Lie theory and separation of variables. 4. The groups $SO(2,1)$ and $SO(3)$

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Winternitz and coworkers have shown that the eigenfunction equation for the Laplacian on the hyperboloid $x_0^2-x_1^2-x_2^2=1$ separates in nine orthogonal coordinate systems, associated with nine symmetric quadratic operators L in the enveloping algebra of $SO(2,1)$. Corresponding to each of the operators L, we employ the standard one-variable model for the principal series of representations of $SO(2,1)$ and compute explicitly an L basis for the Hilbert space as well as the unitary transformations relating different bases. We also compute the associated results for realizations of these representations on the hyperboloid. Three of our bases are related to well-known subgroup reductions of $SO(2,1)$. Of the remaining six, one is related to Bessel functions, two to Legendre functions, and three to Lamé functions. We show that there is virtually a perfect correspondence between the known theory of the Lamé functions and the representation theory of $SO(2,1)$ and $SO(3)$.

1. INTRODUCTION

As is well known, the group $SO(2,1)$ acts on the hyperboloid $x_0^2-x_1^2-x_2^2=1$, $x_0>0$, with induced Lie derivatives K_1, K_2, M_3 given by

$$K_1=-x_0\partial_{x_2}-x_2\partial_{x_0},\quad K_2=-x_0\partial_{x_1}-x_1\partial_{x_0},\qquad M_3=x_1\partial_{x_2}-x_2\partial_{x_1} \tag{1.1}$$

and commutation relations (2.3). Consider the eigenvalue equation

$$Qf(x_0,x_1,x_2)=l(l+1)f(x_0,x_1,x_2), \tag{1.2}$$

where $Q=K_1^2+K_2^2-M_3^2$ is the Casimir operator of the Lie algebra $so(2,1)$ expressed in terms of (1.1) and f is a function on the hyperboloid. Olevsky[1] has shown that Eq. (1.2) separates in nine orthogonal coordinate systems and Winternitz and coworkers[2,3] have shown that these coordinate systems correspond to nine quadratic symmetric operators L in the enveloping algebra U of $SO(2,1)$. Indeed, let S be the space of all symmetric second order elements in U, let C be the center of U and form the factor space $T=S/S\cap C$. (In this case $S\cap C=\{\alpha Q\}$, α any constant). Then $SO(2,1)$ acts on T via the adjoint representation and splits it into nine types of orbits. Choosing an operator L from each orbit, we find that for each such L the pair of equations

$$Qf=l(l+1)f,\quad Lf=\lambda f, \tag{1.3}$$

corresponds to one of the nine coordinate systems in which (1.2) separates. In fact, λ corresponds to a separation constant.

We choose our nine operators L as M_3^2, K_2^2, $(K_1+M_3)^2$, $L_{\rm E}, L_{\rm H}, L_{\rm SH}, L_{\rm EP}, L_{\rm HP}, L_{\rm CP}$, where the last six are given by (3.1). For the explicit derivation of these operators and the orthogonal coordinates to which they correspond see Ref. 2.

In the present paper, rather than study (1.2) directly, we employ the standard one-variable model (2.6) for the principal series representations of $SO(2,1)$ and explicitly compute an L basis for the Hilbert space corresponding to each of our nine L operators. We also compute unitary transformations relating different bases. Our results on the spectral resolutions of the L operators, though determined for the simple one-variable model, are obviously valid for any model of the principal series. The spectral resolutions for the "subgroup operators" M_3^2, K_2^2, and $(K_1+M_3)^2$ are well known, e.g., Refs. 4—6 and partial results for $L_{\rm E}$ and $L_{\rm H}$ can be found in Ref. 3. However, the remaining four cases are treated here for the first time. The operators $L_{\rm E}$, $L_{\rm H}$, $L_{\rm SH}$ lead to expansions in Lamé functions, $L_{\rm CP}$ to Bessel functions and the Hankel transform, and $L_{\rm EP}$, $L_{\rm HP}$ to expansions in Legendre functions.

In Sec. 4 of this paper we construct models of the principal series in terms of solutions of (1.2), thus making explicit the relationship between the above results and separation of variables. This is accomplished via the Gel'fand—Graev transform which maps functions on the unit circle to functions on the hyperboloid and is an intertwining operator for the group action. We obtain a number of new results relating solutions of (1.2) in various bases.

Recently Patera and Winternitz[7] have introduced a new basis for the representations of the rotation group $SO(3)$. Their basis consists of the eigenfunctions of the symmetric operator $E=-4(L_1^2+rL_2^2)$, where $0<r<1$ and $[L_i,L_j]=\epsilon_{ijk}L_k$. In the two-variable model of the irreducible representations of $SO(3)$, functions on a sphere, the eigenfunctions are products of Lamé polynomials. However, the only one-variable model computed in Ref. 7 was one in which the basis functions appeared as complicated Heun polynomials. In Sec. 5 we show that, in fact, by a suitable change of variable and phase, one can construct a one-variable model in which the basis functions are exactly the Lamé polynomials. We show that there is a one-to-one relationship between the results of Ref. 7 and the standard theory of Lamé polynomials as presented in Ref. 8 or Ref. 9. This permits the use of tabulated properties of Lamé polynomials to implement the theory of Ref. 7. In general our results show an intimate relationship between the representation theory of $SO(2,1)$ and $SO(3)$ on the one hand and the theory of Lamé functions on the other.

We have not attempted to compute the matrix elements for the principal series representations of $SO(2,1)$ in any of the nonsubgroup bases. The practical computation of such results awaits the introduction of appropriate coordinates on the group manifold such that variables separate in the differential equations for the matrix elements. Work is in progress on this problem.

This paper is one of a series analyzing the relationship between Lie theory and separation of variables in the partial differential equations of mathematical physics.[10–12]

2. SUBGROUP BASES

In this section we establish notation and review those properties of $SO(2,1)$ that we will need in the sequel.

The group $SO(2,1)$ consists of those proper linear transformations acting on a three-dimensional vector $\mathbf{x}=(x_0,x_1,x_2)$ which preserve the infinitesimal distance

$$ds^2=dx_0^2-dx_1^2-dx_2^2. \tag{2.1}$$

(These are the Lorentz transformations in the plane.) The group $SO(2,1)$ is 2—1 homomorphic to the group $SU(1,1)$ of quasiunitary unimodular matrices

$$g=\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix},\quad |\alpha|^2-|\beta|^2=1. \tag{2.2}$$

The generators of the Lie algebra of $SO(2,1)$ are denoted by K_1, K_2, and M_3. Here K_1, K_2 are the generators of the pure Lorentz transformations along the 1 and 2 axes, respectively, and M_3 is the generator of rotations in the 1, 2 plane. The defining commutation relations of this algebra are

$$[K_1,K_2]=-M_3,\quad [K_2,M_3]=K_1,\quad [M_3,K_1]=K_2. \tag{2.3}$$

All unitary faithful irreducible representations are labeled by the eigenvalue of the Casimir operator Q, where

$$Q=K_1^2+K_2^2-M_3^2=l(l+1). \tag{2.4}$$

All such irreducible representations are infinite dimensional. We now give the spectrum of l corresponding to the unitary irreducible representations and the eigenvalues m of the operator iM_3 in each such representation.

(i) Principal series: $l=-\frac{1}{2}+i\rho$, $0<\rho<\infty$, $m=0,\ \pm1,\ \pm2,\ \cdots$ or $\pm\frac{1}{2},\ \pm\frac{3}{2},\ \cdots$.

(ii) Complementary series: $\operatorname{Im} l=0$, $-1<l<0$, $m=0,\ \pm1,\ \pm2,\ \cdots$.

(iii) Positive discrete series: $2l=$ integer, $m=l+1,\ l+2,\ \cdots$.

(iv) Negative discrete series: $2l=$ integer, $m=-l-1,\ -l-2,\ \cdots$.

For the purposes of this paper we only consider the single valued representations of the principal series. For a more detailed treatment of $SO(2,1)$ we refer to the standard references, 4, 13. The principal series of $SU(1,1)$ can be realized on the Hilbert space $\mathcal{H}$ of square integrable functions f on the unit circle with the scalar product

$$\langle f,h\rangle=\int_0^{2\pi}\overline{f(e^{i\theta})}\ h(e^{i\theta})d\theta. \tag{2.5}$$

The action of a group element g on a function f is specified by

$$T(g)f(e^{i\theta})=|\beta e^{i\theta}+\bar\alpha|^{2l}f\left(\frac{\alpha e^{i\theta}+\bar\beta}{\beta e^{i\theta}+\bar\alpha}\right), \tag{2.6}$$

and the generators of the Lie algebra have the form

$$K_1=l\cos\theta-\sin\theta\frac{d}{d\theta},\qquad K_2=-l\sin\theta-\cos\theta\frac{d}{d\theta},\qquad M_3=\frac{d}{d\theta}. \tag{2.7}$$

Of the nine possible bases for $SO(2,1)$ as given by Winternitz *et al.*[2], three are of the subgroup type and have been treated in some detail in the literature.[4–6] We now give the explicit form of each of these subgroup bases for the principal series. In the section on the two variable model we also give the expansions in the subgroup bases. These results are not new,[6] but we present them here in summarized form in the interest of completeness.

1. *Spherical system*: The explicit form of the principal series in this basis has already been presented in our definition of the principal series. The basis functions of the spherical system are just the eigenfunctions $\exp(im\theta)/\sqrt{2\pi}$ of the operator M_3. This is the canonical or standard basis to which we will relate all subsequent bases.

2. *Equidistant system*: The basis defining operator for this system is K_2.

The representation space of the principal series splits into two spaces. The basis vectors in each space are

$$f^l_{\tau\epsilon}=(\cosh q)^l\exp(i\tau q)\,C_\epsilon,\quad -\infty<\tau<\infty, \tag{2.8}$$

where $\epsilon=+1$ is a reflection label which distinguishes the two spaces and $C_{+1}=\binom{1}{0}$, $C_{-1}=\binom{0}{1}$. The variable q is related to θ by

$$e^q=\tan\tfrac{1}{2}\theta,\qquad 0\le\theta\le\pi,$$
$$e^{-q}=\tan\tfrac{1}{2}(\theta-\pi),\qquad \pi\le\theta\le2\pi. \tag{2.9}$$

On each of the spaces K_2 is essentially the momentum operator with a unitary continuous spectrum, the real line. For further details concerning this basis see Refs. 5, 6.

