Wilson polynomials and the generic superintegrable system on the 2-sphere

E. G. Kalnins
Department of Mathematics, University of Waikato, Hamilton, New Zealand.
W. Miller, Jr. and S. Post
School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 55455, U.S.A.
miller@ima.umn.edu

August 14, 2007

Abstract

For complex two dimensional Riemannian spaces every classical or quantum second order superintegrable system system can be obtained from a single generic 3-parameter potential on the complex two-sphere by delicate limit operations and through Stäckel transforms between manifolds. Here we derive families of finite and infinite dimensional irreducible representations of the corresponding quadratic quantum algebra for the two-sphere and point out their role in explaining the degeneracy of the energy eigenspaces corresponding to bound state and continuous spectra of quantum and wave equation analogs of this system. The algebra is exactly the one that describes the Wilson and Racah polynomials in their full generality.

1 Introduction

For any complex 2D Riemannian manifold we can always find local coordinates x, y such that the classical Hamiltonian takes the form

$$H = \frac{1}{\lambda(x,y)}(p_1^2 + p_2^2) + V(x,y),$$

i.e., the complex metric is $ds^2 = \lambda(x,y)(dx^2 + dy^2)$. This system is **superintegrable** for some potential V if it admits 3 functionally independent constants of the motion (the maximum number possible) that are polynomials in the momenta p_j. It is **second order superintegrable** if the constants of the motion are quadratic, i.e., of the form $L = \sum a^{ji}(x,y)p_jp_i + W(x,y)$. (By taking various real restrictions of the complex system, we can obtain obtain real superintegrable systems, e.g., the complex 2-sphere restricts to the real 2-sphere, to the hyperboloid of two sheets, and the hyperboloid of one sheet.) There is an analogous definition of second order superintegrability for quantum systems with Schrödinger operator

$$\mathcal{H} = \frac{1}{\lambda(x,y)}(\partial_1^2 + \partial_2^2) + V(x,y)$$

and symmetry operators $\mathcal{L} = \sum \partial_j(a^{ji}(x,y))\partial_i + W(x,y)$,, i.e. $[\mathcal{H}.\mathcal{L}] = 0$, and these systems correspond one-to-one. As demonstrated in the literature, these systems have remarkable properties, including multiseparability (which implies multi-integrability, i.e., integrability in distinct ways) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], except for one isolated Euclidean system [14], and the existence of a quadratic algebra of symmetries that closes at order 6. There has been recent intense activity to uncover the structure of second order superintegrable systems in n dimensions and to classify them. For the easiest case, $n = 2$, the classification is complete: [15, 16, 17, 18, 19, 20], and for $n = 3$ it is nearly complete for nondegenerate potentials [21, 22]. A basic result in 2D is that every potential V in a superintegrable system that depends on at least one multiplicative parameter is a restriction of a nondegenerate potential. (This last fact is no longer true for 3D superintegrable systems.) However, upon restriction the structure of the quadratic algebra can change and closure may be achieved at less than 6th order. There is an invertible mapping, called the Stäckel transform [23], or coupling constant metamorphosis [24], that takes a superintegrable system on one manifold to a superintegrable system on another manifold. Thus, although there is a multiplicity of superintegrable systems, it can be shown that all such systems are equivalent under the Stäckel transform to exactly 7 nondegenerate Stäckel inequivalent systems, 6 on complex Euclidean space and 1 on the complex 2-sphere. (There are other nondegenerate superintegrable systems on the 2-sphere, but each is equivalent to a Euclidean system under the Stäckel transform, and some other Euclidean superintegrable systems are equivalent to one of the 6 already mentioned.)

Another important fact about 2D systems is that all systems can be obtained from one generic superintegrable system on the complex 2-sphere by appropriately chosen limit processes, e.g. [25, 26]. The use of these processes in separation of variables methods for wave and Helmholtz equations in n dimensions was pioneered by Bôcher [27]. (We note that for $n = 3$ the situation is much more complicated. It is no longer true that all quadratically superintegrable systems are limit forms or restrictions of one system, and the algebra of symmetries does not always close.)

For $n = 2$ the generic sphere system $\mathcal{S}2$ corresponds to the nondegenerate potential

$$V = \frac{\frac{1}{4} - a^2}{s_1^2} + \frac{\frac{1}{4} - b^2}{s_2^2} + \frac{\frac{1}{4} - c^2}{s_3^2} \tag{1}$$

where $s_1^2 + s_2^2 + s_3^2 = 1$. This nondegenerate superintegrable system is

$$H = J_1^2 + J_2^2 + J_3^2 + V(x,y) = H_0 + V \tag{2}$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of the indices $1, 2, 3$. This system is uniquely characterized by the fact that it admits multiplicative separation in generic Jacobi elliptic coordinates on the 2-sphere in the quantum case (and additive separation in the classical case) and is the only nondegenerate system on the sphere to admit separation in these coordinates. We choose a basis for the 3-dimensional space of second order symmetries in the symmetric form L_1, L_2, L_3 where $L_1 = J_3^2 + W_1,\ L_2 = J_1^2 + W_2,\ L_3 = J_2^2 + W_3,\ H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$. Here, $V = W_1 + W_2 + W_3 + a_1 + a_2 + a_3$ and

$$a_1 = \frac{1}{4} - c^2,\ a_2 = \frac{1}{4} - a^2,\ a_3 = \frac{1}{4} - b^2.$$

and the functions W_j can easily be computed. The algebra generated by these symmetries and their commutators must close at order 6. The structure equations for $\mathcal{S}2$ can be put in the symmetric form [16]

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k), \tag{3}$$

$$\begin{aligned} R^2 = {} & \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 \\ & + \frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 \\ & + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3) \\ & + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3. \end{aligned} \tag{4}$$

Here i, j, k are chosen such that $\epsilon_{ijk} = 1$ where ϵ is the pure skew-symmetric tensor, $R = [L_1, L_2]$ and $\{L_i, L_j\} = L_iL_j + L_jL_i$ with an analogous definition of $\{L_1, L_2, L_3\}$ as a symmetrized sum of 6 terms. In practice we will substitute $L_3 = H - L_1 - L_2 - a_1 - a_2 - a_3$ into these equations.

