

Quadratic algebra contractions and 2nd order superintegrable systems

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Abstract

Quadratic algebras are generalizations of Lie algebras; they include the symmetry algebras of 2nd order superintegrable systems in 2 dimensions as special cases. The superintegrable systems are exactly solvable physical systems in classical and quantum mechanics. We describe a contraction theory for quadratic algebras and show that for constant curvature superintegrable systems, ordinary Lie algebra contractions induce contractions of the quadratic algebras of the superintegrable systems that correspond to geometrical pointwise limits of the physical systems. One consequence is that by contracting function space realizations of representations of the generic superintegrable quantum system on the 2-sphere (which give the structure equations for Racah/Wilson polynomials) to the other superintegrable systems one obtains the full Askey scheme of orthogonal hypergeometric polynomials.

Outline

- 1 Introduction
- 2 Representatives of Nondegenerate Systems
- 3 Representatives of Degenerate Systems
- 4 Models of Superintegrable Systems
- 5 The S_9 Difference Operator Model
- 6 Lie algebra contractions
- 7 Contractions of Superintegrable Systems
 - D2 contractions
- 8 Discussion and Conclusions

Superintegrable Systems: $H\Psi = E\Psi$

- A quantum superintegrable system is an integrable Hamiltonian system on an n -dimensional Riemannian/pseudo-Riemannian manifold with potential:

$$H = \Delta_n + V$$

that admits $2n - 1$ algebraically independent partial differential operators commuting with H , the maximum possible.

$$[H, L_j] = 0, \quad L_{2n-1} = H, \quad n = 1, 2, \dots, 2n - 1.$$

- Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically.
- A system is of order K if the maximum order of the symmetry operators, other than H , is K . For $n = 2$, $K = 1, 2$ all systems are known.

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1st order systems

These are the (zero-potential) Laplace-Beltrami eigenvalue equations on constant curvature spaces.

Simplest examples: **Euclidean Helmholtz equation** $(P_1^2 + P_2^2)\Phi = -\lambda^2\Phi$ (or the **Klein-Gordon equation** $(P_1^2 - P_2^2)\Phi = -\lambda^2\Phi$), and **Laplace -Beltrami eigenvalue equation on the 2-sphere** $(J_1^2 + J_2^2 + J_3^2)\Psi = -j(j+1)\Psi$.

Here the symmetry algebras close under commutation to form the Lie algebras $e(2, \mathbb{R})$, $e(1, 1)$ or $o(3, \mathbb{R})$.

One can find 2-variable differential operator models of the irreducible representations of these Lie algebras in which the basis eigenfunctions are the spherical harmonics ($o(3, \mathbb{R})$) or Bessel functions ($e(2, \mathbb{R})$).

Free 2nd order superintegrable systems, (no potential, $K = 2$)

We apply these ideas to 2nd order systems in 2D ($2n - 1 = 3$). The complex spaces with Laplace-Beltrami operators admitting at least three 2nd order symmetries were classified by Koenigs (1896). They are:

- **The two constant curvature spaces**, six linearly independent 2nd order symmetries and three 1st order symmetries,
- **The four Darboux spaces**, four 2nd order symmetries and one 1st order symmetry,

$$ds^2 = 4x(dx^2 + dy^2), \quad ds^2 = \frac{x^2 + 1}{x^2}(dx^2 + dy^2),$$

$$ds^2 = \frac{e^x + 1}{e^{2x}}(dx^2 + dy^2), \quad ds^2 = \frac{2 \cos 2x + b}{\sin^2 2x}(dx^2 + dy^2),$$

- **Eleven 4-parameter Koenigs spaces**. No 1st order symmetries. Example

$$ds^2 = \left(\frac{c_1}{x^2 + y^2} + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4 \right) (dx^2 + dy^2).$$

2nd order systems with potential, $K = 2$

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of H and their multiplicity
- All the 2nd order superintegrable systems are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere, **S9** in our listing. Analogously all quadratic symmetry algebras of these systems are contractions of **S9**.

$$\mathbf{S9} : \quad H = \Delta_2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad s_1^2 + s_2^2 + s_3^2 = 1,$$

$$L_1 = (s_2 \partial_{s_3} - s_3 \partial_{s_2})^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2, \quad L_3,$$

$$H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$$

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3 types of 2nd order superintegrable systems:

1 Nondegenerate:

$$V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2 V_{(2)}(\mathbf{x}) + a_3 V_{(3)}(\mathbf{x}) + a_4$$

2 Degenerate:

$$V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2$$

3 Free:

$$V = a_1, \text{ no potential}$$

Nondegenerate systems ($2n - 1 = 3$ generators)