3. *Horocyclic system*: The basis defining operator for this system is K_1+M_3. The representation space of the principal series is then spanned by a single set of basis vectors given by

$$f^l_s=[\tfrac{1}{2}(1+z^2)]^l\exp(isz),\quad -\infty<s<\infty, \tag{2.10}$$

where the variable z is related to θ by

$$z=\tan\tfrac{1}{2}\theta. \tag{2.11}$$

This basis has been considered to a limited extent in Ref. 13. The choice of basis operator is more convenient but still equivalent to that used in Ref. 13. (Similar remarks apply to the equidistant system.)

3. NONSUBGROUP BASES

Now we enumerate the six types of orbits in T which do not correspond to subgroup bases. Choosing a standard element on each of the orbits, we obtain the following list of six operators.

(1) Elliptic system: $L_E = M_3^2 + k^2 K_2^2, \quad k \in R,$

(2) Hyperbolic system: $L_H = K_2^2 - r^2 M_3^2, \ 0 < r < 1,$

(3) Semihyperbolic system: $L_{SH} = M_3 K_1 + K_1 M_3 + r K_2^2,$

$$0 < r < \infty, \tag{3.1}$$

(4) Elliptic-parabolic system: $L_{EP} = \gamma K_2^2 + K_1^2 + M_3^2 + K_1 M_3 + M_3 K_1, \quad \gamma > 0,$

(5) Hyperbolic-parabolic system: $L_{HP} = -\gamma K_2^2 + K_1^2 + M_3^2 + K_1 M_3 + M_3 K_1, \quad \gamma > 0,$

(6) Semicircular-parabolic system: $L_{CP} = K_1 K_2 + K_2 K_1 + K_2 M_3 + M_3 K_2.$

We will show that each of these operators corresponds naturally to a symmetric operator on the Hilbert space $\mathcal{H} = L_2[0, 2\pi]$ corresponding to the principal series representations of $SO(2,1)$. Furthermore, we will show that each such symmetric operator has equal deficiency indices and can be extended to one or more self-adjoint operators on $\mathcal{H}$. Finally we will compute the spectral resolutions of these self-adjoint extensions and relate them to the spectral resolution of $L_S = M_3^2$.

Recall that for the principal series the Lie algebra generators are given by (1.7) and $l = -\frac{1}{2} + i\rho, \ \rho > 0$.

A. Elliptic parabolic system

For our first example we consider the operator L_{EP} normalized so that $\gamma = 1$:

$$L_{EP} = 2(1 - \sin\theta)\frac{d^2}{d\theta^2} + (2l-1)\cos\theta \frac{d}{d\theta} + [l(l+1) - l\sin\theta]. \tag{3.2}$$

This operator can be defined on the domain of all C^∞ functions on the circle which vanish near $\theta = \pi/2$. It is straightforward to show that L_{EP} is essentially self-adjoint on this domain and that the self-adjoint extension, which we also call L_{EP}, has continuous spectrum only, covering the negative real axis. The normalized generalized eigenfunctions are

$$F_\xi^{EP}(\theta) = \alpha_\xi (\sin\tfrac{1}{2}\phi)^{-1/2+i\rho} P^{i\xi}_{-1/2+i\rho}(\cos\tfrac{1}{2}\phi),$$
$$\alpha_\xi = \left(\frac{4\pi\xi \sinh \pi\xi}{\cosh\pi\xi + \cosh\pi\rho}\right)^{1/2}, \quad \theta = \tfrac{1}{2}\pi + \phi, \ 0 \le \phi < 2\pi, \tag{3.3}$$

and the orthogonality relations are

$$\int_0^{2\pi} \overline{F_{\xi'}^{EP}(\theta)} F_\xi^{EP}(\theta)\, d\theta = \delta(\xi' - \xi). \tag{3.4}$$

Here, $L_{EP} F_\xi^{EP}(\theta) = -\xi^2 F_\xi^{EP}(\theta)$, $0 < \xi < \infty$, and $P_l^\mu(z)$ is a Legendre function.[8] A tedious computation for the overlap functions between the S and EP bases yields

$$\begin{aligned}
U^{S,EP}_{n,\xi} &= \int_0^{2\pi} \overline{F_n^S(\theta)} F_\xi^{EP}(\theta)\, d\theta \\
&= \frac{\alpha_\xi}{\sqrt{2\pi}}(-i)^n \int_0^{2\pi} \exp(-in\phi)(\sin\tfrac{1}{2}\phi)^l P_l^{i\xi}(\cos\tfrac{1}{2}\phi)\, d\phi \\
&= \alpha_\xi (\tfrac{1}{2}\pi)^{1/2}(-i)^n\, 2^{i\xi}\, \frac{(-1)^n 2^{2(n+1)}\Gamma(n+\frac{1}{2})\Gamma((l+i\xi+1)/2)\Gamma((-2n+l-i\xi+1)/2)}{\Gamma(-n+l+1)\Gamma((l-i\xi+2)/2)\Gamma((-2n-i\xi-l+1)/2)} \\
&\quad \times {}_4F_3\left(\left.\begin{matrix} n-l,\ n+\frac{1}{2},\ n,\ n+\frac{1}{2} \\ (1+2n+i\xi-l)/2,\ (1+2n-i\xi-l)/2,\ 1+2n \end{matrix}\right| 1\right) \pm \frac{i\pi\, 2^{2n}\Gamma((i\xi-l+1)/2)\Gamma((l-2n+i\xi+1)/2)}{\Gamma((2l-2n+3)/2)\Gamma((-l-i\xi)/2)\Gamma((1+2n-l+i\xi)/2)} \\
&\quad \times \frac{1}{(|n|)!}\, {}_4F_3\left(\left.\begin{matrix} n-l-\frac{1}{2},\ n+\frac{1}{2},\ n+1,\ n \\ (1+2n-l+i\xi)/2,\ (1+2n-l-i\xi)/2,\ 2n \end{matrix}\right| 1\right)
\end{aligned} \tag{3.5}$$

where the plus sign applies to the case $n \le 0$ and the minus sign to $n > 0$. The ${}_4F_3$ is a generalized hypergeometric function.[8]

B. Elliptic system

Corresponding to the elliptic system we have

$$L_E = (1 + k^2\cos^2\theta)\frac{d^2}{d\theta^2} + k^2(2l-1)\sin\theta\cos\theta\frac{d}{d\theta} + k^2(l^2\sin^2\theta + l\cos^2\theta). \tag{3.6}$$

Initially we define this operator on the domain of C^∞ functions on the circle. However, it is easy to see that L_E has a unique self-adjoint extension. Indeed, it corresponds to a regular Sturm—Liouville operator on the interval $[0, 2\pi]$ with periodic boundary conditions. Thus the spectrum is discrete. To solve the eigenvalue equation $L_E f_\lambda^E = \lambda f_\lambda^E$, we set

$$f_\lambda(\theta) = (1 + k^2\cos^2\theta)^{l/2} g_\lambda(w),$$

$\theta = \phi - \pi/2$ and $\sin\phi = \mathrm{sn}(w, ik)$, where $\mathrm{sn}(z,k)$ is a Jacobi elliptic function (Ref. 8, Chap. 13). Then the eigenvalue equation becomes

$$\left(\frac{d^2}{dz^2} - r^2 l(l+1)\,\mathrm{sn}^2(z,r) + l(l+1)r^2 - \frac{\lambda}{1+k^2}\right) g_\lambda(z) = 0, \tag{3.7}$$

$$z = (1+k^2)^{1/2} w, \quad r^2 = \frac{k^2}{1+k^2}, \quad -K(r) \le z \le 3K(r),$$

with periodic boundary conditions $g_\lambda(z)|_{-K}^{3K}=0$, $g'_\lambda(z)|_{-K}^{3K}=0$. This is the Lamé equation and the required eigenfunctions are the periodic Lamé functions with period $4K$. We can divide the eigenfunctions into symmetry classes by noting that L_E commutes with the unitary commuting idempotent operators R_1, R_2, where

$$(R_1 f)(\phi)=f(-\phi), \quad (R_2 f)(\phi)=f(\pi-\phi)$$

with ϕ as in (3.3) and $f(\phi)$ a function on the unit circle.

Since the eigenvalues of R_1 and R_2 are ± 1 the eigenfunctions of L_E fall into four classes labeled by these eigenvalues. In terms of the notation given in Ref. 8, Sec. 15.5.1, the results are

$\lambda(1+k^2)^{-1}$	$g_\lambda(z)$	period	R_1	R_2
$a_l^{2m}(r^2)$	$Ec_l^{2m}(z, r^2)$	$2K$	1	1
$a_l^{2m+1}(r^2)$	$Ec_l^{2m+1}(z, r^2)$	$4K$	-1	1
$b_l^{2m+2}(r^2)$	$Es_l^{2m+2}(z, r^2)$	$2K$	1	-1
$b_l^{2m+1}(r^2)$	$Es_l^{2m+1}(z, r^2)$	$4K$	-1	-1

(3.8)

for $m=0,1,2,\cdots$. Here the multiplicity of each eigenvalue is one, and the superscripts m are related to the number of zeros of the corresponding eigenfunctions in a period. We normalize each eigenfunction f_λ^E to have unit length in $\mathcal{H}$, leaving a phase factor undetermined.