Because of the unique role of this system it is important to establish the physically important representations of the quadratic algebra associated with $\mathcal{S}2$. It has already been pointed out by several authors that there are relation between the quadratic algebras of some 2D second order superintegrable systems and special cases of the quadratic Racah algebra QR(3), [28, 29, 30, 10, 31, 11, 12, 32, 33, 34]. Here we show that the the generic 2-sphere system is unique in the fact that its quadratic algebra coincides with the full Racah algebra. Further, since our system is taken in complex form, it also yields the full Wilson polynomial algebraic structure, a structure associated with infinite dimensional representations of the quadratic algebra.

Daskaloyannis, e.g. [31, 12, 32], has developed a deformed oscillator approach to exhibiting the finite dimensional irreducible representations of quadratic algebras. However, rather than adopt this elegant approach here, we construct families of irreducible representations from first principles. This is partly because we are interested in infinite, as well as finite dimensional representations. Why are infinite dimensional representations of the quadratic algebra important? In the usual applications of the quadratic algebra structure one notes that each eigenspace of the Schrödinger operator (corresponding to a discrete eigenvalue) is finite dimensional and invariant under the quadratic algebra, so that finite dimensional irreducible representations of the quadratic algebra can be used to understand and explain the degeneracy of the eigenspace. Little attention is paid to the continuous spectrum because, strictly speaking, the eigenvectors corresponding to an eigenvalue in the continuous spectrum are not normalizable. However, there are other interpretations of the quadratic algebra that make clear the physical and mathematical importance of these, apparently, continuous spectrum cases and their relevance to infinite dimensional irreducible representations of the quadratic algebra.

To clarify the situation let us consider the superintegrable system on the complex 2-sphere in the special case with potential (1) and $a = b = c = \frac{1}{2}$, i.e., $\mathcal{S}2$ with the potential switched off. The structure of the algebra has changed since the first order operators J_1, J_2, J_3 are now symmetries and they generate the algebra. This system has been studied in detail. In the case of the real 2-sphere one usually considers the eigenvalue equation in the form

$$(J_1^2 + J_2^2 + J_3^2)\Psi = -\ell(\ell + 1)\Psi$$

where ℓ is a nonnegative integer, and restricts attention to the $2\ell + 1$ dimensional eigenspace where $H = \ell(\ell + 1)$. This corresponds to the finite dimensional representation $D(\ell)$ of the symmetry algebra $so(3)$ generated by the first order symmetry operators. The basis functions are spherical harmonics in a J_3-basis, and products of Lamé polynomials in an $J_1^2+r^2J_2^2$-basis with $0<r^2<1$. This is a very familiar case [35].

Now take a different real form: $s_1 = ix_1,\ s_2 = ix_2,\ s_3 = x_0$, where the x_1, x_2 are real and $x_0 > 0$. This is the upper sheet of the 2-sheet hyperboloid $x_0^2 - x_1^2 - x_2^2 = 1$. Now the Schrödinger equation can be written as

$$(K_1^2 + K_2^2 - J_3^2)\Psi = \ell(\ell+1)\Psi \tag{5}$$

where a basis for the first order symmetries is

$$K_1 = x_0\partial_{x_1} + x_1\partial_{x_0},\ K_2 = x_0\partial_{x_2} + x_2\partial_{x_0},\ J_3 = x_1\partial_{x_2} - x_2\partial_{x_1}.$$

The first order symmetries generate the Lie algebra $so(2,1)$. In [36] two of the authors studied this equation in the case $\ell = -\frac{1}{2} + i\rho,\ 0 < \rho < \infty$, corresponding to the principle series of single valued representations of $so(2,1)$. If one considers the Schrödinger operator as acting on the hyperboloid with the standard measure then this choice of ℓ corresponds to a value in the continuous spectrum and there are no normalizable solutions. However, we showed that we could construct a model of the (infinite dimensional) principle series in terms of a Hilbert space of functions on the unit circle and an intertwining operator that maps this Hilbert space into the solution space of (5). This effectively induced a Hilbert space structure on this single eigenspace with fixed eigenvalue $\ell(\ell+1)$. A rich structure emerged with 9 types of orthogonal bases, corresponding to 9 types of variable separation.

In paper [37] we studied the equation

$$(K_1^2 + K_2^2 - J_3^2)\Psi = (\nu^2 - \frac{1}{4})\Psi, \quad 0 \le \nu. \tag{6}$$

Again, if one considers the Schrödinger operator as acting on the hyperboloid with the standard measure then this choice of eigenvalue corresponds to the continuous spectrum and there are no normalizable solutions. However, we showed that we could construct a model of the (infinite dimensional) negative discrete series $D^-_{\nu-1/2}$ of the universal covering group of $SL(2,R)$ in terms of a Hilbert space of functions on the positive real line and an intertwining operator that mapped this Hilbert space into the solution space of (6). Again this effectively induced a Hilbert space structure on this single eigenspace with fixed eigenvalue given by ν. Here there were 9 types of orthogonal bases, corresponding to 9 types of variable separation. This equation is of particular physical interest because a change of variables and gauge transforms it to the Euler-Poisson -Darboux (EPD) equation, an equation commonly studied in electromagnetic theory.

$$\left(\partial_{tt} - \partial_{rr} - \frac{1}{r}\partial_r + \frac{\nu^2}{r^2}\right)\Phi = 0. \tag{7}$$

In this same paper the authors showed that one could complexify the EPD equation and just consider locally analytic solutions for fixed ν with no Hilbert space structure. However, raising and lowering operators and separable solutions still existed so the formal algebraic relations still gave useful information. This same idea was used by Viswanathan [38] to derive generating functions for Gegenbauer polynomials.

Now we consider the general potential (1) for $\mathcal{S}2$. This potential was studied in [39] for the real 2-sphere and its n dimensional analog in [40] for the n-sphere. The spectrum is discrete and the eigenspaces all finite dimensional. In [41] the corresponding potential (called the singular oscillator potential) was studied on the upper sheet of the two-sheet hyperboloid for appropriate values of a, b, c. There, bound states were found, but it was remarked that there was also continuous spectrum. In particular the Schrödinger eigenvalue equation separates in spherical coordinates and the separated equations are 1D Schrödinger equations with Pöschl-Teller potential. The bound states correspond to Pöschl-Teller bound states, whereas the continuum spectrum states correspond to Pöschl-Teller scattering states. These Pöschl-Teller states are studied in [42] and, particularly in [43] where the group theoretic description of the scattering states is spelled out. The connection with our problem is that a continuum eigenspace for system $\mathcal{S}2$ on the hyperboloid can be considered as supporting an infinite dimensional irreducible representation of the quadratic algebra.