The symmetry algebra generated by H, L_1, L_2 always closes under commutation. Define 3rd order commutator R by $R = [L_1, L_2]$. Then

$$[L_j, R] = A_1^{(j)} L_1^2 + A_2^{(j)} L_2^2 + A_3^{(j)} H^2 + A_4^{(j)} \{L_1, L_2\} + A_5^{(j)} HL_1 + A_6^{(j)} HL_2$$

$$+ A_7^{(j)} L_1 + A_8^{(j)} L_2 + A_9^{(j)} H + A_{10}^{(j)}, \quad \{L_1, L_2\} = L_1 L_2 + L_2 L_1,$$

$$R^2 = b_1 L_1^3 + b_2 L_2^3 + b_3 H^3 + b_4 \{L_1^2, L_2\} + b_5 \{L_1, L_2^2\} + b_6 L_1 L_2 L_1 + b_7 L_2 L_1 L_2$$

$$+ b_8 H \{L_1, L_2\} + b_9 HL_1^2 + b_{10} HL_2^2 + b_{11} H^2 L_1 + b_{12} H^2 L_2 + b_{13} L_1^2 + b_{14} L_2^2 + b_{15} \{L_1, L_2\}$$

$$+ b_{16} HL_1 + b_{17} HL_2 + b_{18} H^2 + b_{19} L_1 + b_{20} L_2 + b_{21} H + b_{22},$$

This structure is an example of a quadratic algebra. If we know the expansion for R^2 we can compute the other structure relations.

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This structure is an example of a **quadratic algebra**. If we know the expansion for R^2 we can compute the other structure relations.

Degenerate systems ($2n - 1 = 3$)

There are 4 generators: one 1st order X and 3 second order H, L_1, L_2 .

$$[X, L_j] = C_1^{(j)}L_1 + C_2^{(j)}L_2 + C_3^{(j)}H + C_4^{(j)}X^2 + C_5^{(j)}, \quad j = 1, 2,$$

$$[L_1, L_2] = E_1\{L_1, X\} + E_2\{L_2, X\} + E_3HX + E_4X^3 + E_5X,$$

Since $2n - 1 = 3$ there must be an identity satisfied by the 4 generators. It is of 4th order:

$$c_1L_1^2 + c_2L_2^2 + c_3H^2 + c_4\{L_1, L_2\} + c_5HL_1 + c_6HL_2 + c_7X^4 + c_8\{X^2, L_1\} + c_9\{X^2, L_2\} \\ + c_{10}HX^2 + c_{11}XL_1X + c_{12}XL_2X + c_{13}L_1 + c_{14}L_2 + c_{15}H + c_{16}X^2 + c_{17} = 0$$

If we know the 4th order identity, we can compute the other structure relations to within an overall scale factor.

Stäckel Equivalence Classes

All 2nd order 2d superintegrable systems with potential are known. There are 59 types, on a variety of manifolds, but under **the Stäckel transform, an invertible structure preserving mapping**, they divide into 12 equivalence classes with representatives on flat space and the 2-sphere, 6 with nondegenerate 3-parameter potentials

$$\{S9, E1, E2, E3', E8, E10\}$$

and 6 with degenerate 1-parameter potentials

$$\{S3, E3, E4, E5, E6, E14\}$$

Example: S9

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

where $J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

Basis symmetries: ($J_3 = s_2 \partial_{s_1} - s_1 \partial_{s_2}, \dots$)

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \quad L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

Structure equations:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 + \frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1 a_2 + a_2 a_3 + a_3 a_1) + 64a_1 a_2 a_3, \quad R = [L_1, L_2].$$

Interesting 2 variable O.P's: Prorial-Karlin-MacGregor, (right triangle)

Example: S9

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

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Interesting 2 variable O.P.'s: Prorial-Karlin-MacGregor, (right triangle)

Example: E_1

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + \frac{b_1}{x^2} + \frac{b_2}{y^2}$$

Generators:

$$L_1 = \partial_x^2 - \omega^2 x^2 + \frac{b_1}{x^2}, \quad L_2 = \partial_y^2 - \omega^2 y^2 + \frac{b_2}{y^2}, \quad L_3 = (x\partial_y - y\partial_x)^2 + y^2 \frac{b_1}{x^2} + x^2 \frac{b_2}{y^2}$$

Structure relations:

$$[R, L_1] = 8L_1^2 - 8HL_1 - 16\omega^2 L_3 + 8\omega^2,$$

$$[R, L_3] = 8HL_3 - 8\{L_1, L_3\} + (16b_1 + 8)H - 16(b_1 + b_2 + 1)L_1,$$

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Relations between nondegenerate/degenerate systems