Note that the action of R_1 and R_2 on the spherical basis functions $f_m^S(\theta)=\exp(im\theta)/\sqrt{2\pi}=(-i)^m\exp(im\phi)/\sqrt{2\pi}$ is

$$R_1 f_m^S=(-1)^m f_{-m}^S, \quad R_2 f_m^S=f_{-m}^S. \tag{3.9}$$

The overlap functions relating the f_m^S basis to the f_λ^E basis are the coefficients $U_{\lambda,m}^{E,S}$ in the expansion

$$f_\lambda^E=\sum_{m=-\infty}^{\infty} U_{\lambda,m}^{E,S} f_m^S. \tag{3.10}$$

We can obtain recurrence relations for these coefficients by substituting (3.10) into the eigenvalue equation $L_E f_\lambda^E=\lambda f_\lambda^E$ and equating coefficients of f_m^S on both sides of the resulting identity. For example, the basis function $h_m(\phi)=(1+k^2\sin^2\phi)^{1/2}Ec_l^{2m}(z,r^2)$ satisfies $R_1h_m=R_2h_m=h_m$ so that the expansion (3.10) takes the form

$$h_m(\phi)=\tfrac{1}{2}C_0+\sum_{n=1}^{\infty} C_{2n}\cos(2n\phi).$$

Substituting this expression into the eigenvalue equation, we find

$$\begin{aligned}&[k^2l(l+1)-2\lambda]C_0-k^2(l^2-5l-2)C_2=0,\\ &k^2[\tfrac{1}{2}(n-1)(3-2l)+\tfrac{1}{4}l(1-l)]C_{2n-2}\\ &\quad+\{\tfrac{1}{2}k^2[l(l+1)-4n]-(\lambda+4n^2)\}C_{2n}\\ &\quad+k^2[\tfrac{1}{2}(n+1)(1+2l)+\tfrac{1}{4}l(1-l)]C_{2n+2}=0.\end{aligned} \tag{3.11}$$

These expressions are closely related (but not identical) to recurrence formulas derived in Section 15.5.1 of Ref. 8. There are similar formulas for the other three types of periodic Lamé functions.

C. Semicircular parabolic system

The basis defining operator L_{CP} has the form

$$\begin{aligned}L_{CP}&=2\cos\theta(1-\sin\theta)\frac{d^2}{d\theta^2}+(2l-1)(1-\sin\theta)\\ &\quad\times(1+2\sin\theta)\frac{d}{d\theta}+l\cos\theta[1+2(l-1)\sin\theta].\end{aligned} \tag{3.12}$$

Before discussing the self-adjoint extension of L_{CP} it is convenient to use instead of the functions f defined on the unit circle, the functions $g^\epsilon(v)$,

$$f(\theta)=[2v/(1+v^4)]^l g^\epsilon(v), \tag{3.13}$$

where $\epsilon=+1$, $v=\sqrt{\cot\frac{1}{2}\phi}$ $(0<\phi<\pi)$, and $\epsilon=-1$, $v=\sqrt{-\cot\frac{1}{2}\phi}$ $(\pi<\phi<2\pi)$. The space of functions $f(\theta)$ is then replaced by the pair of functions (g^+, g^-), and so we need to consider L_{CP} acting on the direct sum of two Hilbert spaces which we call $\mathcal{H}^+$ and $\mathcal{H}^-$ $(H=\mathcal{H}^+\oplus\mathcal{H}^-)$. On each of these spaces L_{CP} has the form

$$L_{CP}=\frac{1}{4}\left(\frac{d^2}{dv^2}-\frac{l(l+1)}{v^2}\right).$$

This operator has deficiency indices (1,1) on each of the two Hilbert spaces $\mathcal{H}^+$ and $\mathcal{H}^-$. There is thus a two-parameter family of possible self-adjoint extensions of L_{CP} acting on the space of functions defined on H. We choose one of these which immediately suggests itself and relate it to the standard S basis. The normalized generalized eigenfunctions we choose are

$$f_{\lambda\epsilon}^{CP}(\theta)=[2v/(1+v^4)]^l\sqrt{\lambda v}\,J_{l+1/2}(\sqrt{2}\,\lambda v)\,C_\epsilon, \tag{3.14}$$

with C_ϵ as in (2.8). This choice of basis corresponds to the choice of eigenvalue $\epsilon\lambda^2$ $(0<\lambda<\infty)$ for the basis vector $f_{\lambda\epsilon}^{CP}(\theta)$, i.e.,

$$L_{CP}f_{\lambda\epsilon}^{CP}=\epsilon\lambda^2 f_{\lambda\epsilon}^{CP}.$$

The orthogonality relations are

$$\int_0^{2\pi} f_{\lambda'\epsilon'}^{CP\dagger}(\theta) f_{\lambda\epsilon}^{CP}(\theta)\,d\theta=\delta(\lambda'-\lambda)\,\delta_{\epsilon'\epsilon}. \tag{3.15}$$

The relation of this basis to the spherical basis can be readily computed:

$$\begin{aligned}U_{-n,\lambda+}^{S,CP}&=2^{l+1}\sqrt{\lambda}\int_0^\infty v^{l+3/2}J_{l+1/2}(\sqrt{2}\,\lambda v)\\ &\quad\times(v^2+i)^{2n}(1+v^4)^{-n-l-1}\,dv\\ &=\Bigg[2\sqrt{\pi}\left(\frac{\lambda}{2}\right)^{-l-1}\sum_{r=0}^{2n} i^{2n-r}C\binom{2n}{r}\\ &\quad\times\frac{\Gamma(-l-n)}{\Gamma(-l-r)\,\Gamma(r+l+1)}\left(\frac{1}{\lambda}\frac{\partial}{\partial\lambda}\right)^r\left(\frac{1}{16z^3}\frac{\partial}{\partial z}\right)^{n-r}\\ &\quad\times\left(\frac{8z^2}{\lambda^2}\right)^{l+1/2-r}J_{-l-1/2+r}(\lambda z)K_{-l-1/2+r}(\lambda z)\Bigg]_{z=1},\end{aligned} \tag{3.16}$$

where $n>0$ and $K_l(z)$ is a MacDonald function.[8]

For $n<0$ it is only necessary to make the substitution $l\to -l-1$. The only modification of these results for the overlap function $U_{-n,\lambda-1}^{S,CP}$ is the replacement of the i^{2n-r} term in the above expression by $(-i)^{2n-r}$.

D. Hyperbolic system

The basis defining operator L_H has the form

$$\begin{aligned}L_H&=(r^2-\cos^2\theta)\frac{d^2}{d\theta^2}+(1-2l)\sin\theta\cos\theta\frac{d}{d\theta}\\ &\quad-l^2\sin^2\theta-l\cos^2\theta.\end{aligned} \tag{3.17}$$

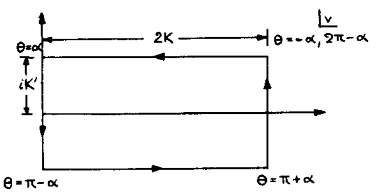

FIG. 1. The θ coordinate in the v plane for the hyperbolic system.

This operator is defined in the domain of all C^∞ functions which vanish near those four points for which $|\cos\theta| = r$ $(r > 0)$. It is convenient at this point to split the space H into a direct sum of four spaces which we label by a discrete index i ($i = 1, 2, 3$ or 4). The splitting is achieved according to the prescriptions $\mathcal{H}^1 \leftrightarrow (-\alpha < \theta < \alpha)$, $\mathcal{H}^2 \leftrightarrow (\alpha < \theta < \pi - \alpha)$, $\mathcal{H}^3 \leftrightarrow (\pi - \alpha < \theta < \pi + \alpha)$, $\mathcal{H}^4 \leftrightarrow (\pi + \alpha < \theta < 2\pi - \alpha)$ so that

$$H = \sum_{i=1}^{4} \oplus \mathcal{H}^i. \tag{3.18}$$

(note: we assume $r = \cos\alpha$, $0 < \alpha < \pi/2$). The functions $f(\theta)$ are then replaced by functions $h_i(v)$, given by

$$f_i(\theta) = [ir'/\mathrm{cn}(v, r)]^l\, h_i(v), \tag{3.19}$$

where $r' = (1 - r^2)^{1/2}$ and $\cos\theta = \mathrm{dn}(v, r)/\mathrm{cn}(v, r)$.

The ranges of the parameters are shown in Fig. 1, and it can be seen that as θ runs from $-\alpha \to 2\pi - \alpha$, the parameter v describes a closed path as indicated in Fig. 1.

On each of the Hilbert spaces $\mathcal{H}^i$ the operator L_{H} has the form

$$L_{\mathrm{H}} = \frac{d^2}{dv^2} - r^2\, l(l+1)\,\mathrm{sn}^2(v, r). \tag{3.20}$$

We are then concerned with four eigenvalue problems each of which is such that the operator L_{H} is singular at each of the two corresponding end points. Let us first consider the choice of basis for $\mathcal{H}^1$. For this space $v \in (iK', iK' + 2K)$. Following Erdelyi,[8] Chap. 15, we choose the boundary conditions for a basis as

$$\text{(i) } [\mathrm{sn}(v, r)]^{1/2}\, \Lambda(v) \text{ bounded at } v = iK', \qquad \Lambda'(K + iK') = 0. \tag{3.21}$$

The corresponding solution is denoted by $\Lambda = F_l^{2m}(v, r)$ and has $2m$ zeros in the interval $(iK', iK' + 2K)$

$$\text{(ii) } [\mathrm{sn}(v, r)]^{1/2}\, \Lambda(v) \text{ bounded at } v = iK', \qquad \Lambda(K + iK') = 0. \tag{3.22}$$

