

A NOTE ON WILSON POLYNOMIALS*

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Abstract. Local symmetry (recurrence relation) techniques are a powerful tool for the efficient derivation of properties associated with families of hypergeometric and basic hypergeometric functions. Here these ideas are applied to the Wilson polynomials, a generalization of the classical orthogonal polynomials, to obtain the orthogonality relations and an elementary evaluation of the norm.

Key words. Wilson polynomials, orthogonal polynomials, q -series, basic hypergeometric functions

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1. Introduction. In [1] Wilson introduced a family of hypergeometric orthogonal polynomials that included as special or limiting cases the classical polynomials and the $6-j$ symbols of angular momentum. In the Memoir [2] Askey and Wilson introduced a still more general class of basic hypergeometric orthogonal polynomials, the most extensive generalization of classical orthogonal polynomials known. The orthogonality proofs in these papers, while not unmotivated, are quite technical and rely on Mellin-Barnes contour integrals and several hypergeometric summation formulas that are unfamiliar to most mathematicians. The Askey-Wilson and Wilson polynomials are important and useful; they deserve to be more widely known. Furthermore, the appropriate algebraic and group theoretic setting for these general families is as yet unclear. The elementary algebraic treatment of Wilson polynomials presented here is offered in the hope that it will help to increase the "audience" for the polynomials as well as to shed some light on their structure.

In [3] the author, with Agarwal and Kalnins, introduced symmetry techniques for the study of families of basic hypergeometric functions, in analogy with the local Lie theory techniques for ordinary hypergeometric functions. The fundamental objects in this study are the recurrence relations obeyed by the families, expressed in terms of difference or q -difference equations. Generating functions and identities for each family are characterized in terms of the recurrence relations. These ideas were applied in [4] to obtain a strikingly simple derivation of the orthogonality relations for the Askey-Wilson q -polynomials. The treatment of the Wilson (ordinary hypergeometric) polynomials presented here is very similar to that in [4]. However several minor complications arise, due to the fact that whereas the first order q -difference equation $f(qz) = f(z)$ has only the solution $f(z) \equiv \text{constant}$, the first order difference equation $f(z+1) = f(z)$ is satisfied by any periodic function f with period 1. Thus the treatment presented here is not entirely algebraic: a few simple facts about Fourier series and the gamma function are required.

2. The results. The (unnormalized) Wilson polynomials are:

$$(2.1) \quad \Phi_n^{(a,b,c,d)}(z^2) = {}_4F_3 \left(\begin{matrix} -n, n+a+b+c+d-1, a+z, a-z \\ a+b, a+c, a+d \end{matrix} ; 1 \right)$$

where $n = 0, 1, 2, \dots$ and $a, b, c, d > 0$. The hypergeometric function ${}_4F_3$ is given by the series

$${}_4F_3 \left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} ; t \right) = \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m (a_3)_m (a_4)_m}{(b_1)_m (b_2)_m (b_3)_m} \frac{t^m}{m!}$$

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where

$$(a)_m = \begin{cases} 1 & \text{if } m = 0, \\ a(a+1) \cdots (a+m-1) & \text{if } m = 1, 2, \dots \end{cases}$$

The functions (2.1) are polynomials of order n in z^2 . (In [1] the parameters a, b, c, d are also permitted to become complex, an important extension; but we shall not consider that case here.)

Two fundamental recurrence relations for the Wilson polynomials are:

$$(2.2a) \quad \tau^{(a,b,c,d)} \Phi_n^{(a,b,c,d)} = \frac{n(n+a+b+c+d-1)}{(a+b)(a+c)(a+d)} \Phi_{n-1}^{(a+1/2,b+1/2,c+1/2,d+1/2)},$$

$$(2.2b) \quad \mu^{(a,b,c,d)} \Phi_n^{(a,b,c,d)} = -(a+b-1) \Phi_n^{(a-1/2,b-1/2,c+1/2,d+1/2)}$$