- Every 1-parameter potential can be obtained from some 3-parameter potential by parameter restriction.
- It is not simply a restriction, however, because the structure of the symmetry algebra changes.
- A formally skew-adjoint 1st order symmetry appears and this induces a new 2nd order symmetry.
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Example: S3 (Higgs oscillator)

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a}{s_3^2}$$

The system is the same as S9 with $a_1 = a_2 = 0$, $a_3 = a$ with the former L_2 replaced by

$$L_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1) - \frac{a s_1 s_2}{s_3^2}$$

and

$$X = J_3 = s_2 \partial_{s_3} - s_3 \partial_{s_2}.$$

Structure relations:

$$[L_1, X] = 2L_2, [L_2, X] = -X^2 - 2L_1 + H - a, [L_1, L_2] = -(L_1 X + X L_1) - \left(\frac{1}{2} + 2a\right)X,$$

$$\frac{1}{3} \left(X^2 L_1 + X L_1 X + L_1 X^2 \right) + L_1^2 + L_2^2 - H L_1 + \left(a + \frac{11}{12} \right) X^2 - \frac{1}{6} H + \left(a - \frac{2}{3} \right) L_1 - \frac{5a}{6} = 0.$$

Interesting 2-variable O.P.'s: Koschmieder, Zerneke, (disk)

Example: E_3 (Harmonic oscillator)

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2)$$

Basis symmetries:

$$L_1 = \partial_x^2 - \omega^2 x^2, \quad L_3 = \partial_{xy} - \omega^2 xy, \quad X = x\partial_y - y\partial_x.$$

Also we set $L_2 = \partial_y^2 - \omega^2 y^2 = H - L_1$.

Structure equations:

$$[L_1, X] = 2L_3, \quad [L_3, X] = H - 2L_1, \quad [L_1, L_3] = 2\omega^2 X,$$

$$L_1^2 + L_3^2 - L_1 H - \omega^2 X^2 + \omega^2 = 0$$

Free triplets

We say that the 2D system without potential,

$$H_0 = \Delta_2$$

and with 3 algebraically independent second-order symmetries is a **2nd order free triplet**. The possible spaces admitting free triplets are just those classified by Koenigs: **2 constant curvature, 4 Darboux, and 11 Koenigs spaces**.

Note that every nondegenerate or degenerate superintegrable system defines a free triplet, simply by setting the parameters $a_j = 0$ in the potential. Similarly, this free triplet defines a **free quadratic algebra**, i.e., a quadratic algebra with all $a_j = 0$.

In general, a free triplet cannot be obtained as a restriction of a superintegrable system and its associated algebra does not close to a free quadratic algebra.

Closure Theorems (Kalnins-Miller, 2014)

Theorem

A free triplet extends to a superintegrable system if and only if it generates a free quadratic algebra \tilde{Q} .

Theorem

A superintegrable system, degenerate or nondegenerate, with quadratic algebra Q , is uniquely determined by its free quadratic algebra \tilde{Q} .

Remark 1: These theorems are true for constant curvature spaces where the free triplets are contained in the enveloping algebras of the Lie symmetry algebras of the spaces as well as for Darboux spaces (1-dimensional Lie symmetry algebra) and Koenig spaces (no symmetry Lie algebra) where this is no longer true.

Remark 2: These theorems are constructive. Given a free quadratic algebra \tilde{Q} one can compute the potential V and the symmetries of the quadratic algebra Q .

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What are models?

- A **representation of a quadratic algebra Q** is a homomorphism of Q into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A **model M** is a faithful representation of Q in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space.
- The dimensions of the finite dimensional irreducible representations determine the multiplicities of the quantum energy eigenspaces and the representations determine the eigenvalues.

What is the significance of models?

- They are easier to work with than the original abstract algebras.
- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
- Racah and Wilson polynomials are related to S9 in this way.

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- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
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The big picture: Special functions and contractions

Connection with special functions

Special functions arise in two distinct ways:

- As separable eigenfunctions of the quantum Hamiltonian.
- As eigenfunctions in the model. Often solutions of difference equations.