The corresponding solutions are denoted by $F_l^{2m+1}(v, r)$. In the above $\Lambda(v)$ is the corresponding solution of the equation $L_{\mathrm{H}}\Lambda = \lambda_m \Lambda$. Here m is the number of zeros of the eigenfunction Λ in the interval $(iK', iK' + 2K)$. These are the finite Lamé or Lamé Wangerin functions. The solution of the corresponding boundary value problem gives these functions as expansion functions with the discrete spectrum of L_{H} labeled by the upper index. [This index is also the number of zeros of the solution in the interval $(iK', iK' + 2K)$.] The problem for the basis of $\mathcal{H}^3$ is exactly similar so that we then have the basis

$$f^{\mathrm{H}}_{m,i}(v) = F_l^m(v, r)\,\lambda_i, \quad i = 1, 3. \tag{3.23}$$

The λ_i are 4×1 column vectors having 1 in the ith row and zero elements elsewhere. For the choice of basis in the spaces $\mathcal{H}^2$ and $\mathcal{H}^4$ the corresponding eigenfunction expansion problem is similar to that considered already but the variable v is now in the range $(iK', -iK')$ or $(2K + iK', 2K - iK')$. The corresponding boundary value problem of interest is now given by the requirement that $(\mathrm{sn} v)^{1/2}\, \Lambda(v)$ be bounded at the end points $v = \pm iK'$ and that $\Lambda'(0) = 0$ or $\Lambda(0) = 0$ according as Λ is even or odd about $v = 0$. The complete set of eigenfunctions are the Lamé Wangerin functions $F_l^m(v, r)$. The corresponding basis functions are then given as in (3.23) with $i = 2, 4$. In particular we have for each eigenfunction $f^{\mathrm{H}}_{m,i}$ ($i = 1, 2, 3, 4$) as θ varies from $-\alpha$ to $2\pi - \alpha$, that v varies continuously around the rectangle drawn in Fig. 1. The corresponding eigenfunction $[ir'/\mathrm{cn}(v, r)]^l f^{\mathrm{H}}_{m,i}$ corresponds to a continuous differentiable function of θ and is therefore an element of the original representation space. This requirement picks out this solution and does not require us to consider the deficiency indices in each subspace. (We have essentially periodic boundary conditions). The latter procedure in general leads to sectionally continuous eigenfunctions on H. The orthogonality of the basis functions is written

$$(f^{\mathrm{H}}_{m,i}, f^{\mathrm{H}}_{m',j}) = \delta_{ij}\,\delta_{mm'}\,N^i_m \tag{3.24}$$

with N^i_m a normalization factor. The eigenfunctions $f^{\mathrm{H}}_{m,i}$ defined as above are nonzero only in the corresponding Hibert space $\mathcal{H}^i$.

We now proceed to calculate a recurrence relation for the overlap functions between hyperbolic and spherical bases.

We consider in detail overlaps associated with the spaces $\mathcal{H}^1$ and $\mathcal{H}^3$. As with the elliptic system it is convenient to consider a number of discrete transformations. The first of these is reflection R about the line $\mathrm{Re}\, v = K$. This corresponds to the transformation $\theta \to -\theta$. We have accordingly

$$R f^{\mathrm{H}}_{m,i}(v) = (-1)^m f^{\mathrm{H}}_{m,i}(v), \quad i = 1, 3. \tag{3.25}$$

In addition, if we consider the reflection $\bar{R}: \theta \to \pi - \theta$, then we have

$$\bar{R} f^{\mathrm{H}}_{m,i}(v) = (-1)^m f^{\mathrm{H}}_{m,j}(v), \quad i \neq j, \; i, j = 1, 3. \tag{3.26}$$

From these equations we can form the linear combinations $F^{\mathrm{H}\pm}_m = f^{\mathrm{H}}_{m,i}(v) \pm f^{\mathrm{H}}_{m,j}(v)$ [with i, j as in (3.23)] having eigenvalues $(-1)^m$, $\pm(-1)^m$ respectively, of the operators R and $\bar{R}$.

It is these functions for which we can form the overlap functions, i.e., instead of relating the normal basis $f^{\mathrm{H}}_{m,i}(v)$ to the spherical basis f^{S}_n via $f^{\mathrm{H}}_{m,i} = \sum_{n=-\infty}^{\infty} U^{\mathrm{H},\mathrm{S}}_{m,n} f^{\mathrm{S}}_n$ we write each $F^{\mathrm{H}\pm}_m$ as a Fourier series in θ and find recurrence relations for the coefficients. This involves extending the domain of the functions $F^{\mathrm{H}\pm}_m$ to be defined on the unit circle, $0 < \theta \leq 2\pi$.

The symmetrized basis function $G^{\mathrm{H}\pm}_{2p} = (r^2 - \cos^2\theta)^{1/2} \times F^{\mathrm{H}\pm}_{2p}$ has eigenvalues $+1$ for the both the reflections R and $\bar{R}$ and so can be represented by the series

$$G^{\mathrm{H}+}_{2p}(\theta) = \tfrac{1}{2}C_0 + \sum_{n=1}^{\infty} C_{2n}\cos(2n\theta) \tag{3.27}$$

for $\alpha < \theta < \pi/2 - \alpha$. Applying the operator $L_{\rm H}$ to both sides, we obtain the recurrence relations

$$
\begin{aligned}
&-[l(l+1)+2\lambda_m]C_0+[2+l(3l-1)]C_2=0,\\
&[\tfrac{1}{2}(p-1)(2l-2p-1)+\tfrac{1}{4}(l-1)]C_{2p-2}\\
&\quad+[2p^2(1-2r^2)-\tfrac{1}{2}l(l+1)-\lambda_m]C_{2p}\\
&\quad+[\tfrac{1}{2}(p+1)(2l+2p+1)+\tfrac{1}{4}l(l-1)]C_{2p+2}=0 \qquad (3.28)
\end{aligned}
$$

for $p \geq 1$.

Similar recurrence relations can be derived for the other symmetrized basis functions. Identical arguments can be applied to overlap functions associated with the Hilbert spaces $\mathcal{H}^2$ and $\mathcal{H}^4$. In this case it is convenient to introduce the same discrete transformations as previously but with θ replaced by $\phi(\theta = \pi/2 + \phi)$. With this change the analysis goes through as before.

E. Semihyperbolic system

The basis defining operator $L_{\rm SH}$ has the form

$$
\begin{aligned}
L_{\rm SH}=&(r\cos^2\theta-2\sin\theta)\frac{d^2}{d\theta^2}+(2l-1)\cos\theta(1+r\sin\theta)\frac{d}{d\theta}\\
&+r(l^2\sin^2\theta+l\cos^2\theta)-l\sin\theta. \qquad (3.29)
\end{aligned}
$$

This operator is defined on the domain of all C^∞ functions which vanish near the two points at which $\sin\theta = 1/r[(1+r^2)^{1/2}-1]$. It is convenient to split the space H into the direct sum of two spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ defined according to the prescription $\mathcal{H}_1 \leftrightarrow (\alpha < \theta < \pi - \alpha)$, $\mathcal{H}_2 \leftrightarrow (\pi - \alpha < \theta < 2\pi + \alpha)$ so that

$$H = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

The functions $f(\theta)$ are then replaced by the pair of functions h_i $(i=1,2)$, where

$$
\begin{aligned}
f(\theta)&=\left(\frac{N\,\mathrm{sn}(v,s)\,dn(v,s)}{[-(1+r^2)^{1/2}+r+1]\,\mathrm{sn}^2(v,s)-2r}\right)^l h_1(v),\\
&\qquad \alpha<\theta<\pi-\alpha,\\
&=\left(\frac{N\,\mathrm{sn}(u,q)\,dn(u,q)}{[(1+r^2)^{1/2}+r-1]\,\mathrm{sn}^2(u,q)-2r}\right)^l h_2(u),\\
&\qquad \pi-\alpha<\theta<2\pi+\alpha, \qquad (3.30)
\end{aligned}
$$

where

$$N^2=\frac{8(1+r^2)^{1/2}}{r^2}\,[(1+r^2)^{1/2}-1]\ ,$$

$$s^2=\frac{(1+r^2)^{1/2}-r}{2(1+r^2)^{1/2}},\quad q^2=\frac{(1+r^2)^{1/2}+r}{2(1+r^2)^{1/2}}$$

and

$$
\begin{aligned}
\sin\theta&=\frac{2[1-(1+r^2)^{1/2}]+[(1+r^2)^{1/2}-1-r]\,\mathrm{sn}^2(v,s)}{[1+r-(1+r^2)^{1/2}]\,\mathrm{sn}^2(v,s)-2r},\\
&\qquad \alpha<\theta<\pi-\alpha,\\
&=\frac{[(1+r^4)^{1/2}-1+r^2]\,\mathrm{sn}^2(u,q)-2[(1+r^4)^{1/2}-1]}{[(1+r^4)^{1/2}-1+r^2]\,\mathrm{sn}^2(u,q)-2r^2},\\
&\qquad \pi-\alpha<\theta<2\pi+\alpha. \qquad (3.31)
\end{aligned}
$$

The corresponding ranges of the variables are $0 < v < 2K(s)$, $0 < u < 2K(q)$. In terms of the new variables the operator $L_{\rm SH}$ assumes the forms

$$
\begin{aligned}
(1+r^2)^{-1/2}L_{\rm SH}&=-\left(\frac{d^2}{dv^2}+l(l+1)\frac{\mathrm{cn}^2(v,s)}{\mathrm{sn}^2(v,s)\,\mathrm{dn}^2(v,s)}\right.\\
&\qquad\left.+\frac{r\,l(l+1)}{(1+r^2)^{1/2}}\right)\\
&=\frac{d^2}{du^2}-l(l+1)\frac{\mathrm{cn}^2(u,q)}{\mathrm{sn}^2(u,q)\,\mathrm{dn}^2(u,q)}\\
&\qquad-\frac{r\,l(l+1)}{(1+r^2)^{1/2}}. \qquad (3.32)
\end{aligned}
$$

It is possible to make further transformations and write $L_{\rm SH}$ in the form of the standard Lamé operator as for instance in (3.20). The resulting elliptic functions then have a complex molulus $k=\exp(i\psi)$ (ψ real) and the range of variation of the new variables is not parallel to either of the directions of periodicity. It is more convenient to consider the operator $L_{\rm SH}$ in one of the forms (3.30). The problem of the self-adjoint extension of $L_{\rm SH}$ on each of the spaces $\mathcal{H}_i$ is exactly analogous to that considered in each of the spaces $\mathcal{H}_i$ of the hyperbolic system. In particular we choose the boundary conditions which require that $[\mathrm{sn}(v,s)]^{-1/2}\Lambda(v,s)$ be bounded in the interval $(0, 2K(s))$. Here $\Lambda(v,s)$ is a solution of $L_{\rm SH}\Lambda = \bar{\lambda}_m\Lambda$. More precisely the boundary conditions are:

(i) $[\mathrm{sn}(v,s)]^{-1/2}\,\Lambda(v,s)$ bounded at $v=0$, $2K(s)$ and $\Lambda'(K,s)=0$. The corresponding solution is denoted by $K_l^{2m}(v,s)$ and has $2m$ zeros in the interval $(0, 2K(s))$.