A final issue is that under a Stäckel transform a superintegrable system on one manifold is mapped to a superintegrable system on another manifold. Under this transform the original quadratic algebra maps to a new quadratic algebra which is isomorphic to the original, except that the parameters a, b, c, H are subject to a linear transformation. Formal eigenfunctions map to formal eigenfunctions. However, the measures on the manifolds are different so that, for example, bound states may not map to bound states.

In the following sections we start from first principles, work out some families of finite and infinite dimensional representations of the algebra $\mathcal{S}2$ and relate them to the Wilson and Racah polynomials in their full generality. In essence, these polynomial families provide one variable models of the quadratic algebra action.

2 The structure equations for $\mathcal{S}2$

We look first for a family of irreducible finite dimensional representations of this quadratic algebra that corresponds to the standard $m+1$-degenerate bound states of the Schrödinger eigenvalue equation for $\mathcal{S}2$. It is easy to show that corresponding to a fixed energy eigenvalue H it is not possible to find nontrivial representations where the eigenvalues of L_1 take the linear form $\lambda_n = An+B, \quad n = 0, 1, \cdots$. Indeed, this is incompatible with equations (3). However, we know that the quantum Schrödinger equation separates in spherical coordinates, and that corresponding to a fixed energy eigenvalue H the eigenvalues of L_1 take the quadratic form

$$\lambda_n = -[2n+B]^2 + \mathcal{K}, \quad n = 0, 1, \cdots, m \tag{8}$$

where $B = a+b+1$ and $\mathcal{K}$ is a constant that we will compute. These are not the only irreducible representations of this algebra but they are of immediate physical relevance.

Indeed in terms of standard spherical coordinates the Schrödinger equation looks like

$$\left[\frac{\partial^2}{\partial_\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\frac{1}{4}-a^2}{\sin^2\theta\cos^2\phi} + \frac{\frac{1}{4}-b^2}{\sin^2\theta\sin^2\phi} + \frac{\frac{1}{4}-c^2}{\cos^2\theta}\right]\Psi = E\Psi.$$

We apply separation of variables and look for solutions $\Psi = \Theta(\theta)\Phi(\phi)$. The separation equations (corresponding to diagonalization of L_1) are

$$\left(\frac{\partial^2}{\partial_\phi^2} + \frac{\frac{1}{4}-a^2}{\cos^2\phi} + \frac{\frac{1}{4}-b^2}{\sin^2\phi}\right)\Phi(\phi) = \lambda\Phi(\phi), \ \left(\frac{\partial^2}{\partial_\theta^2} + \frac{\frac{1}{4}-c^2}{\cos^2\theta} + \frac{\lambda}{\sin^2\theta}\right)\Theta(\theta) = E\Theta(\theta).$$

The finite solutions have the form

$$(\cos\theta)^{c+\frac{1}{2}}(\sin\theta)^{2n+a+b+\frac{3}{2}}P_k^{(2n+a+b+1,c)}(\cos 2\theta)(\sin\phi)^{a+\frac{1}{2}}(\cos\phi)^{b+\frac{1}{2}}P_n^{(a,b)}(\cos 2\phi)$$

where the eigenvalues are

$$\lambda = -(2n+a+b+1)^2, \quad E = \frac{1}{4} - (2(k+n)+a+b+c+1)^2$$

and $P_n^{(a,b)}(z)$ is a Jacobi polynomial. This illustrates (8).

We will use the abstract structure equations to list the corresponding representations and compute the action of L_2 on an L_1 basis. We start, with greater generality, by assuming that there is a basis $\{f_n : n = 0, 1, \cdots\}$, for the representation space such that

$$L_1 f_n = (\mathcal{K} - [2n+B]^2)f_n, \quad L_2 f_n = \sum_\ell C(\ell, n) f_\ell. \tag{9}$$

Here, B is not yet fixed. We do not require that the basis be orthonormal. From these assumptions we can compute the action of R and $[L_1, R]$ on the basis. Indeed,

$$Rf_n = [L_1, L_2]f_n = \sum_\ell 4(n-\ell)(n+\ell+B)C(\ell, n)f_\ell, \tag{10}$$

$$[L_1,R]f_n=\sum_\ell 16(n-\ell)^2(n+\ell+B)^2C(\ell,n)f_\ell. \tag{11}$$

On the other hand, from (3) with $i=1, j=2, k=3$ we have

$$\begin{aligned}[L_1,R]f_n &= 8\sum_\ell\Big([2\ell+B]^2+[2n+B]^2-2\mathcal{K}+2a^2+2b^2-3\Big)C(\ell,n)f_\ell \\ &+8\Big[-([2n+B]^2-\mathcal{K})^2+([2n+B]^2-\mathcal{K})(\frac{9}{4}-a^2-3b^2-c^2-H) \\ &+(\frac{3}{2}-2b^2)(H-\frac{3}{4}+a^2+b^2+c^2)+b^2-a^2\Big]f_n.\end{aligned} \tag{12}$$

Now we equate (11) and (12). For $n\neq\ell$, equating coefficients of f_ℓ in the resulting identity yields the condition

$$C(\ell,n)\left[\frac{1}{8}([2\ell+B]^2-[2n+B]^2)^2-[2\ell+B]^2-[2n+B]^2+3+2(\mathcal{K}-a^2-b^2)\right]=0.$$

We see from this that in order for $C(\ell,n)\neq 0$ we must have $\ell=n,n\pm1$ and $\mathcal{K}=-\frac{1}{2}+a^2+b^2$. Equating coefficients of f_n in the identity, we can solve for $C(n,n)$ and obtain

$$C(n,n)=\frac{w}{2}+\left(\frac{H}{2}+\frac{3}{8}-\frac{a^2}{2}+\frac{b^2}{2}+\frac{c^2}{2}\right)+\frac{Q_1}{w} \tag{13}$$

where $w=(2n+B+1)(2n+B-1)$ and

$$Q_1=\frac{1}{2}(H-\frac{3}{4}+a^2+b^2+c^2)(-a^2+b^2)+\frac{a^4}{2}-\frac{a^2}{4}-\frac{b^4}{2}+\frac{b^2}{4}.$$