where

$$(2.3) \quad \tau^{(a,b,c,d)} = \frac{1}{2z} (E_z^{1/2} - E_z^{-1/2}),$$

$$\mu^{(a,b,c,d)} = \frac{1}{2z} \left[-\left(a+z-\frac{1}{2}\right) \left(b+z-\frac{1}{2}\right) E_z^{1/2} + \left(a-z-\frac{1}{2}\right) \left(b-z-\frac{1}{2}\right) E_z^{-1/2} \right]$$

and $E_z^\alpha f(z) = f(z+\alpha)$. Here (2.2a) follows from

$$\tau(a+z)_k (a-z)_k = -k(a+\frac{1}{2}+z)_{k-1} (a+\frac{1}{2}-z)_{k-1}$$

and (2.2b) follows from

$$\mu(a+z)_k (a-z)_k = -(a+b-k-1)(a-\frac{1}{2}+z)_k (a-\frac{1}{2}-z)_k.$$

The first relation was discussed by Askey and Wilson [2]; I have not found (2.2b) in the literature.

Relation (2.2b) suggests the existence of an operator μ^* mapping $\Phi_n^{(a-1/2,b-1/2,c+1/2,d+1/2)}$ to $\Phi_n^{(a,b,c,d)}$. We find that

$$(2.4) \quad \mu^{(c+1/2,d+1/2,a-1/2,b-1/2)} \Phi_n^{(a-1/2,b-1/2,c+1/2,d+1/2)} = \frac{-(n+c+d)(n+a+b-1)}{(a+b-1)} \Phi_n^{(a,b,c,d)},$$

which follows from

$$\begin{aligned} \mu^{(c+1/2,d+1/2,a-1/2,b-1/2)} (a-\frac{1}{2}+z)_k (a-\frac{1}{2}-z)_k \\ = -(k+c+d)(a+z)_k (a-z)_k + k(a+c+k-1)(a+d+k-1)(a+z)_{k-1} (a-z)_{k-1}. \end{aligned}$$

We try to introduce a pre-Hilbert space structure such that $\mu^* \equiv \mu^{(c+1/2,d+1/2,a-1/2,b-1/2)}$ is the adjoint operator to $\mu \equiv \mu^{(a,b,c,d)}$. Let $w_{a,b,c,d}(z)$ be analytic as a function of the complex variable z in a neighborhood of the imaginary axis $z = iy$, $-\infty < y < \infty$, real analytic in the variables a, b, c, d , of exponential decrease as $|y| \rightarrow \infty$ and such that $w_{a,b,c,d}(iy) \geq 0$. Define an inner product:

$$(g_1, g_2)_{a,b,c,d} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} g_1(z^2) g_2(z^2) w_{a,b,c,d}(z) dz$$

where the contour is a deformation of the imaginary axis and g_1, g_2 are real polynomials in z^2 . Let $S_{a,b,c,d}$ be the space of such polynomials with this inner product. We have

$$\mu : S_{a,b,c,d} \rightarrow S_{a-1/2,b-1/2,c+1/2,d+1/2},$$

$$\mu^* : S_{a-1/2,b-1/2,c+1/2,d+1/2} \rightarrow S_{a,b,c,d}$$

and seek a weight function $w_{a,b,c,d}$ such that

$$(2.5) \quad (f, \mu g)_{a-1/2,b-1/2,c+1/2,d+1/2} = (\mu^* f, g)_{a,b,c,d}$$

for all polynomials $f \in S_{a-1/2,b-1/2,c+1/2,d+1/2}$ and $g \in S_{a,b,c,d}$. A straightforward computation yields the necessary and sufficient condition

$$\frac{w_{a,b,c,d}(z+1)}{w_{a,b,c,d}(z)} = \frac{(z+1)(a+z)(b+z)(c+z)(d+z)}{z(a-z-1)(b-z-1)(c-z-1)(d-z-1)}$$

with general solution

$$\begin{aligned} w_{a,b,c,d}(z) &= \frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)}{\Gamma(2z)\Gamma(-2z)} \\ &\quad \cdot \Gamma(d+z)\Gamma(d-z)h(a,b,c,d,z) \\ &= \hat{w}_{a,b,c,d}(z)h(a,b,c,d,z) \end{aligned}$$

where h satisfies the periodicity properties

$$\begin{aligned} h(a-\tfrac{1}{2}, b-\tfrac{1}{2}, c+\tfrac{1}{2}, d+\tfrac{1}{2}, z+\tfrac{1}{2}) &= h(a-\tfrac{1}{2}, b-\tfrac{1}{2}, c+\tfrac{1}{2}, d+\tfrac{1}{2}, z-\tfrac{1}{2}) \\ &= h(a,b,c,d,z). \end{aligned}$$

