Structure relations for the symmetry algebras of classical and quantum superintegrable systems

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Abstract 1

A quantum superintegrable system is an integrable n-dimensional Hamiltonian system with potential: $H = \Delta_n + V$ that admits $2n-1$ algebraically independent partial differential operators commuting with the Hamiltonian, the maximum number possible. The system is of order ℓ if the maximum order of the symmetry operators other than the Hamiltonian is ℓ. Typically, the algebra generated by the symmetry operators and their commutators has been proven to close.

Abstract 2

Similarly, a classical superintegrable system is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent phase space functions, polynomial in the momenta, whose Poisson brackets with the Hamiltonian vanish.
Superintegrability captures what it means for a Hamiltonian system to be explicitly algebraically and analytically solvable, not just solvable numerically. Until recently there were very few examples of superintegrable systems of order ℓ with $\ell > 3$ and and virtually no structure results.

Absract 3

The situation has changed dramatically in the last two years with the discovery of families of systems depending on a rational parameter $k = p/q$ that are superintegrable for all k and of arbitrarily high order, such as $\ell = p + q + 1$.

We review a method, based on recurrence formulas for special functions, that proves superintegrability of quantum systems, and we introduce the classical counterpart of this method. Just a few months ago, these constructions seemed out of reach.

Maximal Superintegrability (Quantum)

Schrödinger operator

$$H = \Delta + V(\mathbf{x})$$

where $\Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} g^{ij}) \partial_{x_j}$ is the Laplace-Beltrami operator on a Riemannian manifold, expressed in local coordinates x_j. The system admits $2n-1$ algebraically independent differential symmetry operators

$$S_j, \quad j = 1, \cdots, 2n-1$$

with $S_1 = H$ and $[H, S_j] \equiv HS_j - S_j H = 0$.

Order of a System

- A superintegrable system is of order ℓ if ℓ is the maximum order of the generating symmetries other than the Hamiltonian as a differential operator for the quantum system).
- Systems with $\ell = 1, 2$ have been the most tractible due to the association with separation of variables.

Superintegrability (Classical)

Hamiltonian

$$\mathcal{H} = \sum g^{jk} p_j p_k + V(\mathbf{x})$$

on phase space with local coordinates $x_1, \cdots, x_n, p_1, \cdots, p_n$ where $ds^2 = \sum g^{jk} dx_j \, dx_k$ is the Riemannian. Superintegrable if there are $2n-1$ functionally independent functions

$$\mathcal{S}_j(\mathbf{p}, \mathbf{x}), \quad j = 1, \cdots, 2n-1$$

with $\mathcal{S}_1 = \mathcal{H}$, polynomial in the momenta, and such that $\{\mathcal{S}_j, \mathcal{H}\} = 0$, where $\{\cdot, \cdot\}$ is the Poisson bracket.

Order of a Classical System

- A classical superintegrable system is of order ℓ if ℓ is the maximum order of the generating symmetries other than the Hamiltonian as polynomials in the momenta.
- Systems with $\ell = 1, 2$ have been the most tractible due to the association with separation of variables.

Integrability and Superintegrability

- An integrable system has n algebraically independent commuting symmetry operators.
- A superintegrable system has $2n-1$ algebraically independent symmetry operators. For many researchers the definition of superintegrability also requires integrability.
- The symmetry algebra generated by the symmetries of a merely integrable system is abelian. The algebra corresponding to a superintegrable system is nonabelian.

Historical Significance

- The isotropic harmonic oscillator is classically and quantum second order superintegrable.
- Celestial navigation including sending satellites into lunar and earth orbits depends on the superintegrability of the Kepler system (Hohmann transfer).
- The quantum Coulomb or hydrogen atom problem is second order superintegrable. Thus the spectrum can be deduced algebraically. An important reason that the Balmer series was discovered far in advance of quantum mechanics.

Other Significant Links

- Quasi-Exactly Solvable quantum systems.
- Supersymmetry.
- PT-symmetric systems.
- Deep connections with special functions and orthogonal polynomials: 1) Solutions of the Schrödinger eigenvalue equation 2) Irreducible representations of the symmetry algebras.