(ii) $[\mathrm{sn}(v,s)]^{-1/2}\,\Lambda(v,s)$ bounded at $v=0$, $2K(s)$ and $\Lambda(K,s)=0$. The corresponding solution is denoted by $K_l^{2m+1}(v,s)$ and has $2m+1$ zeros in the interval $[0, 2K(s)]$. Similar remarks apply to the related problem on $\mathcal{H}_2$. The corresponding solutions are denoted by $M_l^m(u,q)$. The spectrum in each case is discrete. A complete set of eigenfunctions for the Hilbert space H is then

$$
\begin{aligned}
f_{m,1}^{\rm SH}(v)&=K_l^m(v,s)\,C_+,\\
f_{n,2}^{\rm SH}(v)&=M_l^m(v,q)\,C_-.
\end{aligned} \qquad (3.33)
$$

Satisfying the normalization conditions, we have

$$(f_{m,\eta}^{\rm SH}, f_{m',\eta'}^{\rm SH})=\delta_{mm'}\,\delta_{\eta\eta'},\quad \eta,\eta'=1,2.$$

The functions $K_l^m(v,s)$ and $M_l^m(u,q)$ that we have introduced are closely related to the Lamé Wangerin functions which appear in the hyperbolic basis. In fact if we take the operator $L_{\rm SH}$ in the standard Lamé form we have in the space $\mathcal{H}_1$

$$
\begin{aligned}
&[r+(r^2+1)^{1/2}]^{1/2}L_{\rm SH}\\
&\quad=\frac{d^2}{dw^2}-k^2l(l+1)\,\mathrm{sn}^2(w,k)+\frac{r\,l(l+1)}{[r+(r^2+1)^{1/2}]^{1/2}} \qquad (3.34)
\end{aligned}
$$

where $k=[q-i(1-q^2)^{1/2}]/[q+i(1-q^2)^{1/2}]$ and $w=[q+i(1-q^2)^{1/2}]v-iK'(k)$.

The corresponding eigenfunctions of this operator are then Lamé Wangerin functions. These solutions can be represented in a series as Erdeyli has done for the case of complex k, e.g.,

$$F_l^{2m}(w,k)=\sum_{r=0}^{\infty}A_r\exp[-i(l+1+2r)\xi], \tag{3.35}$$

where $\cos\xi=\mathrm{sn}(w,k)$ and the coefficients A_r satisfy the recurrence relations

$$\begin{aligned}&[H-(l+1)^2(2-k^2)]A_0+(2l+3)k^2A_1=0,\\ &(2r-1)(l+r)k^2A_{r-1}+[H-(l+1+2r)^2(2-k^2)]A_r\\ &\quad+(r+1)(2l+2r+3)k^2A_{r+1}=0,\\ &r\geq 1 \quad\text{and}\quad H=2\bar{\lambda}_{2m}-l(l+1)k^2.\end{aligned} \tag{3.36}$$

In this way we can write a series expansion for each of our basis functions K_l^m and M_l^m. It is again straightforward to calculate recurrence relations for the overlap functions between the semihyperbolic system and the spherical or canonical basis. This again depends on the fact that a given basis function consisting of two components represents a continuous function of θ for $\theta\in[0,2\pi]$. We merely note here that this can be done and omit the calculation which leads to rather lengthy recurrence relations.

F. The hyperbolic parabolic system

The operator $L_{\rm HP}$ has the form

$$\begin{aligned}L_{\rm HP}&=2\sin\theta(\sin\theta-1)\frac{d^2}{d\theta^2}+(2l-1)\cos\theta(1-2\sin\theta)\frac{d}{d\theta}\\ &\quad-2l^2\sin^2\theta-2l\cos^2\theta-l\sin\theta,\quad \gamma=1.\end{aligned} \tag{3.37}$$

We consider this operator to be defined initially on the C^∞ functions of θ which vanish near the points $\theta=\pi/2$, π, $3\pi/2$, where $L_{\rm HP}$ is singular. It is convenient to consider the space H divided into four subspaces $\mathcal{H}^i$ as with the hyperbolic system, i.e., $H=\sum_{i=1}^4\oplus\mathcal{H}^i$. Each of these subspaces corresponding to functions of θ defined over an interval of length $\pi/2$, e.g., $\mathcal{H}^1\leftrightarrow(0<\theta<\pi/2)$ etc. It is then convenient to consider the operator $L_{\rm HP}$ acting on new functions h_i in each of these spaces where

$$\begin{aligned}f_i(\theta)&=[\sqrt{2}\sinh b/(1+\cosh^2 b)]^l h_i(b),\quad i=1,2,\\ &=[\sqrt{2}\sin\psi/(1+\cos^2\psi)]^l h_i(\psi),\quad i=3,4.\end{aligned} \tag{3.38}$$

The variables b and ψ are given by

$$\begin{aligned}[(1+\sin\theta)/2\sin\theta]^{1/2}&=\coth b \quad\text{if}\quad 0<\theta<\pi\\ &=i\cot\psi\quad\text{if}\quad \pi<\theta<2\pi.\end{aligned} \tag{3.39}$$

For $i=1,2$, $L_{\rm HP}$ acting on the functions $h_i(b)$ has the form

$$L_{\rm HP}=\frac{d^2}{db^2}-\frac{l(l+1)}{\sinh^2 b}$$

and for $i=3,4$ it is just required to make the substitution $b\to i\psi$. For $i=1,2$ the solutions of the eigenvalue equation $L_{\rm HP}\,h=\mu^2h$ are the functions $(\sinh b)^{1/2}$ $P^{-l-1/2}_{-1/2-\mu}(\cosh b)$. From this observation it is immediately seen that a complete set of basis functions does exist if we take $\mu=-i\rho$ (ρ real and positive). The corresponding completeness properties follow from the properties of the generalized Mehler transform. A complete set of orthonormal basis functions is then

$$\begin{aligned}f^{\rm HP}_{\rho,i}(b)&=[(\rho\sinh\pi\rho/\pi)\,\Gamma(1+l+i\rho)\,\Gamma(1+l-i\rho)]^{1/2}\\ &\quad\times(\sinh b)^{1/2}P^{-l-1/2}_{-1/2+i\rho}(\cosh b),\end{aligned} \tag{3.40}$$

$i=1,2$, satisfying the orthogonality relations

$$\langle f^{\rm HP}_{\rho,i}, f^{\rm HP}_{\rho',i}\rangle=\delta(\rho-\rho').$$

The spaces $\mathcal{H}_3$ and $\mathcal{H}_4$ can be combined by defining the variable ψ as in (3.37) with $0<\psi<\pi$ but now taking into account the sign of the square root. The corresponding eigenvalue problem is singular at both ends of the interval $\psi\in(0,\pi)$. There is a two-parameter family of self-adjoint extensions of $L_{\rm HP}$ since the deficiency indices are (2, 2).

Each linearly independant solution is square integrable so that the spectrum is discrete for each self-adjoint extension. The computation of an orthonormal basis of eigenfunctions is straightforward but complicated and unenlightening and so we omit it. Also, the integrals relating these bases to the standard spherical basis appear intractable.

4. THE TWO VARIABLE MODEL

The group $SO(2,1)$ acts on 3-space according to $\mathbf{x}\to L(g)\mathbf{x}$, where $\mathbf{x}=(x_0,x_1,x_2)$ is a column 3-vector and $L(g)$ is the 3×3 matrix representation of $SU(1,1)$ defined as in Ref. 13, p. 289. This action induces a representation of $SU(1,1)$ on the space $\mathcal{F}$ of C^∞ functions in 3-space, defined by operators $\mathbf{T}(g)$:

$$[\mathbf{T}(g)F](\mathbf{x})=F(L(g^{-1})\mathbf{x}),\quad F\in\mathcal{F}. \tag{4.1}$$

To be precise, we choose the action so that the corresponding Lie derivatives are as in (1.1). Clearly the quadratic form $x_0^2-x_1^2-x_2^2$ is preserved by this action. In this section we will construct models of the principal series representations of $SO(2,1)$ in which the Hilbert space consists of functions $F(\mathbf{x})$ defined on the hyperboloid $x_0^2-x_1^2-x_2^2=1$, $x_0>0$, and the group acts via (4.1). In particular we will explicitly construct in this space the various basis functions listed above. Furthermore, we will use the Gel'fand—Graev transform to expand an arbitrary function, square integrable on the hyperboloid, in terms of each type of basis. We note that the basis functions are exactly those which appear when one uses separation of variable methods to find solutions of the wave equation

$$\left(\frac{\partial^2}{\partial y_0^2}-\frac{\partial^2}{\partial y_1^2}-\frac{\partial^2}{\partial y_2^2}\right)\phi(\mathbf{y})=0, \tag{4.2}$$

which are homogeneous in y_0, y_1, y_2.

We use the Gel'fand—Graev transform[14] to map functions on the unit circle corresponding to a principal series representation of $SO(2,1)$ to functions on the hyperboloid. Thus, corresponding to $f\in\mathcal{H}$ and the representation $l=-\frac{1}{2}+i\rho$, we define a function $F(\mathbf{x})$ on the hyperboloid by the integral

$$F(\mathbf{x})=\int_0^{2\pi}(x_0+x_1\sin\theta-x_2\cos\theta)^{-l-1}f(\theta)d\theta=I[f]. \tag{4.3}$$

It is easy to check that the operator $\mathbf{T}(g)$, (2.6), acting on f induces the operator $T(g)$, (4.1), acting on F:

$$T(g)F=I[\mathbf{T}(g)f].$$

It follows that the Lie derivatives (2.7) acting on f induce the Lie derivatives (1.1) acting on F.