It is straightforward to show that the action of $[L_2,R]$ on the basis is

$$[L_2,R]f_n=\sum_{\ell,j}K(n,j,\ell)C(j,\ell)C(\ell,n)f_j \tag{14}$$

where

$$K(n,j,\ell)=[2n+B]^2+[2j+B]^2-2[2\ell+B]^2.$$

For fixed n there are 8 nonzero terms in the double sum:

j	ℓ	$K(n,j,\ell)$
$n+2$	$n+1$	8
$n-2$	$n-1$	8
$n+1$	n	$8n+4+4B$
$n+1$	$n+1$	$-8n-4-4B$
$n-1$	n	$-8n+4-4B$
$n-1$	$n-1$	$8n-4+4B$
n	$n+1$	$-16n-8-8B$
n	$n-1$	$16n-8+8B.$

On the other hand, the structure equation for $[L_2, R]$ is

$$[L_2, R] = 8(L_1L_2 + L_2L_1) + 8L_2^2 - 8(H - \frac{3}{4} + a^2 + b^2 + c^2)(L_2 - \frac{3}{2} + 2b^2) \quad (15)$$

$$+16(\frac{3}{2} - b^2 - c^2)L_1 + 16(\frac{3}{2} - b^2)L_2 + 8(-b^2 + c^2).$$

Comparing (14) and (15) and equating coefficients of $f_{n\pm2},\ f_{n\pm1}$, respectively, on both sides of the resulting identities, we do not obtain new conditions. However, equating coefficients of f_n results in the condition

$$-(2n+B+2)C(n,n+1)C(n+1,n) + (2n+B-2)C(n-1,n)C(n,n-1) = \quad (16)$$

$$C(n,n)^2 + (-2w - H - \frac{3}{4} + a^2 - b^2 - c^2)C(n,n) + 2(-\frac{3}{2} + b^2 + c^2)w + Q_2$$

where

$$Q_2 = (H - \frac{3}{4} + a^2 + b^2 + c^2)(-\frac{3}{2} + 2b^2) - \frac{9}{2} + 3a^2 + 5b^2 + 4c^2 - 2b^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2.$$

We can regard this as an inhomogeneous recurrence relating F_n and F_{n-1} for the sequence

$$F_n = C(n, n+1)C(n+1, n),\ n = 0, 1, \cdots m.$$

Using (13) we find that the general solution is

$$F_n = C(n, n+1)C(n+1, n) = \quad (17)$$

$$\frac{A_8(2n+B+1)^8 + A_6(2n+B+1)^6 + A_4(2n+B+1)^4 + A_2(2n+B+1)^2 + A_0}{(2n+B+2)(2n+B+1)^2(2n+B)}$$

where A_2 is arbitrary and

$$4A_0 = \left[\frac{1}{2}(H - \frac{3}{4} + a^2 + b^2 + c^2)(b^2 - a^2) + \frac{a^4}{2} - \frac{a^2}{4} - \frac{b^4}{2} + \frac{b^2}{4}\right],$$

$$A_8 = \frac{1}{16},\ A_6 = \frac{1}{8}H - \frac{1}{32} - \frac{1}{8}(a^2 + b^2 + c^2),\ A_4 = \frac{1}{256} + \frac{H^2}{16} -$$

$$\frac{H}{32}(1 + 8a^2 + 8b^2 - 8c^2) + \frac{1}{16}(a^2 + b^2 - \frac{1}{2}c^2 + a^4 + b^4 + c^4) + \frac{1}{4}(-\frac{1}{2}a^2b^2 + a^2c^2 + b^2c^2).$$

We determine A_2 by the requirement $F_{-1} \equiv 0$, so that

$$A_8(B-1)^8 + A_6(B-1)^6 + A_4(B-1)^4 + A_2(B-1)^2 + A_0 = 0.$$

At this point we can already see that there are parameter-dependent raising and lowering relations. Indeed

$$\left(R \mp 4(2n+B \mp 1)L_2 \pm 4C(n,n)\frac{2n+B\mp 1}{-1/2+a^2+b^2-(2n+B)^2}L_1\right) f_n \tag{18}$$
$$= \mp 8(2n+B)\ C(n \pm 1,n)\ f_{n \pm 1}.$$

For the quantum superintegrable system these operators provide differential recurrences for products of Jacobi polynomials, the two variable orthogonal polynomials of Karlin and McGregor.

3 The Casimir operator

Before proceeding to solve the structure equations we clarify the significance of equation (4). Consider the algebra $\mathcal{S}2'$ generated by linear operators $L_1,\ L_2, H$, such that $R=[L_1,L_2]$, $[H,L_i]=0$ and conditions (3) hold. This will not be the quadratic algebra $\mathcal{S}2$ unless condition (4) also holds. Let S be the operator in $\mathcal{S}2'$ defined by the right-hand side of (4), and define the operator C for $\mathcal{S}2'$ by $C=S-R^2$, i.e.,

$$C = \frac{8}{3}\{L_1,L_2,L_3\} - (16a_1+12)L_1^2 - (16a_2+12)L_2^2 - (16a_3+12)L_3^2 \tag{19}$$
$$+\frac{52}{3}(\{L_1,L_2\}+\{L_2,L_3\}+\{L_3,L_1\}) + \frac{1}{3}(16+176a_1)L_1$$
$$+\frac{1}{3}(16+176a_2)L_2 + \frac{1}{3}(16+176a_3)L_3 + \frac{32}{3}(a_1+a_2+a_3)$$
$$+48(a_1a_2+a_2a_3+a_3a_1) + 64a_1a_2a_3 - R^2.$$

We know that the equations defining the quadratic algebra $\mathcal{S}2$ are consistent. In particular there is a realization of $\mathcal{S}2$ by functionally independent differential operators L_1, L_2, H. If we compute the commutators $[L_i, C]$

in $\mathcal{S}2'$ we can write them in the form

$$[L_i, C] = RK_i(L_1,L_2,H), \quad i=1,2,$$

where K_i is a symmetric polynomial in its arguments. Similarly we can compute the commutator $[R,C]$ in $\mathcal{S}2'$ to obtain $[R,C]=K_3(L_1,L_2,H)$ where K_3 is a symmetric polynomial in its arguments, and $[H,C]=0$. Since the K_j must vanish for the superintegrable system $\mathcal{S}2$ it follows that $K_j\equiv 0$ in $\mathcal{S}2'$, i.e., C is a Casimir operator: $[L_i,C]=[L_i,H]=0$. Now suppose we have constructed a finite dimensional irreducible representation of $\mathcal{S}2'$ by operators L_1, L_2, H. Then H and C must be multiples of the identity operator I. In particular, $C = \mu I$. Then we will have a representation of the quadratic algebra $\mathcal{S}2$ if and only if $\mu = 0$. Similar remarks hold for the infinite dimensional representations that we construct.