Connection with contractions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $\mathfrak{o}(3, \mathbb{C})$ and $\mathfrak{e}(2, \mathbb{C})$.
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)

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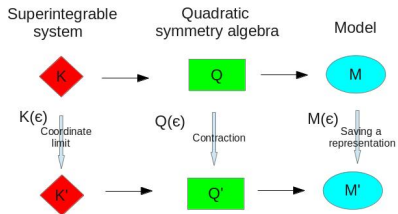
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Model interplay



The Contraction of Models

The S9 difference operator model. 1

$$L_2 f_{n,m} = (-4t^2 - \frac{1}{2} + B_1^2 + B_3^2) f_{n,m},$$

$$L_3 f_{n,m} = (-4\tau^* \tau - 2[B_1 + 1][B_2 + 1] + \frac{1}{2}) f_{n,m},$$

$$H = L_1 + L_2 + L_3 + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2) = -4(m+1)(B_1 + B_2 + B_3 + m + 1) - 2(B_1 B_2 + B_1 B_3 + B_2 B_3) + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2).$$

Here $n = 0, 1, \dots$, m if m is a nonnegative integer and $n = 0, 1, \dots$ otherwise. Also

$$a_j = \frac{1}{4} - B_j^2, \quad \alpha = -(B_1 + B_3 + 1)/2 - m, \quad \beta = (B_1 + B_3 + 1)/2, \quad \gamma = (B_1 - B_3 + 1)/2, \quad \delta = (B_1 + B_3 - 1)/2 + B_2 + m + 2,$$

$$E^A F(t) = F(t+A), \quad \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}), \quad \tau^* = \frac{1}{2t} \left[(\alpha + t)(\beta + t)(\gamma + t)(\delta + t)E^{1/2} - (\alpha - t)(\beta - t)(\gamma - t)(\delta - t)E^{-1/2} \right],$$

$$w_n(t^2) = (\alpha + \beta)_n (\alpha + \gamma)_n (\alpha + \delta)_n {}_4F_3 \left(\begin{matrix} -n, & \alpha + \beta + \gamma + \delta + n - 1, & \alpha - t, & \alpha + t \\ \alpha + \beta, & \alpha + \gamma, & \alpha + \delta & \end{matrix} ; 1 \right)$$

$$= (\alpha + \beta)_n (\alpha + \gamma)_n (\alpha + \delta)_n \Phi_n^{(\alpha, \beta, \gamma, \delta)}(t^2), \quad \Phi_n \equiv f_{n,m},$$

$$\tau^* \tau \Phi_n = n(n + \alpha + \beta + \gamma + \delta - 1) \Phi_n,$$

where $(a)_n$ is the Pochhammer symbol and ${}_4F_3(1)$ is a hypergeometric function of unit argument. The polynomial $w_n(t^2)$ is symmetric in $\alpha, \beta, \gamma, \delta$. For the finite dimensional representations the spectrum of t^2 is $\{(\alpha + k)^2, k = 0, 1, \dots, m\}$ and the orthogonal basis eigenfunctions are Racah polynomials. In the infinite dimensional case they are Wilson polynomials.

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The S9 difference operator model. 2

The action of L_2 and L_3 on an L_3 eigenbasis is

$$L_2 f_{n,m} = -4K(n+1, n)f_{n+1,m} - 4K(n, n)f_{n,m} - 4K(n-1, n)f_{n-1,m} + (B_1^2 + B_3^2 - \frac{1}{2})f_{n,m},$$

$$L_3 f_{n,m} = -(4n^2 + 4n[B_1 + B_2 + 1] + 2[B_1 + 1][B_2 + 1] - \frac{1}{2})f_{n,m},$$

$$K(n+1, n) = \frac{(B_1 + B_2 + n + 1)(n - m)(-B_3 - m + n)(B_2 + n + 1)}{(B_1 + B_2 + 2n + 1)(B_1 + B_2 + 2n + 2)},$$

$$K(n-1, n) = \frac{n(B_1 + n)(B_1 + B_2 + B_3 + m + n + 1)(B_1 + B_2 + m + n + 1)}{(B_1 + B_2 + 2n)(B_1 + B_2 + 2n + 1)},$$

$$K(n, n) = \left[\frac{B_1 + B_2 + 2m + 1}{2} \right]^2 - K(n+1, n) - K(n-1, n),$$

Lie algebra contractions

Let $(A; [\cdot; \cdot]_A)$, $(B; [\cdot; \cdot]_B)$ be two complex Lie algebras. We say B is a *contraction* of A if for every $\epsilon \in (0; 1]$ there exists a linear invertible map $t_\epsilon : B \rightarrow A$ such that for every $X, Y \in B$,

$$\lim_{\epsilon \rightarrow 0} t_\epsilon^{-1} [t_\epsilon X, t_\epsilon Y]_A = [X, Y]_B.$$

Thus, as $\epsilon \rightarrow 0$ the 1-parameter family of basis transformations can become nonsingular but the structure constants go to a finite limit.

Contractions of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$

These are the symmetry algebras of free systems on constant curvature spaces. Their contractions have long since been classified. There are 6 nontrivial contractions of $e(2, \mathbb{C})$ and 4 of $o(3, \mathbb{C})$.

