \right\} \ell \right) \Lambda(\ell) = -\lambda_r \Lambda(\ell)$$

where $\Lambda = U, V$ and $\ell = u, v$. There are typically $r + 1$ solutions

$$\Lambda(\ell) = \exp \left(4k_1 \ell^3 + 2k_2 \ell^2 - 2 \left(\frac{k_2^2}{k_1} + k_3 \right) \ell \right) \prod_{i=0}^r (\ell - \theta_i)$$

of this equation, with eigenvalues $\lambda_r^{(s)}$, $s = 1, \dots, r + 1$.

A new QES system!

Note that for this last coordinate system we have given an example of a QES problem with a quartic potential.

It is clear that in higher dimensions there are many examples which generalize the hitherto known examples. In fact we can find QES polynomial potentials of arbitrarily high order. The utility of the use of partial differential operators, rather than ordinary differential operators, is evident.

QES and n -dim superintegrable systems.

We show how to get QES systems from superintegrable systems with nondegenerate potentials.

A Euclidean space quantum system $H\Psi = E\Psi$ where $H = \Delta_n + V(\mathbf{x})$ is second order **superintegrable** if there are $2n - 1$ independent second order differential operators (in Cartesian coordinates x_s), $L_j = \sum_{k,\ell} a_{(j)}^{k\ell} \partial_{k\ell}^2 +$ lower order terms, $j = 1, \dots, 2n - 1$ that commute with $H = L_1$.

Nondegenerate potentials

The potential is **nondegenerate** if it is the general solution of a system

$$V_{jj} - V_{11} = \sum_{\ell=1}^n A^{jj,\ell}(\mathbf{x})V_{\ell}, \quad j = 2, \dots, n,$$

$$V_{kj} = \sum_{\ell=1}^n A^{kj,\ell}(\mathbf{x})V_{\ell}, \quad 1 \leq k < j \leq n,$$

where all of the integrability conditions for this system of partial differential equations are identically satisfied. Thus there is an $n + 2$ dimensional solution space of potentials V . Nondegenerate potentials are those with the maximum number of parameters possible.

Generic Jacobi ellipsoidal coordinates 1

Consider the Hamiltonian

$$H = \sum_{i=1}^n \left(\partial_i^2 + \alpha x_i^2 + \frac{\beta_i}{x_i^2} \right) + \delta, \quad \partial_i = \partial_{x_i}.$$

It is superintegrable with nondegenerate potential and a basis of $n(n+1)/2$ second order symmetry operators given by

$$P_i = \partial_i^2 + \alpha x_i^2 + \frac{\beta_i}{x_i^2}, \quad J_{ij} = (x_i \partial_j - x_j \partial_i)^2 + \beta_i \frac{x_j^2}{x_i^2} + \beta_j \frac{x_i^2}{x_j^2}, \quad i \neq j.$$

Although there appear to be “too many” symmetry operators, all are functionally dependent on a subset of $2n - 1$ functionally independent symmetries.

Generic Jacobi ellipsoidal coordinates 2

1. The equation $H\Psi = E\Psi$ admits multiplicative separation in n generic ellipsoidal coordinates
 $x_i^2 = c^2 \prod_{j=1}^n (u_j - e_i) / \prod_{k \neq i} (e_k - e_i)$, simultaneously for all values of the parameters with $e_i \neq e_j$.
2. Thus the equation is multiseparable and separates in a continuum of ellipsoidal coordinate systems (and in many others besides).
3. The n commuting symmetries characterizing a fixed elliptic separable system are polynomial functions of the e_i , and requiring separation for all e_i simultaneously sweeps out the full $n(n+1)/2$ space of symmetries and uniquely determines the nondegenerate potential.

Generic Jacobi ellipsoidal coordinates 3

1. The infinitesimal distance in Jacobi elliptical coordinates u_j is

$$ds^2 = -\frac{c^2}{4} \sum_{i=1}^n \frac{\prod_{j \neq i} (u_i - u_j)}{P(u_i)} du_i^2,$$

where $P(\lambda) = \prod_{k=1}^n (\lambda - e_k)$.

2. This is a flat space separable metric for any polynomial $P(\lambda)$ of order $\leq n$.
3. The distinct cases are labeled by the degree of the polynomial and the multiplicities of its distinct roots.

Generic Jacobi ellipsoidal coordinates 4

1. If for each distinct case we determine the most general potential that admits separation for all e_i compatible with the multiplicity structure of the roots, we obtain a unique generic superintegrable system with nondegenerate potential and $n(n + 1)/2$ second order symmetries.
2. Thus, for $n = 3$ there are 7 distinct cases for $-\frac{1}{4} P(\lambda)$:

$$(\lambda - e_1)(\lambda - e_2)(\lambda - e_3), (\lambda - e_1)(\lambda - e_2)^2, (\lambda - e_1)^3,$$

$$(\lambda - e_1)(\lambda - e_2), (\lambda - e_1)^2, (\lambda - e_1), 1,$$

where $e_i \neq e_j$ for $i \neq j$. The first case corresponds to Jacobi elliptic coordinates.