If $\{f_n^G\}$ is a basis for $\mathcal{H}$ corresponding to the operator L_G, then

$$(K_1^2+K_2^2-M_3^2)f_n^G=l(l+1)f_n^G,$$
$$L_Gf_n^G=\lambda_nf_n^G. \tag{4.4}$$

It follows that the functions $F_n^G=I(f_n^G)$ satisfy the equations

$$(K_1^2+K_2^2-M_3^2)F_n^G=l(l+1)F_n^G,$$
$$L_GF_n^G=\lambda_nF_n^G, \tag{4.5}$$

where now the operators K_1, K_2, M_3 are given by (1.1) and L_G is expressed in terms of these operators by one of the Eqs. (3.1). We shall see that each choice of L_G in (3.1) corresponds to a separation of variables in the first equation (4.5).

We can now employ any one of our bases $\{F_n^G\}$ to expand functions on the hyperboloid. Thus, if $H(\mathbf{x})$ is square integrable on the hyperboloid $x_0^2-x_1^2-x_2^2=1$, $x_0>0$, with respect to the measure dx_1dx_2/x_0, then the Gel'fand—Graev integral transform yields the expansion

$$H(\mathbf{x})=\frac{1}{8\pi^2i}\int_{-1/2-i\infty}^{-1/2+i\infty}I[f_l]\,l\cot\pi l\,dl, \tag{4.6}$$

where $f_l(\theta)$ is a function on the circle defined by

$$f_l(\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(\mathbf{x})\,(x_0+x_1\sin\theta-x_2\cos\theta)^l\,\frac{dx_1dx_2}{x_0}. \tag{4.7}$$

Since $f_l(\theta)$ can be expanded in a $\{f_n^G\}$ basis, we obtain

$$f_l(\theta)=\sum_n A_l^{G,n}f_n^G,\quad A_l^{G,n}=\langle f_n^G,f_l\rangle,$$

or

$$H(\mathbf{x})=\frac{1}{8\pi^2i}\int_{-1/2-i\infty}^{-1/2+i\infty}l\cot\pi l\,dl\sum_n A_l^{G,n}\,F_n^G(\mathbf{x}),$$
$$A_l^{G,n}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(\mathbf{x})\,\overline{F_n^G(\mathbf{x})}\,\frac{dx_1dx_2}{x_0}. \tag{4.8}$$

Formulas (4.8) apply directly in the case L_G has discrete spectrum. When L_G has continuous spectrum, it is necessary to replace the sum over n by an integral.

Note: In the usual treatments of the Gel'fand—Graev integral transforms, our $I[f_l]$ is replaced by an integral over an arbitrary contour Γ on the cone $x_0^2-x_1^2-x_2^2=0$, which intersects every generator once. In this paper that contour is always chosen to be the circle $(x_0,x_1,x_2)=(1,-\sin\theta,\cos\theta)$.

We can view the transform (4.4) in another way: namely as the inner product of the functions $h_{\mathbf{x}}(\theta)$, $f(\theta)\in\mathcal{H}$,

$$F(\mathbf{x})=\langle h_{\mathbf{x}},f\rangle, \tag{4.9}$$
$$h_{\mathbf{x}}(\theta)=(x_0+x_1\sin\theta-x_2\cos\theta)^l\in\mathcal{H}.$$

Then the formula $F_n^G=\langle h_{\mathbf{x}},f_n^G\rangle$ yields immediately the expansion

$$h_{\mathbf{x}}(\theta)=\sum_n\overline{F}_n^G(\mathbf{x})f_n^G(\theta) \tag{4.10}$$

for the kernel function $h_{\mathbf{x}}(\theta)$. Furthermore, a direct computation yields the result

$$\langle h_{\mathbf{x}},h_{\mathbf{y}}\rangle=2\pi P_l(x_0y_0-x_1y_1-x_2y_2), \tag{4.11}$$

where $P_l(z)$ is a Legendre function. Substituting (4.10) into (4.11), we find

$$2\pi P_l(x_0y_0-x_1y_1-x_2y_2)=\sum_n\overline{F}_n^G(\mathbf{x})F_n^G(\mathbf{y}). \tag{4.12}$$

Finally, if two $\mathcal{H}$ bases $\{f_n^G\}$, $\{f_m^K\}$ are related by overlap functions $U_{n,m}^{G,K}$,

$$f_n^G=\sum_m U_{n,m}^{G,K}f_m^K,$$

it follows immediately that

$$F_n^G=\sum_m U_{n,m}^{G,K}F_m^K. \tag{4.13}$$

We now list the functions F_n^G for each choice of G. In several cases the integral $I[f_n^G]$ appears not to be known, and we have to make explicit use of the fact that, in each of the appropriate coordinates tabulated in Ref. 2, $I[f_n^G]$ satisfies a simple second order ordinary differential equation. Thus F_n^G can be expressed as products of solutions of such equations with coefficients determined by evaluating the integral for special values of the parameters $\mathbf{x}$. We now give explicit expressions for seven of the nine bases discussed.

A. Spherical system

$$F_m^S(a,\psi)=\int_0^{2\pi}[\cosh a-\sinh a\sin\theta\sin\psi-\sinh a\cos\theta\cos\psi]^{-l-1}\times\exp(im\theta)\,d\theta \tag{4.14}$$
$$=\frac{-1}{2\sqrt{2\pi}}\,\frac{\Gamma(l+1-m)}{\Gamma(l+1)}\,P_l^m(\cosh a)\exp(im\phi)$$

with $(x_0,x_1,x_2)=(\cosh a,\ -\sinh a\sin\psi,\ \sinh a\cos\psi)$, $0\le a<\infty$, $0\le\psi\le 2\pi$.

B. Equidistant system

$$F_{\tau\epsilon}^{\mathrm{E}q}(a,b)=\int_{-\infty}^{\infty}[\cosh a\cosh b\cosh q-\cosh a\sinh b\sinh q-\epsilon\sinh a]^{-l-1}\exp(i\tau q)\,dq$$
$$=\frac{4}{(\cosh a)^{1/2}}\exp[-i\pi(l+1/2)/4]\,\frac{\Gamma(l+1+i\tau)\Gamma(l+1-i\tau)}{\Gamma(l+1)}\times P_{-1/2+i\tau}^{-(l+1/2)}(-\epsilon\tanh a)\exp(i\tau b) \tag{4.15}$$

with $(x_0,x_1,x_2)=(\cosh a\cosh b,\ -\sinh a,\ \cosh a\sinh b)$, $-\infty<a<\infty$, $-\infty<b<\infty$.

C. Horicyclic system

$$F_s^0(a,r)=\int_0^{2\pi}[\tfrac{1}{2}(\exp(-a)+(r^2+1)\exp(a))-re^a\cos\theta-\tfrac{1}{2}(\exp(-a)+(r^2-1)\exp(a))\sin\theta]^{-l-1}\times(2\cos^2\tfrac{1}{2}\theta)^l\times\exp(is\tan\tfrac{1}{2}\theta)\,d\theta$$
$$=\frac{2\sqrt{\pi}}{\Gamma(l+1)}\left|\frac{s}{2}\right|^{l+1/2}\exp(-a/2)K_{l+1/2}(e^{-a}|s|)\exp(isr) \tag{4.16}$$

with

$$(x_0,x_1,x_2)=(\tfrac{1}{2}[\exp(-a)+(r^2+1)e^a],\ -\tfrac{1}{2}[\exp(-a)+(r^2-1)e^a],\ re^a),\quad 0<r<\infty,\ -\infty<a<\infty.$$

D. Elliptic-parabolic system

$$F_\xi^{\mathrm{EP}}(a,\theta)=\alpha_\xi[2\cosh a\cos\theta]^{l+1}$$
$$\times\int_0^{2\pi}[\cosh^2 a+\cos^2\theta-\cos\phi(\cosh^2 a+\cos^2\theta-2)$$
$$-2\sin\phi\sinh a\sin\theta]^{-l-1}(\sin\tfrac{1}{2}\phi)^l$$
$$\times P_l^{i\xi}(\cos\tfrac{1}{2}\phi)\,d\phi. \qquad (4.17)$$

Here,

$$x_0=\frac{1}{2}\left(\frac{\cosh^2 a+\cos^2\theta}{\cosh a\cos\theta}\right),$$

$$x_1=\frac{1}{2}\left(\frac{\sin^2\theta-\sinh^2 a}{\cosh a\cos\theta}\right),$$

$$x_2=-\frac{\sinh a\sin\theta}{\cosh a\cos\theta}.$$

Using Ref. 2 and symmetry in a and $i\theta$, we have

$$F_\xi^{\mathrm{EP}}(a,\theta)=A\,P_l^{i\xi}(\tanh a)\,P_l^{i\xi}(i\tan\theta)$$
$$+B\,(P_l^{i\xi}(\tanh a)\,Q_l^{i\xi}(i\tan\theta)+Q_l^{i\xi}(\tanh a)$$
$$\times P_l^{i\xi}(i\tan\theta))+C\,Q_l^{i\xi}(\tanh a)\,Q_l^{i\xi}(i\tan\theta). \qquad (4.18)$$

Setting $P_0=P_l(0)$, $P_0'=[dP_l(x)/dx]_{x=0}$, etc., (these values are listed explicitly in 8, Vol. 1), and computing $F_\xi^{\mathrm{EP}}(0,0)$, $\partial_a F_\xi^{\mathrm{EP}}(0,0)$, and $\partial_a\partial_\theta F_\xi^{\mathrm{EP}}(0,0)$ directly from (4.17) and from (4.18), we obtain the equations

$$\begin{pmatrix} P_0P_0 & P_0Q_0+Q_0P_0 & Q_0Q_0\\ P_0'P_0 & P_0'Q_0+Q_0'P_0 & Q_0'Q_0\\ P_0'P_0' & P_0'Q_0'+Q_0'P_0' & Q_0'Q_0'\end{pmatrix}\begin{pmatrix}A\\B\\C\end{pmatrix}=\begin{pmatrix}E_1\\0\\E_3\end{pmatrix}, \qquad (4.19)$$

where

$$E_1=\frac{\alpha_\xi\,2^{2i\xi+1}\pi\,\Gamma((l+1+2i\xi)/2)\,\Gamma((l+1-2i\xi)/2)}{\Gamma(l+1)\,\Gamma(\frac{1}{2})\,\Gamma(l-2i\xi+2)/2)\,\Gamma((-l-2i\xi+1)/2)},$$

$$E_2=\frac{-\alpha_\xi\,2^{l+3+2i\xi}\pi\,\Gamma((l+2-2i\xi)/2)\,\Gamma((l+2+2i\xi)/2)}{\Gamma(l+1)\,\Gamma(\frac{1}{2})\,\Gamma((l-2i\xi+1)/2)\,\Gamma((-l-2i\xi)/2)}.$$

Equations (4.19) can be solved via Cramer's rule to give explicit values for the constants A, B, C.