Example: A Wigner-Inönü contraction of $o(3, \mathbb{C})$. We use the classical realization for $o(3, \mathbb{C})$ acting on the 2-sphere, with basis

$J_1 = s_2 p_3 - s_3 p_2$, $J_2 = s_3 p_1 - s_1 p_3$, $J_3 = s_1 p_2 - s_2 p_1$, commutation relations

$$[J_2, J_1] = J_3, \quad [J_3, J_2] = J_1, \quad [J_1, J_3] = J_2,$$

and Hamiltonian $H = J_1^2 + J_2^2 + J_3^2$. Here $s_1^2 + s_2^2 + s_3^2 = 1$.

Basis change: $\{J'_1, J'_2, J'_3\} = \{\epsilon J_1, \epsilon J_2, J_3\}$, $0 < \epsilon \leq 1$

coordinate implementation $x = \frac{s_1}{\epsilon}$, $y = \frac{s_2}{\epsilon}$, $s_3 \approx 1$, $J = J_3$

New structure relations: $[J'_2, J'_1] = \epsilon^2 J'_3$, $[J'_3, J'_2] = J'_1$, $[J'_1, J'_3] = J'_2$,

Let $\epsilon \rightarrow 0$: $[J'_2, J'_1] = 0$, $[J'_3, J'_2] = J'_1$, $[J'_1, J'_3] = J'_2$, get $e(2, \mathbb{C})$

Contractions of nondegenerate systems. 1

Suppose we have a nondegenerate superintegrable system with generators $H, L_1, L_2, R = [L_1, L_2]$ and the usual structure equations, defining a quadratic algebra Q . If we make a change of basis to new generators $\tilde{H}, \tilde{L}_1, \tilde{L}_2$ and parameters $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ such that

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{H} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,3} \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ H \end{pmatrix} + \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

for some 3×3 constant matrices $A = (A_{i,j}), B, C$ such that $\det A \cdot \det C \neq 0$, we will have the same system with new structure equations of the same form for $\tilde{R} = [\tilde{L}_1, \tilde{L}_2], [\tilde{L}_j, \tilde{R}], \tilde{R}^2$, but with transformed structure constants.

Contractions of nondegenerate systems. 2

- Choose a continuous 1-parameter family of basis transformation matrices $A(\epsilon), B(\epsilon), C(\epsilon)$, $0 < \epsilon \leq 1$ such that $A(1) = C(1)$ is the identity matrix, $B(1) = 0$ and $\det A(\epsilon) \neq 0$, $\det C(\epsilon) \neq 0$.
- Now suppose as $\epsilon \rightarrow 0$ the basis change becomes singular, (i.e., the limits of A, B, C either do not exist or, if they exist do not satisfy $\det A(0) \det C(0) \neq 0$) but the structure equations involving $A(\epsilon), B(\epsilon), C(\epsilon)$, go to a limit, defining a new quadratic algebra Q' .
- We call Q' a contraction of Q in analogy with Lie algebra contractions.

There is a similar definition of a contraction of a degenerate superintegrable system.

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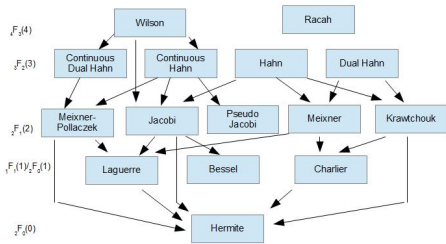
Lie algebra contractions \Rightarrow quadratic algebra contractions

Constant curvature spaces:

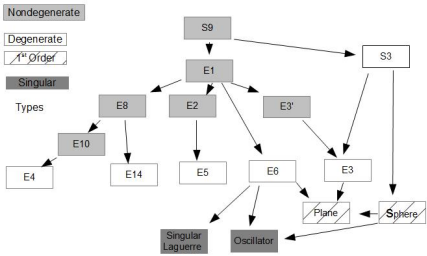
Theorem

(Kalnins-Miller, 2014) Every Lie algebra contraction of $A = e(2, \mathbb{C})$ or $A = o(3, \mathbb{C})$ induces uniquely a contraction of a free quadratic algebra \tilde{Q} based on A , which in turn induces uniquely a contraction of the quadratic algebra Q with potential. This is true for both classical and quantum algebras.

A complication: **Darboux spaces:** The Lie symmetry algebra is only 1-dimensional so Wigner contractions don't apply and geometrical free quadratic algebra contractions must be determined on a case-by-case basis.