Generic Jacobi ellipsoidal coordinates 5

The number of distinct generic superintegrable systems for each integer $n \geq 2$ is

$$\sum_{j=0}^n p(j),$$

where $p(j)$ is the number of integer partitions of j , given by the Euler generating function

$$\frac{1}{\prod_{k=1}^{\infty} (1 - t^k)} = \sum_{j=0}^{\infty} p(j) t^j.$$

Generic Jacobi ellipsoidal coordinates 6

1. Although we cannot write down master canonical expressions for all such generic systems in Cartesian coordinates, it is easy to take these limits and write down a master equation for the separated ordinary differential equations in the elliptic coordinates, u_i .
2. The separation equation for each of the coordinates $x = u_i$ is essentially the same:

The master equation 1

$$\begin{aligned} & \sqrt{P(x)} \frac{d}{dx} \left(\sqrt{P(x)} \frac{dF}{dx} \right) + (a_{n+p_0} x^{n+p_0} + \dots + a_n x^n + \hat{a}_{n-1} x^{n-1} + \dots + \hat{a}_0) \\ & + \frac{b_1^{(1)}}{(x - e_1)} + \frac{b_2^{(1)}}{(x - e_1)^2} + \dots + \frac{b_{p_1}^{(1)}}{(x - e_1)^{p_1}} + \dots + \frac{b_1^{(r)}}{(x - e_r)} + \frac{b_2^{(r)}}{(x - e_r)^2} + \dots \\ & + \frac{b_{p_r}^{(r)}}{(x - e_r)^{p_r}} \Big) F = 0, \end{aligned}$$

Here,

$$P(x) = -4(x - e_1)^{p_1} \dots (x - e_r)^{p_r}$$

and $p_0 + p_1 + \dots + p_r = n$.

The master equation 2

1. The n constants \hat{a}_s , $s = 0, \dots, n - 1$ are the separation constants for the superintegrable system in these elliptic coordinates. In particular, $E = \hat{a}_{n-1}$.
2. The other $n + 1$ constants depend on the $n + 1$ parameters in the potential and can be assigned arbitrarily by specifying the appropriate potential.

The master equation 2

First we look for explicit solutions of a single master equation, ignoring the fact that it is a separation equation for a superintegrable system. To obtain polynomials, we perform a gauge transformation and find solutions of the form

$$\Psi = \exp \left(\ell_1 x + \ell_2 x^2 + \cdots + \ell_{p_0+1} x^{p_0+1} + \frac{c_1^{(1)}}{(x - e_1)} + \frac{c_2^{(1)}}{(x - e_1)^2} + \cdots + \frac{c_{p_1-1}^{(1)}}{(x - e_{p_1})^{p_1-1}} + \cdots + \frac{c_1^{(r)}}{(x - e_r)} + \frac{c_2^{(r)}}{(x - e_r)^2} + \cdots + \frac{c_{p_r-1}^{(r)}}{(x - e_r)^{p_r-1}} \right) \\ \times \prod_{t=1}^r (x - e_t)^{q_t} \prod_{i=1}^s (x - \theta_i) = f(x) \prod_{i=1}^s (x - \theta_i) = f(x) \Phi(x).$$

The master equation 3

We require that the differential equation satisfied by the polynomial $\Phi(x)$ is of the form

$$(r_{n-p_0}x^{n-p_0} + \dots + r_0) \frac{d^2\Phi}{dx^2} + (s_nx^n + \dots + s_0) \frac{d\Phi}{dx} + (t_{n-1}x^{n-1} + t_0)\Phi = 0$$

This is always possible.

The master equation 4

The zeros θ_i of these polynomials satisfy relations of the form

$$\sum_{j \neq i} \frac{1}{\theta_i - \theta_j} + \ell_1 \theta_i + 2\ell_2 \theta_i^2 + \cdots + (p_0 + 1)\ell_{p_0+1} \theta_i^{p_0} + \frac{\frac{1}{4}p_1 + q_1}{(\theta_i - e_1)} - \frac{c_1^{(1)}}{(\theta_i - e_1)^2} + \frac{(p_1 - 1)c_{p_1-1}^{(1)}}{(\theta_i - e_1)^{p_1}} \cdots + \frac{\frac{1}{4}p_r + q_r}{(\theta_i - e_r)} - \frac{c_r^{(1)}}{(\theta_i - e_r)^2} + \cdots + \frac{(p_r - 1)c_{p_r-1}^{(r)}}{(\theta_i - e_r)^{p_r}} = 0.$$

This is easy to see from the original differential equation evaluated at $x = \theta_i$.

The master equation 5

We relate these finite solutions to the superintegrable system in n dimensional Euclidean space from which they are obtained by separation of variables. The admissible values of the separation constants for obtaining polynomial solutions in the variables u_i can be determined for the n -space Hamiltonian and the results translated back to the separated master equations.

The master equation 6

1. The same procedure works for generically separable systems for n -dimensional constant curvature spaces.
2. Many of these polynomial solutions are not normalizable with respect to the standard inner product that makes the separated Hamiltonian formally self-adjoint. However, they are often understandable as CP-invariant Hamiltonians.

Final Message

1. 1D QES problems arise by separation of variables in higher-dimensional Schrödinger equations with superintegrable (multiseparable) potentials.
2. The quantized values of the parameters that permit explicit solutions are the energy eigenvalues of the higher-dimensional problem. The “energy” for the 1D QES problem is an associated separation constant.
3. Apparently distinct 1D QES and ES problems can arise from the same higher-dimensional problem by separation in distinct coordinate systems. In this case the quantized energy is the same.
4. Superintegrability theory is a powerful tool for the analysis and classification of ES and QES systems.