A 2D Example 1

Here $n = 2$, so $2n - 1 = 3$. Hamiltonian:

$$H = \partial_x^2 + \partial_y^2 - \alpha^2(4x^2 + y^2) + bx + \frac{\frac{1}{4} - c^2}{y}.$$

Generating symmetry operators: H, L_1, L_2 where

$$L_1 = \partial_x^2 - 4\alpha^2 x^2 - bx$$

$$L_2 = \frac{1}{2}\{M, \partial_y\} + y^2(\frac{b}{4} - x\alpha^2) - (\frac{1}{4} - c^2)\frac{x}{y^2}.$$

Here $M = x\partial_y - y\partial_x$ and $\{A, B\} = AB + BA$.

A 2D Example 2

Add the commutator $R=[L_1,L_2]$ to the symmetry algebra. Then

$$[L_1,R]=2bH+16\alpha^2L_2-2bL_1$$

$$[L_2,R]=8L_1H-6L_1^2-2H^2+2bL_2-8\alpha^2(1-c^2)$$

$$R^2+4L_1^3+4L_1H^2-8L_1^2H+16\alpha^2L_2^2+4bL_2H-2b\{L_1,L_2\}$$

$$+16\alpha^2(3-c^2)L_1-32\alpha^2H-b^2(1-c^2)=0$$

A 2D Example 3

- We have a second order superintegrable system whose symmetry algebra closes at level 6.
- If Ψ is an eigenvector of H with eigenvalue E, i.e., $H\Psi = E\Psi$, and L is a symmetry operator then also $H(L\Psi) = E(L\Psi)$, so the symmetry algebra preserves eigenspaces of H.
- Thus we can use the irreducible representations of the symmetry algebra to explain the "accidental" degeneracies of the eigenspaces of H.

A 2D Example 4

By classifying the finite dimensional irreducible representations of the symmetry algebra we can show that the possible bound state energy levels must take the form

$$E_m = 4\alpha(m - 2c) + \frac{b^2}{16\alpha^2}$$

where $m = 0, 1, \cdots$ and the multiplicity of the energy level E_m is $m + 1$.

Basic 2D Structure Results

If $\mathcal{H}$ is the Hamiltonian of a 2D second order superintegrable system with nondegenerate (3 parameter) potential then

- 3D space of 2nd order symmetries.
- 1D space of 3rd order symmetries.
- 6D space of 4th order symmetries and spanned by symmetric polynomials in 2nd order symmetries.
- 10D space 6th order symmetries and spanned by symmetric polynomials in 2nd order symmetries.

More Examples 8

$$H = \partial_x^2 + \partial_y^2 + \hbar^2\omega_1^2 P_1(\omega_1 x) + \hbar^2\omega_2^2 P_1(\omega_2 y)$$

Here, P_1 is the first Painlevè transcendent. Third order quantum superintegrable but not classically superintegrable. Other examples with the transcendents P_2, P_3, P_4.

2-sphere Example 1

The quantum system on the 2-sphere, the spherical analog of the hydogen atom, is determined by

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{\alpha s_3}{\sqrt{s_1^2 + s_2^2}},$$

where, J_1, J_2, J_3 are the angular momentum operators, $s_1^2 + s_2^2 + s_3^2 = 1$ and $H\Psi = E\Psi$ where $\Psi(s_1, s_2, s_3)$ is a function on the unit sphere, square integrable with respect to the area measure on the sphere.

2-sphere Example 2

The quantum basis for the symmetry operators is

$$L_1 = J_1J_3+J_3J_1-\frac{\alpha s_1}{\sqrt{s_1^2+s_2^2}},\ L_2 = J_2J_3+J_3J_2-\frac{\alpha s_2}{\sqrt{s_1^2+s_2^2}},$$

$X = J_3$. The structure relations are $[H, L_1] = [H, L_2] = [H, X] = 0$, $[X, L_1] = -L_2$, $[X, L_2] = L_1$, $[L_1, L_2] = 4HX - 8X^3 + X$. The Casimir relation is

$$L_1^2 + L_2^2 + 4X^4 - 4HX^2 + H - 5X^2 - \alpha^2 = 0.$$

2-sphere Example 3

- The symmetry operators leave each eigenspace of H invariant.
- Thus the degeneracies can be explained in terms of dimensions of the irreducible representations of the symmetry algebra.
- Use a highest weight vector argument to find these finite dimensional representations.
- Energy levels are

$$E_m = -\frac{1}{4}(2m+1)^2 + \frac{1}{4} + \frac{\alpha^2}{(2m+1)^2},$$

with multiplicity $m+1$

2-sphere Example 4

- Classical analogs of Kepler's 3 laws hold, due to superintegrability.
- Hohmann transfer for obital manuvers in a spherical universe hold.
- Quantum spectrum follows from irreducible representations of the symmetry algebra. Also solutions with real spectrum

$$E_n = -\frac{1}{4}(2\mu_n + 1)^2 + \frac{1}{4} - \frac{a^2}{(2\mu_n + 1)^2},$$

where $2\mu_n = n + \sqrt{(n+1)^2 + 2a}$ and $\alpha = ia$ is real. This is an example of a PT-symmetry. Superintegrability theory provides a means of discovery of these systems.