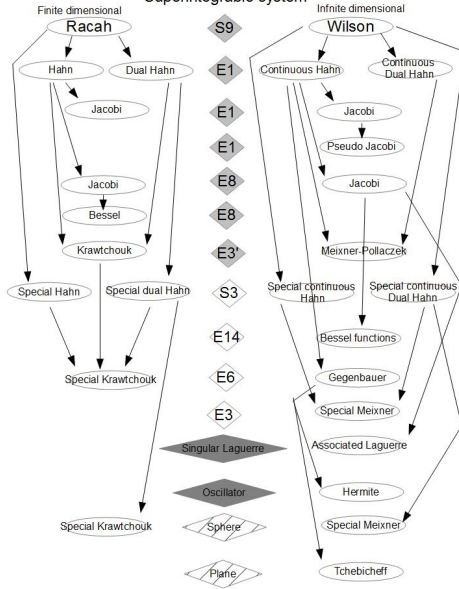


Askey Scheme of Hypergeometric Orthogonal Polynomials



Partial list of contractions of superintegrable systems

Superintegrable system



Example: free nondegenerate Darboux 2 systems

The space Darboux 2 (D_2) has free degenerate Hamiltonian

$$\mathcal{H} = \frac{x^2}{x^2 + 1} (p_x^2 + p_y^2)$$

Killing vector: $\mathcal{K} = p_y$. Basis, $\{\mathcal{H}, \mathcal{K}^2, \mathcal{X}_1, \mathcal{X}_2\}$ for the 4-dimensional space of 2nd order Killing tensors.

$$\mathcal{X}_1 = 2xp_x p_y + \frac{2y}{x^2 + 1} (p_y^2 - x^2 p_x^2), \quad \mathcal{X}_2 = 2xyp_x p_y + \frac{(y^2 - x^4)p_y^2 + x^2(1 - y^2)p_x^2}{x^2 + 1}.$$

Functional relation

$$4\mathcal{H}\mathcal{X}_2 + \mathcal{X}_1^2 - 4\mathcal{K}^2\mathcal{X}_2 - 4\mathcal{H}^2 = 0.$$

Possible free systems:

① **D2C**: $\mathcal{L}_1 = \mathcal{X}_2, \mathcal{L}_2 = \mathcal{X}_1, \mathcal{R}^2 = 4\mathcal{L}_1\mathcal{L}_2^2 + 16\mathcal{L}_1^2\mathcal{H} - 16\mathcal{L}_1\mathcal{H}^2.$

② **D2B**: $\mathcal{L}_1 = \mathcal{X}_2, \mathcal{L}_2 = \mathcal{K}^2, \mathcal{R}^2 = 16\mathcal{L}_1\mathcal{L}_2^2 - 16\mathcal{L}_1\mathcal{L}_2\mathcal{H} + 16\mathcal{L}_2\mathcal{H}^2.$

③ **D2A**: $\mathcal{L}_1 = \mathcal{X}_1, \mathcal{L}_2 = \mathcal{K}^2, \mathcal{R}^2 = 16\mathcal{L}_2^3 - 32\mathcal{L}_2^2\mathcal{H} + 16\mathcal{L}_2\mathcal{H}^2.$

Extension of $\tilde{D}2C$ to a nondegenerate superintegrable system.

$$\mathcal{H} = \frac{x^2}{x^2 + 1}(p_x^2 + p_y^2) + V(x, y),$$

$$V(x, y) = \frac{x^2}{2\sqrt{x^2 + y^2}(x^2 + 1)} \left(c_1 + \frac{c_2}{y + \sqrt{x^2 + y^2}} + \frac{c_3}{-y + \sqrt{x^2 + y^2}} \right) + c_4.$$

The Casimir is

$$\begin{aligned} \mathcal{R}^2 - 4\mathcal{L}_1\mathcal{L}_2^2 - 16\mathcal{L}_1^2\mathcal{H} + 16\mathcal{L}_1\mathcal{H}^2 - 4(c_2 + c_3)\mathcal{H}^2 + 16c_4\mathcal{L}_1^2 + (c_2 + c_3)\mathcal{L}_2^2 - 32c_4\mathcal{H}\mathcal{L}_1 \\ + (8c_2c_4 - c_1^2 + 4c_2c_3 + 8c_3c_4)\mathcal{H} + (16c_4^2 + c_1^2)\mathcal{L}_1 + c_1(-c_3 + c_2)\mathcal{L}_2 \\ + \left(\frac{1}{4}c_1^2c_2 + \frac{1}{4}c_1^2c_3 + c_1^2c_4 - 4c_3c_4^2 - 4c_2c_3c_4 - 4c_2c_4^2\right). \end{aligned}$$

Example: D2 contractions

1 Let

$$x = x' + \frac{1}{\epsilon}, \quad y = y', \quad \mathcal{H}' = \mathcal{H}.$$

As $\epsilon \rightarrow 0$ we have

$$\mathcal{H}' = p_{x'}^2 + p_{y'}^2, \quad \epsilon \mathcal{X}_1 \approx 2p_{x'} p_{y'}, \quad \epsilon^2 \mathcal{X}_2 \approx -p_{y'}^2.$$

Contractions constructed from such limits have flat space as the target manifold.