E. Elliptic system

$$F_{P,m}^{\mathrm{E}}(\alpha,\beta)=\int_0^{2\pi}[\mathrm{dn}\alpha\,\mathrm{dn}\beta-\mathrm{cn}\alpha\,\mathrm{cn}\beta\sin\theta$$
$$+(i/\sqrt{2})\,\mathrm{sn}\alpha\,\mathrm{sn}\beta\cos\theta]^{-l-1}$$
$$\times\ (1+\cos^2\theta)^{l/2}\,Ep_l^m(z)\,d\theta. \qquad (4.20)$$

Here for simplicity the moduli of all elliptic and Lamé functions are chosen to be r, where $r=r'=1/\sqrt{2}$, and we have introduced coordinates α, β on the hyperboloid via the expressions

$$x_0=\sqrt{2}\ \mathrm{dn}\alpha\ \mathrm{dn}\beta,\quad x_1=-\mathrm{cn}\alpha\,\mathrm{cn}\beta,\quad x_2=-(i/\sqrt{2})\,\mathrm{sn}\alpha\,\mathrm{sn}\beta,$$
$$0\le\alpha\le 4K,\ 0\le\beta<iK'$$

(see Ref. 3). The letter p in $Ep_l^m(z)$ stands for either c or s from expressions (3.8). Finally,

$$\mathrm{sn}(z,r)=\frac{-(1+k^2)^{1/2}\cos\theta}{(1+k^2\cos^2\theta)^{1/2}},\quad r=\frac{1}{\sqrt{2}},\quad k=1.$$

Making use of the facts that $F_{P,m}^{\mathrm{E}}(\alpha,\beta)$ is symmetric in α and β, that it satisfies the Lamé equation in α, and that $F_{P,m}^{\mathrm{E}}(\alpha,\beta)=F_{P,m}^{\mathrm{E}}(\alpha+4K,\beta)$, we easily obtain

$$F_{P,m}^{\mathrm{E}}(\alpha,\beta)=C_{P,m}Ep_l^m(\alpha)\,Ep_l^m(\beta), \qquad (4.21)$$

where the constant $C_{P,m}$ can be determined by evaluating the integral for a fixed choice of α and β.

Substituting this result into (4.12) and using the orthogonality relations for the elliptic basis, we obtain the integral

$$A_{P,m}Ep_l^m(\alpha')\,\overline{Ep_l^m(\beta)}\,Ep_l^m(\beta')=2\pi\int_0^{4K}P_l(2\,\mathrm{dn}\alpha\,\mathrm{dn}\alpha'\,\mathrm{dn}\beta\,\mathrm{dn}\beta'$$
$$-\mathrm{cn}\alpha\,\mathrm{cn}\alpha'\,\mathrm{cn}\beta\,\mathrm{cn}\beta'$$
$$-\tfrac{1}{2}\,\mathrm{sn}\alpha\,\mathrm{sn}\alpha'\,\mathrm{sn}\beta\,\mathrm{sn}\beta')$$
$$\times\ Ep_l^m(\alpha)\,d\alpha, \qquad (4.22)$$

where $A_{P,m}$ is a constant.

F. Semicircular parabolic system

$$F_{\lambda+}^{\mathrm{CP}}(\xi,\eta)$$
$$=2\sqrt{\lambda}\,(2\xi\eta)^{l+1}\int_0^\infty\frac{J_{l+1/2}(\sqrt{2}\,\lambda v)\,v^{l+3/2}\,dv}{\{[v^2+(\xi-i\eta)^2][v^2+(\xi+i\eta)^2]\}^{l+1}}$$
$$=\frac{2^{-2l}\lambda^{l+1}(2\pi\xi\eta)^{1/2}}{\Gamma(l+1)}J_{l+1/2}(\lambda\xi)K_{l+1/2}(\lambda\eta). \qquad (4.23)$$

The remaining integral is given by interchanging ξ and η, i.e.,

$$F_{\lambda-}^{\mathrm{CP}}(\xi,\eta)=F_{\lambda+}^{\mathrm{CP}}(\eta,\xi);$$

the coordinates on the hyperboloid are

$$x_0=\frac{(\xi^2+\eta^2)^2+4}{8\xi\eta},\quad x_1=\frac{1}{2}\left(\frac{\eta}{\xi}-\frac{\xi}{\eta}\right),\quad x_2=\frac{(\xi^2+\eta^2)^2-4}{8\xi\eta}$$

with $\xi,\eta>0$.

G. Hyperbolic system

$$F_{m,i}^{\mathrm{H}}(\alpha,\beta)$$
$$=(ir')^{l+1}\int_A^B F_l^m(v,r)\left(\frac{ir}{r'}\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\,\mathrm{cn}v+rr'\,\mathrm{sn}\alpha\,\mathrm{sn}\beta\,\mathrm{sn}v\right.$$
$$\left.+\frac{i}{r'}\,\mathrm{dn}\alpha\,\mathrm{dn}\beta\,\mathrm{dn}v\right)^{-l-1}dv$$
$$=\lambda_m^i F_l^m(\alpha,r)\,F_l^m(\beta,r), \qquad (4.24)$$

where the integration region is over the appropriate side of the rectangle in Fig. 1 corresponding to the Hilbert space $\mathcal{H}^i$, e.g., if $i=1$, $(A,B)=(iK'+2K,\ iK')$.

The coordinates on the hyperboloid are

$$x_0=(ir/r')\,\mathrm{cn}(\alpha,r)\,\mathrm{cn}(\beta,r),$$
$$x_1=-ir\,\mathrm{sn}(\alpha,r)\,\mathrm{sn}(\beta,r),$$
$$x_2=(i/r')\,\mathrm{dn}(\alpha,r)\,\mathrm{dn}(\beta,r),$$

where $\alpha\in(iK',iK'+2K)$, $\beta\in(iK',\ -iK')$. The constants appearing in (4.24) are numbers which can in principle be determined by calculation in special cases of the integrand.

5. THE ROTATION GROUP IN AN ELLIPTIC BASIS

There has recently been an investigation by Patera and Winternitz[7] of the rotation group in a basis alternate to the usual one in which the component of angular momentum in a fixed direction is diagonalized. If the components of angular momentum are denoted by L_i $(i=1,2,3)$, satisfying the usual commutation relations $[L_i, L_j]=\epsilon_{ijk}L_k$, the operator which is diagonalized is $E=-4(L_1^2+r^2L_2^2)$, where $0<r^2<1$. In their work Patera and Winternitz examined the two variable realization on the sphere of $SO(3)$ and showed that in this basis the corresponding basis functions are ellipsoidal harmonics or products of Lamé polynomials as opposed to the conventional spherical harmonics in the canonical basis. The two-variable realization was discussed in detail in that paper together with the properties of the matrix relating the two bases. In that paper the authors were not, however, able to produce a realization of the single-variable model in which the basis functions were single Lamé polynomials. It is the purpose of this section to show that this can be done in a quite straightforward way. We also show how to relate the overlap coefficients to the coefficients of the Lamé polynomials.

The one-parameter model of the representations of the rotation group is realized on the space of polynomials $f(z)$ of order less than or equal to $2J$ (J = angular momentum) in the complex variable z. The invariant scalar product is so defined that

$$(z^{J-M}, z^{J-N}) = (J-M)!\,(J+M)!\,\delta_{MN}. \tag{5.1}$$

A canonical basis in this realization (i.e., one in which L_3 is diagonal) is

$$f_M^J = \frac{z^{J-M}}{[(J-M)!\,(J+M)!\,]^{1/2}}, \quad -J\le M\le J. \tag{5.2}$$

The generators of $SO(3)$ are

$$L_1=\tfrac12 i(1-z^2)\frac{d}{dz}+iJz, \quad L_2=\tfrac12(1+z^2)\frac{d}{dz}-Jz,$$

$$L_3 = iz\frac{d}{dz} - iJ. \tag{5.3}$$

The operator E can then be written

$$E=[(1-r)z^2-(1+r)][(1+r)z^2-(1-r)]\frac{d^2}{dz^2} + (2J-1)\,2z[1+r^2-z^2(1-r^2)]\frac{d}{dz} + 2J[1+r^2+(1-r^2)(2J-1)z^2]. \tag{5.4}$$

If we now write the eigenfunctions f of E in terms of new functions h, where

$$f(z)=(r')^J[(b-z^2)(1-bz^2)]^{J/2}h(z), \quad b=\frac{1+r}{1-r}, \tag{5.5}$$

and make the change of variable

$$\mathrm{sn}(w,r)=\frac{-i(1+b)z}{[(b-z^2)(1-bz^2)]^{1/2}}, \tag{5.6}$$

the operator E acting on the h functions has the form

$$\tfrac14 E = \frac{d^2}{dw^2} - r^2J(J+1)\,\mathrm{sn}^2(w,r). \tag{5.7}$$

The eigenvalue equation for E acting on the h functions is then the Lamé equation. The corresponding solutions are the Lamé polynomials. There are two cases to consider, viz., when J is even or odd.