The Problem

- For possible superintegrable systems of order > 2 it is much harder to verify superintegrability and to compute the symmetry algebra.
- How can we compute efficiently the commutators and products of symmetry operators of arbitrarily high order?

Recurrence Approach 1

IDEA:

- Requires a 2nd order symmetry that determines a separation of variables.
- Formal eigenspaces of Hamiltonian are invariant under action of any symmetry operator, so the operator must induce recurrence relations for the basis of separated eigenfunctions.
- Use the known recurrence relations for hypergeometric functions to reverse this process and determine a symmetry operator from the recurrence relations.

Recurrence Approach 2

- We can compute the symmetry operators and structure equations by restricting ourselves to a formal “basis” of separated eigenfunctions.
- Then we appeal to our theory of canonical forms for symmetry operators to show that results obtained on restriction to a formal eigenbasis actually hold as true identities for purely differential operators defined independent of “basis” functions.

The TTW System 1

Recent winner of the Journal of Physics A Best Paper Prize! The quantum Tremblay, Turbiner, Winternitz system (in polar ccordinates in the plane) is

$$H = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 - \omega^2 r^2 + \frac{1}{r^2}\left(\frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)}\right)$$

where $k = \frac{p}{q}$ is rational. Solution of $H\Psi = E\Psi$ is

$$\Psi = e^{-\frac{\omega}{2}r^2} r^{k(2n+a+b+1)} L_m^{k(2n+a+b+1)}(\omega r^2)(\sin(k\theta))^{a+\frac{1}{2}}$$

$$(\cos(k\theta))^{b+\frac{1}{2}} P_n^{a,b}(\cos(2k\theta)),$$

The TTW System 2

Here $\alpha = k^2(\frac{1}{4} - a^2)$ and $\beta = k^2(\frac{1}{4} - b^2)$. The L_B^A are associated Laguerre functions and the $P_n^{a,b}$ are Jacobi functions.

For $k = 1$ this is the caged isotropic oscillator, for $k = 2$ it is a Calogero system on the line, etc.

Use gauge transformation to get eigenfunctions

$$\Pi = e^{-\frac{\omega}{2}r^2} r^{k(2n+a+b+1)} L_m^{k(2n+a+b+1)}(\omega r^2) P_n^{a,b}(x)$$

$$= Y_m^A(r) X_n^{a,b}(x)$$

where $x = \cos(2k\theta)$.

The TTW System 3

$$\text{Energy}: \quad E = -2\omega\left[2(m+nk)+1+(a+b+1)k\right]$$

$$\text{Separation}: \; L_2\Psi = (\partial_\theta^2 + \frac{\alpha}{\sin^2(k\theta)} + \frac{\beta}{\cos^2(k\theta)})\Psi = -A^2\Psi$$

where $A = k(2n+a+b+1)$, $k = p/q$.

Look for recurrences that change m, n but fix $u = m+nk$. The two transformations

$(1) \quad n \rightarrow n+q, \quad m \rightarrow m-p,$

$(2) \quad n \rightarrow n-q, \quad m \rightarrow m+p$

each achieve this.

The TTW System 4

Consider $X_n^{a,b}(x)$. Raise or lower n with

$$(1): \quad J_n^+ X_n^{a,b} = (2n+a+b+2)(1-x^2)\partial_x X_n^{a,b}$$

$$+(n+a+b+1)(-(2n+a+b+2)x-(a-b))X_n^{a,b}$$
$$= 2(n+1)(n+a+b+1)X_{n+1}^{a,b},$$

$$(2): \quad J_n^- X_n^{a,b} = -(2n+a+b)(1-x^2)\partial_x X_n^{a,b}$$

$$-n((2n+a+b)x-(a-b))X_n^{a,b}$$
$$= 2(n+a)(n+b)X_{n-1}^{a,b}.$$

The TTW System 5

For $\mathcal{Y}^A_m(R) = \omega^{A/2} Y^A_m(r)$ where $R = r^2$ change m by

$$(1): \quad K^+_{A,m}\mathcal{Y}^A_m = \left\{(A+1)\partial_R - \frac{E}{4} - \frac{1}{2R}A(A+1)\right\}\mathcal{Y}^A_m$$