4 Solution of the structure equations

We see from the last section that we will obtain a model of the superintegrable system $\mathcal{S}2$ from equations (13) and (17) if and only if the eigenvalue μ of the Casimir operator C vanishes. To determine μ for finite dimensional irreducible representations it is enough to compute $Cf_0 = \mu f_0$, i.e., to evaluate C on the lowest weight vector f_0. A straightforward computation, using the fact that $C(-1,0)C(0,-1) = 0$ leads to the result

$$\mu = -\frac{(B-1+a+b)(B-1+a-b)(B-1-a+b)(B-1-a-b)}{(B-1)^2} \times$$

$$(2H+3+4c-2a^2-2b^2-4B-4cB+2B^2)(2H+3-4c-2a^2-2b^2-4B+4cB+2B^2).$$

Thus, in order to achieve a model of the superintegrable system, we must have B equal to one of the four roots: $B = 1 \pm a \pm b$. To be definite, we make the standard choice $B = 1 + a + b$.

Now we return to the solution of the structure equations. The final requirement that uniquely determines the sequence F_n is the highest weight vector condition $F_m \equiv 0$, i.e.,

$$A_8(2m+B+1)^8+A_6(2m+B+1)^6+A_4(2m+B+1)^4+A_2(2m+B+1)^2+A_0 = 0$$

where m is a fixed nonnegative integer. This last equation is quadratic in the energy eigenvalue H. If we solve this equation for H with general B we get two complicated solutions for the quadratic algebra $\mathcal{S}2'$ that involve square roots and for which in general $\mu \neq 0$. However, for the case that mainly concerns us, namely the superintegrable case $B = a + b + 1$, the solution simplifies considerably. The quantization condition is just

$$H = -\frac{1}{4}(4m + 2a + 2b + 2c + 5)(4m + 2a + 2b + 2c + 3). \qquad (20)$$

There is a second solution with c replaced by $-c$. Taking the first solution as standard, we obtain the following values for the expansion coefficients:

$$C(n,n) = \frac{1}{2}(2n+a+b+2)(2n+a+b) - \frac{1}{2}\left[(2m+a+b+c+2)^2 + a^2 - b^2 - c^2 - 1\right] \qquad (21)$$

$$+\frac{1}{2}\frac{(a^2-b^2)(a+b+2m+2)(a+b+2c+2m+2)}{(2n+a+b+2)(2n+a+b)},$$

$$C(n,n+1)C(n+1,n) = \qquad (22)$$

$$16(n+1)(n-m)(n-c-m)(n+b+1)(n+a+1)(n+a+b+1)\times$$
$$\frac{(n+m+a+b+2)(n+m+a+b+c+2)}{(2n+a+b+3)(2n+a+b+2)^2(2n+a+b+1)}.$$

Note that only the product $C(n,n+1)C(n+1,n)$ is determined uniquely. The values of the individual factors depend on the normalization of the basis vectors f_n. This result is in basic agreement with the expansion formula for products of Lamé or Heun polynomials in terms of products of Jacobi polynomials [44]. In that paper the coefficients were derived using recurrence formulas for Jacobi polynomials (and there were some typographical errors in the formulas). Here the derivation is directly from the structure formulas for the quadratic algebra. The condition that there is an inner product with respect to which the f_n form an orthogonal basis is $C(n,n+1)C(n+1,n) > 0$ for $n = 0, 1, \cdots, m-1$, and this is satisfied, for example, if $a, b, c > 0$. Although these representations are finite dimensional for m a positive integer in the expression (20) for H (the bound state energy levels) we can view (20) as a parameterization for H corresponding to arbitrary values of m. In these cases our representation is infinite dimensional but bounded below.

Now suppose we have an irreducible representation of $\mathcal{S}2'$ of the above form that is unbounded both above and below. Then conditions (17) and the expressions for A_8, A_6, A_4, A_0 will still hold and n will run over all of the integers: $n = 0, \pm1, \pm2, \cdots$. We write the parameter A_2 in the form

$$A_2 = \zeta/64 - \left(A_8(B-1)^8 + A_6(B-1)^6 + A_4(B-1)^4 + A_0\right)/(B-1)^2, \quad (23)$$

so that $\zeta \neq 0$, since otherwise we would have a representation bounded below. Now we must have $Cf_n = \mu f_n$ for all n, and comparing the coefficient of f_n on both sides of the identity we see that the action of C on the constant A_2 term is to multiply it by 64, so

$$\mu = \zeta - \frac{(B-1+a+b)(B-1+a-b)(B-1-a+b)(B-1-a-b)}{(B-1)^2}\times$$

$$(2H+3+4c-2a^2-2b^2-4B-4cB+2B^2)(2H+3-4c-2a^2-2b^2-4B+4cB+2B^2).$$

Thus, in order to achieve a model of the superintegrable system $\mathcal{S}2$, we must choose ζ so that $\mu = 0$.

Due to the symmetry of the structure equations, it follows that the corresponding eigenvalues of the operator L_2 for finite dimensional representations of $\mathcal{S}2$ must be

$$\xi_n = -(2n+b+c+1)^2 - \frac{1}{2} + b^2 + c^2, \ n = 0, 1, \cdots, m,$$

whereas the eigenvalues of the operator L_3 must be

$$\eta_n = -(2n+a+c+1)^2 - \frac{1}{2} + a^2 + c^2, \ n = 0, 1, \cdots, m.$$

5 Wilson polynomials and a one variable model for the quadratic algebra

The bounded below representations of the generic superintegrable system on the 2-sphere are intimately connected with the Wilson polynomials. The connection between these polynomials and the representation theory is the three term recurrence formula for the action of L_2 on an L_1 basis:

$$L_2 f_n = C(n+1,n) f_{n+1} + C(n,n) f_n + C(n-1,n) f_{n-1}$$

where the coefficients are given by (21) and (22). To understand the relationship we recall some facts about the Wilson polynomials [45]. They are given by the expressions

$$p_n(t^2) \equiv p_n(t^2, \alpha, \beta, \gamma, \delta) = (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n \times \tag{24}$$