Arscott[9] has shown that there are eight species of Lamé polynomials, four corresponding to even J and four to odd J. We shall consistently use his notation for the Lamé polynomials as it is very suggestive of the corresponding expansion of the Lamé polynomials in terms of Jacobi elliptic functions. In each case (J even or odd) the four corresponding polynomials form a complete basis for representation space. We now make these statements explicit.

Case 1, $J=2N$ $(N=1,2,\cdots)$

The complete basis set is

$$\Lambda_{Jm}^{++}=F^{2N}uE_{2N+2}^m(w), \quad \Lambda_{Jm}^{-+}=F^{2N}scE_{2N+2}^m(w),$$
$$\Lambda_{Jm}^{--}=F^{2N}sdE_{2N+2}^m(w), \quad \Lambda_{Jm}^{+-}=F^{2N}cdE_{2N+2}^m(w), \tag{5.8}$$

where $F=r'[(b-z^2)(1-bz^2)]^{1/2}$.

F can also be expressed in terms of w via Eq. (5.6), but we not do this here. The pair of discrete indices labeling the Λ functions are the eigenvalues of two discrete operators. The first of these is the reflection operator R which acts on functions f according to

$$Rf(z)=f(-z)$$

so that $R\,\Lambda_{Jm}^{pq}=p\,\Lambda_{Jm}^{pq}$. The second discrete label is related to the inversion operation I which acts on functions f according to

$$If(z)=z^{2J}f(1/z)$$

so that $I\Lambda_{Jm}^{pq}=q\Lambda_{Jm}^{pq}$. This method of labeling basis functions has been employed by Patera and Winternitz. The index m in each case labels the number of zeros of each Lamé polynomial appearing in the basis and hence also labels the basis vectors of a given type. For the basis function Λ_{Jm}^{++}, m lies in the range $0\le m\le N+1$; for all other basis functions we have the range $0\le m\le N$.

Case 2, $J=2N+1$ $(N=1,2,\cdots)$

The complete basis set is

$$\Lambda_{Jm}^{++}=F^{2N+1}cE_{2N+3}^m(w), \quad \Lambda_{Jm}^{+-}=F^{2N+1}dE_{2N+3}^m(w),$$
$$\Lambda_{Jm}^{--}=F^{2N+1}scdE_{2N+3}(w), \quad \Lambda_{Jm}^{-+}=F^{2N+1}sE_{2N+3}^m(w). \tag{5.9}$$

Here m varies between $0\le m\le N$ for Λ_{Jm}^{--} but varies between $0\le m\le N+1$ otherwise.

The calculation of the nonzero elements of the overlap matrix relating the E or Lamé basis to the canonical basis can be achieved by writing down the equation

$$\Lambda_{Jm}^{pq}=\sum_{M>0}(X_J^p)_{m,M}\frac{1}{[(J-M)!\,(J+M)!\,]^{1/2}}(z^{J-M}+pz^{J+M}), \tag{5.10}$$

where the summation extends over those M for which $(-1)^{J+M}=q$. All that is required is then the writing out of the left-hand side as a polynomial in z and equating coefficients. We shall illustrate this calculation in the particular case of the coefficient $(X_{2N}^+)_{m,2q}$ corresponding to the basis function Λ_{2Nm}^{++} on the left-hand side of (5.10).

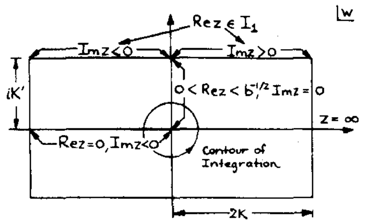

FIG. 2. The mapping $\mathrm{sn}w = -i(1+b)z/[(1-bz^2)(b-z^2)]^{1/2}$ in the w plane. In order to make this a single valued map, the z plane has two cuts along the intervals $I_1=[b^{-1/2},\ b^{1/2}]$ and $I_2=[-b^{-1/2},\ -b^{1/2}]$. The lines $w=2K+iv$ and $w=-2K+iv$ with $-K'<\mathrm{Im}v<K'$ are identified.

Written in terms of the variable z the basis function Λ^{++}_{2Nm} can be expressed in the form

$$\Lambda^{++}_{2N,m}=r'^{2N}\sum_{p=0}^{N}(-1)^p(1+b)^{2p}a^m_{2p}\times[(b-z^2)(1-bz^2)]^{N-p}z^{2p}, \qquad (5.11)$$

where $uE^m_{2N+2}(w)=\sum_{p=0}^{N}a^m_{2p}\,sn^{2p}w$ and the coefficients satisfy the recurrence relations

$$\lambda^{++}_m a^m_0+2a^m_2=0,$$

$$(2N-2p+2)(2N+2p-1)r^2a^m_{2p-2}+[4(1+r^2)p^2-\lambda^{++}_m]a^m_{2p}-(2p+1)(2p+2)a^m_{2p+2}=0, \qquad (5.12)$$

where $4\lambda^{++}_m$ is the eigenvalue of the operator E. Equating coefficients on both sides of (5.10), we obtain

$$(X^+_{2N})_{m,2q}=[(2N-2q)!\,(2N+2q)!\,]^{1/2}\sum_{p=0}^{N}2^{2p}a^m_{2p}\times\sum_{u,v}(-1)^{p+u+v}\,C\binom{N-p}{u}\binom{N-p}{v}(1+r)^{2N-p-u+v}(1-r)^{u-p-v}. \qquad (5.13)$$

For $0\leq p<N-q$ the u,v summation is over integers u,v such that $0\leq u+v\leq N-q-p$. For $N-q\leq p\leq N$, $u=v=0$. This expression then relates the overlap matrix to the coefficients a^m_{2p} of the expansion of Lamé polynomials in terms of Jacobi elliptic functions as given by Arscott. Similar calculations can be made for the other nonzero elements of the matrix $(X^p_J)_{m,M}$.

It is also possible to map the one-variable model we have examined thus far, into the two variable model of the rotation group realized as square integrable functions on the three-dimensional sphere. This is achieved by the following means. With each function $f(z)$ we associate a function on the sphere given by

$$F_J(\mathbf{x})=\frac{J!}{2\pi i}\int_C\left(\frac{\mathbf{x}\cdot\mathbf{v}}{z^2}\right)^J f(z)\,\frac{dz}{z}. \qquad (5.14)$$

Here $\mathbf{x}$ is a point on the two-dimensional unit sphere, i.e., $\mathbf{x}=(x_1,x_2,x_3)$, $x_1^2+x_2^2+x_3^2=1$ and $\mathbf{v}=[\frac{1}{2}i(z^2-1),\ \frac{1}{2}i(z^2+1),\ z]$. The contour of integration is any closed path around the origin.

1. *Canonical basis*: Substituting the basis vector f^J_M in this expression, we get

$$F_{JM}(\theta,\phi)=\frac{(J!)^2}{(J-M)!\,(J+M)!}\,i^M P^J_{M0}(\cos\theta)\exp(-iM\phi), \qquad (5.15)$$

where $P^J_{MN}(\cos\theta)$ is the matrix element of a rotation about the x axis in the canonical basis. The point $\mathbf{x}$ on the sphere is parametrized as

$$\mathbf{x}=(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

2. *The elliptic basis*: In this case it is convenient to make the change of variable indicated in Eq. (5.6). The resulting integral is then

$$F^{pq}_{Jm}(\alpha,\beta)=\frac{J!}{2\pi i}\int_C(1-r)^J[iK\,\mathrm{sn}\alpha\,\mathrm{sn}\beta\,\mathrm{sn}w-\mathrm{dn}\alpha\,\mathrm{dn}\beta\,\mathrm{dn}w-r\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\,\mathrm{cn}w]^J(\mathrm{sn}w)^{-2J}E^{pq}_{Jm}(w)\,\frac{dw}{\mathrm{sn}\,w}, \qquad (5.16)$$

where $E^{pq}_{Jm}(w)$ is one of the Lamé polynomials which form the particular basis for given J, e.g., $E^{++}_{2Nm}(w)=uE^m_{2N+2}(w)$. The integration is over a contour which encloses the origin in the w plane and lies strictly inside the square in the complex w plane with vertices $(2K,\ \pm iK')$ and $(2K,\ \pm iK')$. The situation is illustrated in Fig. 2, where the details of the mapping are shown together with a possible contour. The coordinates on the sphere are given by the relations

$$\mathbf{x}=((1/r')\,\mathrm{dn}(\alpha,r)\,\mathrm{dn}(\beta,r),\ -(ir/r')\,\mathrm{cn}(\alpha,r)\,\mathrm{cn}(\beta,r),\ -r\,\mathrm{sn}(\alpha,r)\,\mathrm{sn}(\beta,r))$$

with $\alpha\in(-2K,\ 2K)$, $\beta\in(-K,\ -K+2iK')$.

In each case the integral (5.16) and hence $F^{pq}_{Jm}(\alpha,\beta)$ is expressible in terms of a product of Lamé polynomials of the type appearing in the integral, e.g.,

$$F^{++}_{2Nm}(\alpha,\beta)=\lambda^m_N uE^m_{p2N+2}(\alpha,\beta)=\lambda\ uE^m_{2N+2}(\alpha)\,uE^m_{2N+2}(\beta),$$

where we have used the notation of Arscott for the product of two Lamé polynomials. In each case λ is a constant of proportionality which can in principle be calculated. This result can readily be obtained by considering the properties of the integral under the discrete operators R and I as well as using the fact that the integral satisfies the Laplace equation and is symmetric in α and β.

In order to make this a single valued map, the z plane has two cuts along the intervals $I_1=[b^{-1/2},b^{1/2}]$ and $I_2=[-b^{-1/2},\ -b^{1/2}]$. Because of the periodicity of the elliptic functions the lines $2K+iv$ and $-2K+iv$, where $-K'<v<K'$ are identified.

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