2 Let

$$x = \epsilon x', \quad y = \epsilon y', \quad \mathcal{H}' = \mathcal{H}.$$

As $\epsilon \rightarrow 0$ we have $\mathcal{H}' = x'^2(p_{x'}^2 + p_{y'}^2)$ and

$$\epsilon \mathcal{X}_1 \approx 2x' p_{x'} p_{y'} + 2y' p_{y'}^2, \quad \epsilon \mathcal{X}_2 = 2x' y' p_{x'} p_{y'} + y'^2 p_{y'}^2 + x'^2 p_{x'}^2, \quad \epsilon \mathcal{K} = p_{y'}.$$

Contractions constructed from such limits have the complex 2-sphere as the target manifold.

Example: Contraction 2 applied to D2C

In the limit, and in Cartesian coordinates s_1, s_2, s_3 on the complex 2-sphere, where $s_1^2 + s_2^2 + s_3^2 = 1$ and $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are defined by $\mathcal{J}_3 = s_1 p_2 - s_2 p_3$ and cyclic permutations, we have

$$\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2 + \frac{k_1}{(s_1 - is_2)^2} + \frac{k_2 s_3}{\sqrt{s_1^2 + s_2^2}} + \frac{k_3}{(s_1 - is_2)\sqrt{s_1^2 + s_2^2}}.$$

This system is real on a real hyperboloid.

In general, constant curvature superintegrable systems cannot contract to Darboux or Koenig systems but contractions in the reverse order occur.

Table of Darboux 2 contractions

	Darboux 2 to flat space	Darboux 2 to 2-sphere
$\tilde{D}2A$	$\tilde{E}3'$	$\tilde{S}1$
$\tilde{D}2B$	$\tilde{E}2$	$\tilde{S}2$
$\tilde{D}2C$	$\tilde{E}3'$	$\tilde{S}4$
$\tilde{D}2D^*$	$\tilde{E}5^*$	$\tilde{S}5^*$

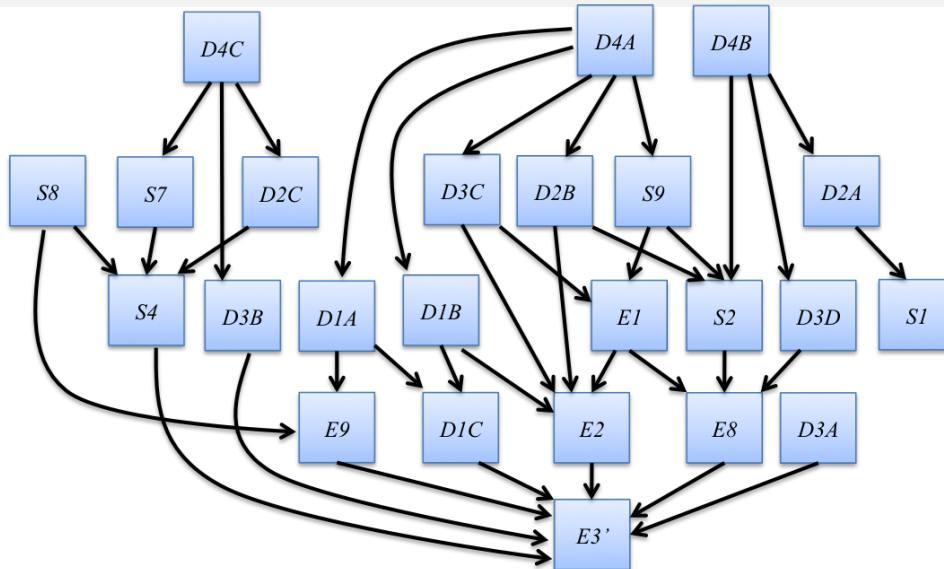
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$\tilde{D}2D^*$	$\tilde{E}5^*$	$\tilde{S}5^*$