$${}_4F_3\left(\begin{matrix} -n, & \alpha+\beta+\gamma+\delta-n-1, & \alpha-t, & \alpha+t \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta & \end{matrix} ; 1 \right)$$

where $(a)_n$ is the Pochhammer symbol and ${}_4F_3(1)$ is a generalized hypergeometric function of unit argument. The polynomial $p_n(t^2)$ is symmetric in $\alpha, \beta, \gamma, \delta$. For fixed $\alpha, \beta, \gamma, \delta > 0$ the Wilson polynomials are orthogonal with respect to the inner product

$$< p_n, p_{n'} > = \frac{1}{2\pi} \int_0^\infty p_n(-t^2) p_{n'}(-t^2) \left| \frac{\Gamma(\alpha+it)\Gamma(\beta+it)\Gamma(\gamma+it)\Gamma(\delta+it)}{\Gamma(2it)} \right|^2 dt \tag{25}$$

$$= \delta_{nn'} n! (\alpha+\beta+\gamma+\delta+n-1)_n \times$$

$$\frac{\Gamma(\alpha+\beta+n)\Gamma(\alpha+\gamma+n)\Gamma(\alpha+\delta+n)\Gamma(\beta+\gamma+n)\Gamma(\beta+\delta+n)\Gamma(\gamma+\delta+n)}{\Gamma(\alpha+\beta+\gamma+\delta+2n)}.$$

The Wilson polynomials satisfy the three term recurrence formula

$$t^2 p_n(t^2) = K(n+1,n)p_{n+1}(t^2) + K(n,n)p_n(t^2) + K(n-1,n)p_{n-1}(t^2) \quad (26)$$

where

$$K(n+1,n) = \frac{\alpha+\beta+\gamma+\delta+n-1}{(\alpha+\beta+\gamma+\delta+2n-1)(\alpha+\beta+\gamma+\delta+2n)},$$

$$K(n-1,n) = \frac{n(\alpha+\beta+n-1)(\alpha+\gamma+n-1)(\alpha+\delta+n-1)}{(\alpha+\beta+\gamma+\delta+2n-2)(\alpha+\beta+\gamma+\delta+2n-1)}\times$$

$$(\beta+\gamma+n-1)(\beta+\delta+n-1)(\gamma+\delta+n-1),$$

$$K(n,n) = \alpha^2 - K(n+1,n)(\alpha+\beta+n)(\alpha+\gamma+n)(\alpha+\delta+n)$$

$$-\frac{K(n-1,n)}{(\alpha+\beta+n-1)(\alpha+\gamma+n-1)(\alpha+\delta+n-1)}.$$

This formula, together with $p_{-1}=0,\ p_0=1$, determines the polynomials uniquely.

We define the operator L_4 on the representation space of the superintegrable system by the action

$$L_4 f_n = K(n+1,n)f_{n+1} + K(n,n)f_n + K(n-1,n)f_{n-1}. \quad (27)$$

Note that with the choices

$$\alpha = -\frac{a+c+1}{2} - m,\ \beta = \frac{a+c+1}{2},\ \gamma = \frac{a-c+1}{2},\ \delta = \frac{a+c-1}{2} + b + m + 2,$$

we have a perfect match with

$$C(n+1,n) = 4K(n+1,n), \quad C(n-1,n) = 4K(n-1,n).$$

The diagonal elements are related by

$$C(n,n) = 4K(n,n) - \lambda_n + H + \frac{5}{4} - 2a^2 - b^2 - 2c^2$$

where H is given by (20). Thus $L_2 = 4L_4 - L_1 + 5/4 - 2a^2 - b^2 - 2c^2$, or more simply

$$L_3 = -4L_4 - \frac{1}{2} + a^2 + c^2.$$

Now we can construct a one variable model for the realization of these representations. The L_1 basis functions are the Wilson polynomials $f_n = p_n(t)$

and $L_4 = t^2$ is multiplication by the transform variable. We can use the divided difference operator eigenvalue equation for the Wilson polynomials

$$\tau^* \tau p_n = n(n + \alpha + \beta + \gamma + \delta - 1)p_n$$

where

$$E^A F(t) = F(t + A),\ \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}),$$

$$\tau^* = \frac{1}{2t}\left[(\alpha + t)(\beta + t)(\gamma + t)(\delta + t)E^{1/2} - (\alpha - t)(\beta - t)(\gamma - t)(\delta - t)E^{-1/2}\right]$$

to express the action of L_1: $L_1 = -4\tau^*\tau - 2(a+1)(b+1) + 1/2$. See [46] for a simple derivation. The inner product is (25).

When m is a nonnegative integer then $\alpha + \beta = -m < 0$ so that the above continuous Wilson orthogonality does not apply. The representation becomes finite dimensional and the orthogonality is a finite sum

$$\frac{(\alpha - \gamma + 1)_m(\alpha - \delta + 1)_n}{(2\alpha + 1)_m(1 - \gamma - \delta)_m} \sum_{k=0}^{m} \frac{(2\alpha)_k(\alpha + 1)_k(\alpha + \beta)_k(\alpha + \gamma)_k(\alpha + \delta)_k}{(1)_k(\alpha)_k(\alpha - \beta + 1)_k(\alpha - \gamma + 1)_k(\alpha - \delta + 1)_k}$$

$$\times p_n((\alpha + k)^2)p_{n'}((\alpha + k)^2) = \delta_{nn'} \times \tag{28}$$

$$\frac{n!(n + \alpha + \beta + \gamma + \delta - 1)_n(\alpha + \beta)_n(\alpha + \gamma)_n(\alpha + \delta)_n(\beta + \gamma)_n(\beta + \delta)_n(\gamma + \delta)_n}{(\alpha + \beta + \gamma + \delta)_{2n}}.$$

Thus, the spectrum of $L_4 = t^2$ is the set $\{(\alpha + k)^2 :\ k = 0, \cdots, m\}$. In the original quantum mechanics eigenvalue problem the eigenfunctions of L_1 and L_4 each separate in suitable versions of spherical coordinates to give Karlin-McGregor polynomials. It follows from this derivation that the expansion coefficients relating one eigenbasis to the other are just the general Racah polynomials.

These relations are derived in equations (3.4) and (4.2) of Wilson's paper [45]. These finite discrete polynomials, suitably renormalized, are called the Racah polynomials. Thus the Racah polynomials are those associated with the bound state energy levels of the $\mathcal{S}2$ Schrödinger eigenvalue equation, whereas the continuous Wilson polynomials are those associated with the continuous (but infinitely degenerate) spectrum of the Schrödinger operator.