$$= -\omega\mathcal{Y}^{A+2}_{m-1},$$

$$(2): \quad K^-_{A,m}\mathcal{Y}^A_m = \left\{(-A+1)\partial_R - \frac{E}{4} + \frac{1}{2R}A(1-A)\right\}\mathcal{Y}^A_m$$

$$= -\omega(m+1)(m+A)\mathcal{Y}^{A-2}_{m+1}.$$

The TTW System 6

We construct the two operators

$$\Xi_+ = K^+_{A+2(p-1),m-(p-1)} \cdots K^+_{A,m} J^+_{n+q-1} \cdots J^+_n$$

$$\Xi_- = K^-_{A-2(p-1),m+p-1} \cdots K^-_{A,m} J^-_{n-q+1} \cdots J^-_n .$$

For fixed $u = m + kn$, we have

$$\Xi_+ \Psi_n = 2^q (-1)^p \omega^p (n+1)_q (n+a+b+1)_q \Psi_{n+q},$$

$$\Xi_- \Psi_n = 2^q \omega^p (-n-a)_q (-n-b)_q (u-kn+1)_p$$
$$(-u-k[n+a+b+1])_p \Psi_{n-q}.$$

The TTW System 7

These are basis dependent operators. However, under the transformation $n \to -n-a-b-1$ we have $\Xi_+ \to \Xi_-$ and $\Xi_- \to \Xi_+$. Thus $\Xi = \Xi_+ + \Xi_-$ is a polynomial in $(2n+a+b+1)^2$ and u is unchanged under this transformation.

Therefore we can replace $(2n+a+b+1)^2$ by L_2/k^2 and E by H everywhere they occur, and express Ξ as a pure differential symmetry operator.

The TTW System 8

Note that under the transformation $n \rightarrow -n-a-b-1$ the operator $\Xi_+ - \Xi_-$ changes sign, hence the operator $\tilde{\tilde{\Xi}} = (1/(2n+a+b+1))(\Xi_+ - \Xi_-)$ is unchanged under this transformation.

This defines $\tilde{\tilde{\Xi}}$ as a symmetry operator. We set $L_3 = \Xi$, $L_4 = \tilde{\tilde{\Xi}}$.

Remark

- We have shown that L_3, L_4 commute with H on any formal eigenbasis.
- In fact, we have constructed pure differential operators which commute with H, independent of basis.
- This takes some proof and we have verified this in general using a canonical form for higher order differential operators.
- Indeed, we can prove that operator relations verified on formal eigenbases of separated solutions must actually hold identically.

The TTW System 9

Applying a product of raising and lowering operators to a basis function we obtain

$$\Xi_-\Xi_+\Psi_n = (-1)^p 4^q \omega^{2p}(n+1)_q(n+a+1)_q(n+b+1)_q$$

$$\times(n+a+b+1)_q(-u+kn)_p(u+k[n+a+b+1]+1)_p\Psi_n$$

$$= \xi_n\Psi_n,$$

$$\Xi_+\Xi_-\Psi_n = (-1)^p 4^q \omega^{2p}(-n)_q(-n-a)_q(-n-b)_q$$

$$\times(-n-a-b)_q(u-kn+1)_p(-u-k[n+a+b+1])_p\Psi_n$$

$$= \eta_n\Psi_n.$$

The TTW System 10

Thus $\Xi^{(+)} = \Xi_-\Xi_+ + \Xi_+\Xi_-$ multiplies any basis function by $\xi_n + \eta_n$.

However, the transformation $n \longrightarrow -n - a - b - 1$ maps $\Xi_-\Xi_+ \leftrightarrow \Xi_+\Xi_-$ and $\xi_n \leftrightarrow \eta_n$.

Thus $\Xi^{(+)}$ is an even polynomial operator in $(2n + a + b + 1)$, polynomial in u, and $\xi_n + \eta_n$ is an even polynomial function in $(2n + a + b + 1)$, polynomial in u.

The TTW System 11

Furthermore, each of $\Xi_-\Xi_+$ and $\Xi_+\Xi_-$ is unchanged under $u \longrightarrow -u-(a+b+1)-1$, hence a polynomial of order p in $[2u+(a+b+1)k+1]^2$.