Here $\Gamma(z)$ is the gamma function [5, Chap. XII]. From Stirling's series for the gamma function, $\hat{w}_{a,b,c,d}(z) = (z)^{2(a+b+c+d)-3} O(e^{-2\pi|y|})$ as $|y| \rightarrow \infty$, where $z = x + iy$. Thus we must require that $h(z) = o(e^{2\pi|y|})$ as $|y| \rightarrow \infty$ in order that $\hat{w}h$ be a suitable weight function. Furthermore, the "integration by parts" formula (2.5) will not be valid unless $h(z)$ is analytic in an open set containing the strip $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Since $h(z) = h(z+1)$ it follows that h can be analytically continued to an entire periodic function of z :

$$h(z) = \sum_{m=-\infty}^{\infty} c_m(y) e^{2\pi i m x}, \quad c_m(y) = \frac{1}{2\pi i} \int_0^{2\pi} h(z) e^{-2\pi i n x} dx.$$

Using the Cauchy-Riemann conditions for analytic functions we find that $c_m(y) = a_m e^{-2\pi m y}$ where a_m is independent of z . Since $h(iy) = o(e^{2\pi|y|})$ we have that $|a_m e^{-2\pi m y}| = o(e^{2\pi|y|})$ as $|y| \rightarrow \infty$, so $a_m = 0$ for $m \neq 0$. Thus h is independent of z and, without loss of generality, we can set $h \equiv 1$:

$$(2.6) \quad w_{a,b,c,d}(z) = \frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)}{\Gamma(2z)\Gamma(-2z)}.$$

Since (2.5) holds, $\mu^* \mu$ is formally selfadjoint:

$$(2.7) \quad (\mu^* \mu g_1, g_2)_{a,b,c,d} = (g_1, \mu^* \mu g_2)_{a,b,c,d}.$$

From recurrence relations (2.2b) and (2.4) it follows that the Wilson polynomials are eigenfunctions of $\mu^* \mu$:

$$(2.8) \quad \mu^* \mu \Phi_n^{(a,b,c,d)} = \lambda_n \Phi_n^{(a,b,c,d)}, \quad \lambda_n = (a+b+n-1)(c+d+n).$$

Note that $\lambda_n = \lambda_m$ iff $n = m$. Since the eigenfunctions corresponding to distinct eigenvalues are orthogonal, we have

$$(\Phi_n^{(a,b,c,d)}, \Phi_m^{(a,b,c,d)})_{a,b,c,d} = 0 \quad \text{if } n \neq m.$$

The operator $\mu^* \mu$ and the weight function are symmetric with respect to the interchange $a \leftrightarrow b$. Thus the polynomials $\{\Phi_n^{(b,a,c,d)}(z^2)\}$ are also orthogonal in $S_{a,b,c,d}$ and are eigenfunctions of $\mu^* \mu$. This means that there exists a constant K_n such that

$$\Phi_n^{(b,a,c,d)}(z^2) = K_n \Phi_n^{(a,b,c,d)}(z^2).$$

Equating coefficients of z^{2n} on both sides of this expression to obtain K_n , we find that

$$(2.9) \quad {}_4F_3 \left(\begin{matrix} -n, n+b+a+c+d-1, b+z, b-z \\ b+a, b+c, b+d \end{matrix} ; 1 \right) = \frac{(a+c)_n (a+d)_n}{(b+c)_n (b+d)_n} {}_4F_3 \left(\begin{matrix} -n, n+a+b+c+d-1, a+z, a-z \\ a+b, a+c, a+d \end{matrix} ; 1 \right).$$

This is a transformation formula due to Bailey [6, p. 56] and, as Wilson pointed out [1], it essentially contains the symmetries of the 6- j symbols. It follows from this result that the renormalized polynomials

$$(a+b)_n (a+c)_n (a+d)_n \Phi_n^{(a,b,c,d)}(z^2)$$

are symmetric in all four parameters a, b, c, d .