6 Heun type operators

We can now obtain information about the spectrum of the Heun-type operator $Q = L_3 + kL_1$, where $k \neq 0, 1$, by using standard linear algebra arguments for the finite dimensional representations. This operator yields separable solutions of the Schrödinger eigenvalue problem in terms of ellipsoidal coordinates. The solutions are expressed as products of Heun polynomials. Computation of the expansion coefficients $C(m,n)$ is the essential step in the expansion of these Heun solutions in a Karlin-McGregor basis.

If η is an eigenvalue of Q then, by considering the action of Q on an $\{f_n\}$ basis, and using the relation $L_3 = -4L_4 - \frac{1}{2} + a^2 + c^2$, we see that η must be a root of the eigenvalue equation $\det(Q - \eta I) = 0$, where I is the $(m+1) \times (m+1)$ identity matrix and Q is the $(m+1) \times (m+1)$ matrix with elements

$$Q(j,j) = -4K(j,j) - \frac{1}{2} + a^2 + c^2 + k\lambda_j, \ j = 0, \cdots, m,$$

$$Q(h,h+1) = -4K(h,h+1), \ Q(h+1,h) = -4K(h+1,h), \ h = 0, \cdots, m-1,$$

on the diagonal, superdiagonal and subdiagonal, respectively, and all other elements zero. Alternatively, we could use the one variable model and relations (28) to express the action of Q on an L_3-basis and then write down the determanental condition.

7 Conclusions and outlook

We have demonstrated explicitly the isomorphism between the quadratic algebra of the generic quantum superintegrable system on the 2-sphere and the quadratic algebra generated by the Wilson polynomials, and have worked out the basic theory for infinite as well as finite dimensional representations of the algebra. It follows from our analysis that the quadratic algebras for all 13 equivalence classes of 2D second order quantum superintegrable systems should be obtainable by appropriate limit processes from the quadratic algebra associated with the generic superintegrable system on the 2-sphere, namely that generated by the Wilson polynomials. However these limit processes are very intricate and each equivalence class exhibits unique structure, so each class is important for study by itself. Moreover, within each class of Stäckel equivalent systems the structure of the quadratic algebra remains unchanged but the spectral analysis of the generators for the algebra can change. Since the the algebra QR(3) is itself a limit as $q \to 1$ of the algebra associated with the Askey-Wilson polynomials, this suggests the existence of a q-version of second order quantum superintegrability [28].

Another important issue concerns the quadratic algebras associated with 3D second order nondegenerate quantum superintegrable systems. In 2D there are 3 functionally independent generators for the algebra of symmetries and the algebra is isomorphic to QR(3) or one of its limiting cases. In 3D there are 5 functionally independent, but 6 linearly independent, generators. The algebra again closes at 6th order in the momenta, but in addition there is an identity at 8th order that relates the 6 functionally dependent generators. The representation theory of such quadratic algebras is much more complicated and remains to be studied, as does the detailed relationship with multivariable orthogonal polynomials. Similarly for nD nondegenerate systems there are $2n-1$ functionally independent but $n(n+1)/2$ linearly independent generators for the quadratic algebra.

References

[1] Friš J., Mandrosov V., Smorodinsky Ya.A, Uhlír M. and Winternitz P., On Higher Symmetries in Quantum Mechanics. *Phys. Lett.* 1965, V.16, 354–356.

[2] Makarov A.A., Smorodinsky Ya.A., Valiev Kh. and Winternitz P., A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries. *Nuovo Cimento*, 1967, V. 52, 1061–1084.

[3] Friš J., Smorodinskii Ya.A., Uhlír M. and Winternitz P., Symmetry Groups in Classical and Quantum Mechanics; *Sov. J. Nucl. Phys.*, 1967, V.4, 444–450.

[4] Cisneros A. and McIntosh H.V., Symmetry of the Two-Dimensional Hydrogen Atom. *J. Math. Phys.* 1969, V.10, 277–286.

[5] Calogero F., Solution of a Three-Body Problem in One Dimension. *J. Math. Phys.* 1969, V.10, 2191–2196.

[6] Wojciechowski S., Superintegrability of the Calogero-Moser System. *Phys. Lett.*, 1983, V. A 95, 279–281.

[7] Mardoyan L. G., Pogosyan G. S., Sissakian A. N. and Ter-Antonyan V. M.. Elliptic Basis for a Circular Oscillator. *Nuovo Cimento*, 1985 **B 88**, 43. Two-Dimensional Hydrogen Atom: I. Elliptic Bases; *Theor. Math. Phys.* **61**, 1021 (1984); Hidden symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom. *J. Phys.*, **A 18**, 455 (1985).

[8] Evans N.W., Super-Integrability of the Winternitz System; *Phys. Lett.* 1990, V.A 147, 483–486.

[9] Evans N.W., Superintegrability in Classical Mechanics; *Phys. Rev.* 1990, V. A 41, 5666–5676; Group Theory of the Smorodinsky-Winternitz System; *J. Math. Phys.* 1991, V. 32, 3369.

[10] Granovsky Ya. A., Zhedanov A. S. and Lutzenko I. M. Quadratic Algebra as a 'Hidden' Symmetry of the Hartmann Potential; *J. Phys.* 1991, **A 24**, 3887.

[11] Bonatos D., Daskaloyannis C. and Kokkotas K., Deformed Oscillator Algebras for Two-Dimensional Quantum Superintegrable Systems; *Phys. Rev.*, 1994, V.A 50, 3700–3709.

[12] Daskaloyannis C., Quadratic Poisson algebras of two-dimensional classical superintegrable systems and quadratic associate algebras of quantum superintegrable systems. *J. Math. Phys.*, 2001, V.42, 1100–1119.

[13] Superintegrability in Classical and Quantum Systems, P. Tempesta, P. Winternitz, W. Miller, G. Pogosyan editors, AMS, vol. 37, 2005.

[14] Kalnins E.G., Kress J.M, and Miller W.Jr., Second order superintegrable systems in conformally flat spaces. I: 2D classical structure theory. *J. Math. Phys.*, 2005, V.46, 053509.

[15] Kalnins E. G., Miller W.Jr. and Pogosyan G. S. Completeness of multiseparable superintegrability in $E_{2,C}$. *J. Phys. A: Math Gen.*, 2000, **33**, 4105.