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Contraction hierarchy 1



Contraction hierarchy 2

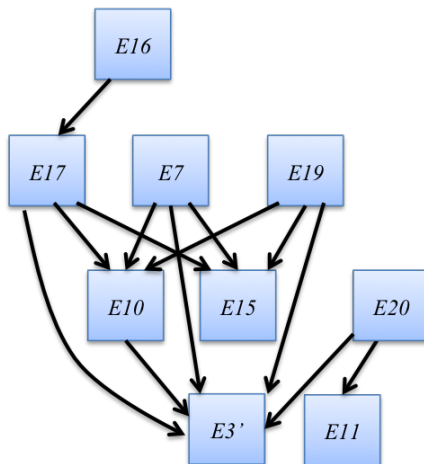
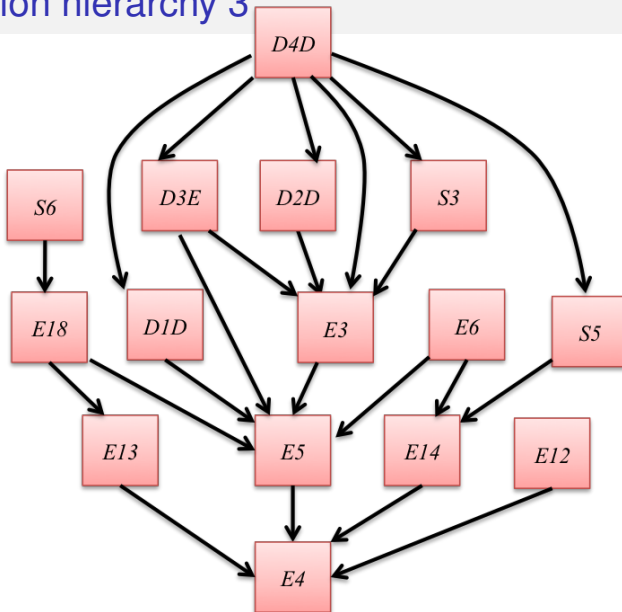


Figure : Second part of nondegenerate contraction diagram. This can in principle be joined with the previous.

Contraction hierarchy 3



Abstract contractions

We are in the process of classifying all abstract contractions of free degenerate and nondegenerate quadratic algebras. **Every Lie algebra induced contraction is also an abstract contraction, but the converse is false.** There are many abstract contractions that cannot be obtained by taking pointwise coordinate limits of quantum superintegrable systems. For example: $\tilde{E}7$ to $\tilde{E}2$ and $\tilde{S}7$ to $\tilde{E}15$. The significance of these non-physical limits is unclear to us at present.

Abstract contractions between nondegenerate free systems

Figure 1: All known abstract nondegenerate free quadratic contractions, classified up to Stäckel equivalence.

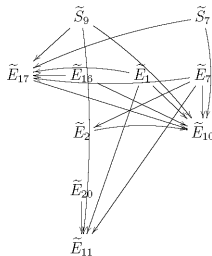
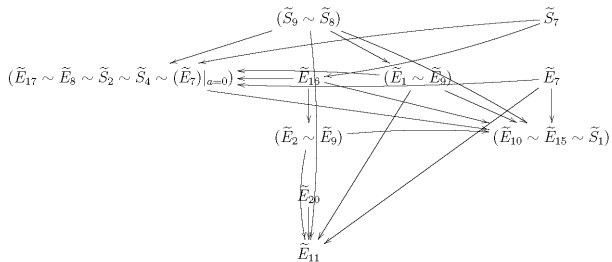


Figure 2: All free nondegenerate quadratic algebra contractions induced by Lie algebras contractions.



Wrap-up. 1

- Free quadratic algebras uniquely determine associated superintegrable systems with potential.
- A contraction of a free quadratic algebra to another uniquely determines a contraction of the associated superintegrable systems.
- For a 2D superintegrable systems on a constant curvature space these contractions can be induced by Lie algebra contractions of the underlying Lie symmetry algebra.
- Every 2D superintegrable system is obtained either as a sequence of contractions from S_9 or is Stäckel equivalent to a system that is so obtained.

Wrap-up. 2

- Taking contractions step-by-step from the $S9$ model we can recover the Askey Scheme. However, the contraction method is more general. It applies to all special functions that arise from the quantum systems via separation of variables, not just polynomials of hypergeometric type, and it extends to higher dimensions.
- The special functions arising from the models can be described as the coefficients in the expansion of one separable eigenbasis for the original quantum system in terms of another separable eigenbasis.
- The functions in the Askey Scheme are just those hypergeometric polynomials that arise as the expansion coefficients relating two separable eigenbases that are *both* of hypergeometric type. Thus, there are some contractions which do not fit in the Askey scheme since the physical system fails to have such a pair of separable eigenbases.

Wrap-up. 3

- Even though 2nd order 2D nondegenerate superintegrable systems admit no group symmetry, their structure is determined completely by the underlying symmetry of constant curvature spaces.