Setting $f(z^2) = g(z^2) = 1$ in (2.5) and using (2.2b) and (2.1), we obtain the following relationship between the norms on $S_{a-1/2,b-1/2,c+1/2,d+1/2}$ and $S_{a,b,c,d}$:

$$(2.10) \quad \|1\|_{a-1/2,b-1/2,c+1/2,d+1/2}^2 = \frac{c+d}{a+b-1} \|1\|_{a,b,c,d}^2.$$

The symmetry of the weight function in a, b, c, d yields 5 more such relations.

Now consider the recurrence (2.2a):

$$\tau^{(a,b,c,d)} : S_{a,b,c,d} \rightarrow S_{a+1/2,b+1/2,c+1/2,d+1/2}.$$

We seek the adjoint τ^* to $\tau \equiv \tau^{(a,b,c,d)}$:

$$(2.11) \quad (f, \tau g)_{a+1/2,b+1/2,c+1/2,d+1/2} = (\tau^* f, g)_{a,b,c,d}$$

for all $f \in S_{a+1/2,\dots,d+1/2}, g \in S_{a,\dots,d}$. A simple computation using (2.11) yields

$$(2.12) \quad \tau^* \equiv \tau^{*(a+1/2,b+1/2,c+1/2,d+1/2)} = \frac{1}{2z} [(a+z)(b+z)(c+z)(d+z)E_z^{1/2} - (a-z)(b-z)(c-z)(d-z)E_z^{-1/2}].$$

From (2.11) and the orthogonality relations it follows that

$$(2.13) \quad \tau^* \Phi_{n-1}^{(a+1/2,b+1/2,c+1/2,d+1/2)} = H_n \Phi_n^{(a,b,c,d)}.$$

Comparing coefficients of z^{2n} on both sides of this expression we find that

$$H_n = (a+b)(a+c)(a+d).$$

Thus $\tau^* \tau$ is selfadjoint on $S_{a,b,c,d}$ and the eigenvalue equation is:

$$(2.14) \quad \tau^* \tau \Phi_n^{(a,b,c,d)} = n(n+a+b+c+d-1) \Phi_n^{(a,b,c,d)}.$$

We also have the Rodrigues formula

$$(2.15) \quad \Phi_n^{(a,b,c,d)} = J_n \tau^{*(a+1/2, \dots, d+1/2)} \tau^{*(a+1, \dots, d+1)} \dots \tau^{*(a+n/2, \dots, d+n/2)}(1),$$

$$J_n = (a+b)_n (a+c)_n (a+d)_n.$$

Substituting $f = \Phi_{n-1}^{(a+1/2, \dots, d+1/2)}$, $g = \Phi_n^{(a, \dots, d)}$ in (2.11), we obtain the recurrence

$$(2.16) \quad \|\Phi_n^{(a, \dots, d)}\|_{a, \dots, d}^2 = \frac{n(n+a+b+c+d-1)}{(a+b)^2(a+c)^2(a+d)^2} \|\Phi_{n-1}^{(a+1/2, \dots, d+1/2)}\|_{a+1/2, \dots, d+1/2}^2$$

which enables us to compute the norm of any Wilson polynomial once the norm $\|1\|_{a, \dots, d}^2$ is determined for all $a, \dots, d > 0$. We now turn to this last task.

From the orthogonality relation $(\Phi_1^{(a, \dots, d)}, \Phi_0^{(a, \dots, d)})_{a, \dots, d} = 0$ and the explicit expression (2.1) for Wilson polynomials we find that

$$(2.17) \quad \|1\|_{a, \dots, d}^2 = \frac{(a+b+c+d)}{(a+b)(a+c)(a+d)} \|1\|_{a+1, b, c, d}^2.$$

Here we have used the evident relation

$$(g_m^a, 1)_{a, \dots, d} = \|1\|_{a+m, b, c, d}^2, \quad g_m^a(z^2) = (a+z)_m (a-z)_m.$$