We conclude that

$$\Xi^{(+)} = P^{(+)}(H^2, L_2, \omega^2, a, b).$$

Similarly

$$\Xi^{(-)} = (\Xi_-\Xi_+ - \Xi_+\Xi_-)/(2n+a+b+1)$$
$$= P^{(-)}(H^2, L_2, \omega^2, a, b)$$

The TTW System 12

Setting $R = -4k^2qL_3 - 4k^2q^2L_4$, we find the structure equations

$$[L_2, L_4] = R, \quad [L_2, R] = -8k^2q^2\{L_2, L_4\} - 16k^4q^4L_4,$$

$$[L_4, R] = 8k^2q^2L_4^2 - 8k^2qP^{(-)}(H^2, L_2, \omega^2, a, b),$$

$$\frac{3}{8k^2q^2}R^2 = -22k^2q^2L_4^2 + \{L_2, L_4, L_4\}$$

$$+4k^2qP^{(-)}(H^2, L_2, \omega^2, a, b) + 12k^2P^{(+)}(H^2, L_2, \omega^2, a, b).$$

Thus H, L_2, L_4 generate a closed algebra.

Classical TTW 1

In terms of polar coordinates with $r = e^R$ the TTW Hamiltonian is

$$\mathcal{H} = e^{-2R}(p_R^2 + p_\theta^2 - \omega^2 e^{4R} + \frac{\alpha}{\cos^2(k\theta)} + \frac{\beta}{\sin^2(k\theta)}).$$

The second order constant of the motion corresponding to separation of variables in polar coordinates, is

$$\mathcal{L}_2 = p_\theta^2 + \frac{\alpha}{\cos^2 k\theta} + \frac{\beta}{\sin^2 k\theta}.$$

Classical TTW 2

To demonstrate superintegrability we need to exhibit a third polynomial constant of the motion, functionally independent of $\mathcal{H}, \mathcal{L}_2$. The method for constructing this symmetry is based on action angle variables and the existence of an additively separable coordinate system for the Hamilton-Jacobi equation,

Classical TTW 3

Using separation in polar coordinates we can exploit an action angle construction and find “raising and lowering" symmetries $\mathcal{L}^+, \mathcal{L}^-$ such that

$$\{\mathcal{L}_2, \mathcal{L}^{\pm}\} = \mp 4\sqrt{\mathcal{L}_2}\mathcal{L}^{\pm}$$

where $\{., .\}$ is the Poisson bracket. Here, $\mathcal{L}^{\pm}$ are not polynomials in the momenta.

Classical TTW 7

Now let $\mathcal{L}_4 = \frac{1}{i\sqrt{\mathcal{L}_2}}(\mathcal{L}^- - \mathcal{L}^+),\ \mathcal{L}_3 = \mathcal{L}^+ + \mathcal{L}^-,$
Then $\mathcal{L}_3, \mathcal{L}_4$ ARE polynomial constants of the motion and generate the symmetry algebra

Classical TTW 8

Setting $\mathcal{R} = 4p\mathcal{L}_3$ we derive

$$\{\mathcal{L}_2, \mathcal{L}_4\} = \mathcal{R}, \ \{\mathcal{L}_2, \mathcal{R}\} = -16p^2\mathcal{L}_2\mathcal{L}_4,$$

$$\{\mathcal{L}_4, \mathcal{R}\} = 16p^2\mathcal{L}_4^2 - 32p^2\frac{\partial P}{\partial \mathcal{L}_2},$$

$$\mathcal{R}^2 = -16p^2\mathcal{L}_2\mathcal{L}_4^2 + 64p^2 P(\mathcal{L}_2, \mathcal{H}),$$

where

$$P(\mathcal{L}_2, \mathcal{H}) = ((\mathcal{L}_2 - \alpha - \beta)^2 - 4\alpha\beta)^q(\mathcal{H}^2 + 4\omega^2\mathcal{L}_2)^p.$$

Final Comments

- The approach generalizes to higher dimensions, for both quantum and classical systems.
- There are quantum superintegrable systems in which the separated eigenfunctions are not of hypergeometric type. How do we treat these systems?
- In many cases in dimensions $n \geq 3$ the symmetry algebra closes rationally but not polynomially.
- Joint work with E.G. Kalnins (Hamilton, NJ) and J. M. Kress (Sydney).