[16] Kalnins E. G., Miller W. Jr. and Pogosyan G. S. Completeness of multiseparable superintegrability on the complex 2-sphere. *J. Phys. A: Math Gen.*, 2000, **33**, 6791-6806.

[17] Kalnins E.G., Kress J.M., Miller W.Jr. and Pogosyan G.S., *Completeness of superintegrability in two-dimensional constant curvature spaces.* *J. Phys. A: Math Gen.*, 2001, **34**, 4705–4720.

[18] Kalnins E.G., Kress J.M, and Miller W.Jr., Second order superintegrable systems in conformally flat spaces. II: The classical 2D Stäckel transform. *J. Math. Phys.*, 2005, V.46, 053510.

[19] Daskaloyannis C. and Ypsilantis K. Unified treatment and classification of superintegrable systems with integrals quadratic in momenta on a two dimensional manifold. *J. Math. Phys.*, 2006, **47**, 042904.

[20] Kalnins E. G., Kress J. M, and Miller W. Jr., Nondegenerate 2D complex Euclidean superintegrable systems and algebraic varieties, *J. Phys. A: Math. Theor.*, 2007, **40**, 3399-3411.

[21] Kalnins E.G., Kress J.M, and Miller W.Jr., Second order superintegrable systems in conformally flat spaces. IV: The classical 3D Stäckel transform and 3D classification theory. *J. Math. Phys.*, 2006, V.47, 043514.

[22] Kalnins E. G., Kress J. M, and Miller W. Jr., Nondegenerate 3D complex Euclidean superintegrable systems and algebraic varieties, 2007 (submitted).

[23] Boyer C.P., Kalnins E.G., and Miller W., *Stäckel - equivalent integrable Hamiltonian systems. SIAM J. Math. Anal.*, 1986, V.17, 778–797.

[24] Hietarinta J., Grammaticos B., Dorizzi B. and Ramani A., *Coupling-constant metamorphosis and duality between integrable Hamiltonian systems. Phys. Rev. Lett.*, 1984, V.53, 1707–1710.

[25] Kalnins E. G., Miller W. Jr. and Reid G. K., Separation of variables for Riemannian spaces of constant curvature. I. Orthogonal separable coordinates for S_c and E_{nC}. *Proc R. Soc. Lond. A*, 1984, **39**, 183–206.

[26] Kalnins E. G. , Kress J. M., Miller W. Jr. and Pogosyan G. S. Non degenerate superintegrable systems in n dimensional complex Euclidean spaces. (To appear in Physics of Atomic Nuclei)

[27] M. Bôcher. Über die Riehenentwickelungen der Potentialtheory. *Teubner*, Leipzig, 1894.

[28] Zhedanov A.S., “Hidden symmetry” of Askey-Wilson polynomials, *Theoret. and Math. Phys.*, 1991, **89** , 1146–1157.

[29] Granovskii Ya. I., Zhedanov A. S., and Lutsenko I. M., Quadratic algebras and dynamics in curved spaces. I. Oscillator, *Theoret. and Math. Phys.*, 1992, **91**, 474-480.

[30] Granovskii Ya. I., Zhedanov A. S., and Lutsenko I.M., Quadratic algebras and dynamics in curved spaces. II. The Kepler problem, *Theoret. and Math. Phys.*, 1992, **91**, 604-612.

[31] Daskaloyannis C., Generalized deformed oscillator and nonlinear algebras, *J. Phys. A: Math. Gen.* 1991, **24** , L789-L794.

[32] Daskaloyannis C., Quadratic Poisson algebras of two-dimensional classical superintegrable systems and quadratic associative algebras of quantum superintegrable systems, *J. Math. Phys*, 2001, **42**, 1100-1119, math-ph/0003017.

[33] Quesne C., Generalized deformed parafermions, nonlinear deformations of so(3) and exactly solvable potentials, *Phys. Lett. A*, 1994, **193**, 245-250.

[34] Quesne C., Quadratic Algebra Approach to an Exactly Solvable Position-Dependent Mass Schrödinger Equation in Two Dimensions. *SIGMA* , 2007, **3** , 067, 14 pages.

[35] Patera J. and Winternitz P., A new basis for the representations of the rotation group: Lamé and Heun polynomials. *J. Math. Phys.*, 1973, V. 14, 1130–1139.

[36] Kalnins E. G. and Miller W. Jr., Lie theory and separation of variables. IV. The groups SO(2,1) and SO(3) , *J. Math. Phys.* 1974, **15** , 1263-74.

[37] Kalnins E. G. and Miller W. Jr., Lie theory and separation of variables. XI. The Euler-Poisson-Darboux equation, *J. Math. Phys.*, 1976, **17**, 369-377.

[38] Viswanathan B., Generating functions for ultraspherical functions, *Can. J. Math.* 1968, **20**, 120–134 .

[39] Kalnins E.G., Miller W.Jr. and Pogosyan G.S., Superintegrability and associated polynomial solutions. Euclidean space and the sphere in two dimensions. *J.Math.Phys.*, 1996, V.37, 6439–6467.

[40] Kalnins E.G., Miller W. and Tratnik M.V., Families of orthogonal and biorthogonal polynomials on the n-sphere. *SIAM J. Math. Anal.*, 1991, V.22, 272–294.

[41] Kalnins E.G., Miller W.Jr., and Pogosyan G.S., Superintegrability on the two dimensional hyperboloid, *J. Math. Phys.*, 1997, **38**, 5416–5433.

[42] Barut A. O., Inomata A. and Wilson R., A new realisation of dynamical groups and factorisation method, *J. Phys. A: Math. Gen.*, 1987, **20**, 4075-4082.

[43] Frank A. and Wolf K. B., Lie algebras for systems with mixed spectra. I. The scattering Pöschl-Teller potential. *J. Math. Phys.*, 1985, **26**, 977–983.

[44] Kalnins E. G. and Miller W. Jr., Hypergeometric expansions of Heun polynomials, *SIAM J. Math. Anal.* 1991, **22**, 1450-1459.

[45] Wilson J., Some hypergeometric orthogonal polynomials. *SIAM J. Math. Anal.*, 1980, **11**, 690–701.

[46] Miller W. Jr., A note on Wilson polynomials, *SIAM J. Math. Anal.*, 1987, **18** 1221–1226.