From (2.10) and (2.17) and the obvious invariance of $\|1\|_{a, \dots, d}$ with respect to a permutation of a, b, c, d we find:

$$(2.18) \quad \|1\|_{a, b, c, d}^2 = \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)} M(a, b, c, d)$$

where M satisfies the periodicity properties

$$M(a, b, c, d) = M(a + \frac{1}{2}, b + \frac{1}{2}, c + \frac{1}{2}, d + \frac{1}{2}) = M(a+1, b, c, d)$$

and is invariant under any permutation of a, \dots, d . Now replace a by $a+k$, k a positive integer, in (2.18) and write this expression in the following form:

$$(2.19) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\Gamma(a+k+iy)\Gamma(a+k-iy)\Gamma(a+k+b+c+d)}{\Gamma(a+k+b)\Gamma(a+k+c)\Gamma(a+k+d)} \right) \cdot \left| \frac{\Gamma(b+iy)\Gamma(c+iy)\Gamma(d+iy)}{\Gamma(2iy)} \right|^2 dy$$

$$= \Gamma(b+c)\Gamma(b+d)\Gamma(c+d)M(a, b, c, d).$$

From Stirling's series

$$\Gamma(z+k) = \sqrt{2\pi}(k)^{z+k-1/2} e^{-k} \left(1 + O\left(\frac{1}{k}\right) \right)$$

as $k \rightarrow +\infty$, so

$$\lim_{k \rightarrow +\infty} \frac{\Gamma(a+k+iy)\Gamma(a+k-iy)\Gamma(a+k+b+c+d)}{\Gamma(a+k+b)\Gamma(a+k+c)\Gamma(a+k+d)} = 1$$

and

$$(2.20) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\Gamma(b+iy)\Gamma(c+iy)\Gamma(d+iy)}{\Gamma(2iy)} \right|^2 dy = \Gamma(b+c)\Gamma(b+d)\Gamma(c+d)M(a, b, c, d).$$

(The passage to the limit under the integral sign is easily justified since $|\Gamma(a+iy)| \leq \Gamma(a)$ for $a > 0$.) It is evident from (2.20) that M is independent of a . By symmetry, M is

constant. To evaluate the constant we set $b = 0$, $c = d = \frac{1}{2}$ in (2.20). Using the multiplication and reflection theorems for the gamma function, we reduce (2.20) to

$$2 \int_{-\infty}^{\infty} \frac{dy}{\cosh(\pi y)} = M$$

or $M = 2$. Thus

$$(2.21) \quad \begin{aligned} \|1\|_{a,b,c,d}^2 &= \frac{1}{\pi} \int_0^{\infty} \left| \frac{\Gamma(a+iy)\Gamma(b+iy)\Gamma(c+iy)\Gamma(d+iy)}{\Gamma(2iy)} \right|^2 dy \\ &= \frac{2\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}. \end{aligned}$$

Note that this integral, and special cases of it, were originally derived by contour integration, evaluation of the residues at the poles of the integrand in the right half plane and use of known summation theorems to sum the resulting infinite series. Wilson [1] used Bailey's Theorem [6, p. 27]

$$\begin{aligned} {}_5F_4 \left(\begin{matrix} 2a, a+1, a+b, a+c, a+d \\ a, a-b+1, a-c+1, a-d+1 \end{matrix}; 1 \right) \\ = \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(-a-b-c-d+1)}{\Gamma(2a+1)\Gamma(-b-c+1)\Gamma(-b-d+1)\Gamma(-c-d+1)} \end{aligned}$$

to compute (2.21). Since we have independently obtained the value of this integral we can consider the usual contour integral technique as a derivation of Bailey's ${}_5F_4$ summation.

For the Racah polynomials (discrete orthogonality), [1], the recurrence relation methods of this paper yield a purely algebraic derivation of the orthogonality, including as a byproduct the terminating version of Bailey's Theorem: $a+b = -N$.

Note added in proof. Recurrence techniques similar to those used in this paper have been employed by Nikiforov, Suslov and Uvarov [7] and Nikiforov and Suslov [8], but these authors have apparently not applied them to the computation of contour integrals and summation formulas.

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