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0.1 Applications of the Alternating Series Estimation Theorem

An alternating series is one that can be written in the form

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

where $a_n \geq 0$ for all n. In your textbook and in class the Alternating Series Convergence Theorem was proved:

Theorem 1 *The alternating series* $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ *converges provided*

1. $a_n \geq 0$ *for all* n
2. $a_{n+1} \leq a_n$ *for all* n, *and*
3. $\lim_{n\to\infty} a_n = 0$.

Indeed, if we denote the Nth partial sum of the series by $S_N = \sum_{n=1}^{N} (-1)^{n-1} a_n$, the proof consists in showing that the sequence of even partial sums is increasing:

$$S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_{2N} \leq \cdots$$

the sequence of odd partial sums is decreasing:

$$S_1 \geq S_3 \geq S_5 \geq \cdots \geq S_{2N-1} \geq S_{2N+1} \geq \cdots,$$

and that each odd partial sum is greater than each even partial sum. In particular $S_{2N+1} - S_{2N} = a_{2N+1} \geq 0$, and since $a_{2N+1} \to 0$ as $N \to \infty$ the difference between S_{2N+1} and S_{2N} (or the difference between S_{2N} and S_{2N-1} goes to zero as $N \to \infty$. This implies that the limit S exists and that it lies between S_{2N} and $S_{2N=1}$ for any N:

$$S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_{2N-2} \leq S_{2N} \leq S \leq S_{2N+1} \leq S_{2N-1} \cdots \leq S_5 \leq S_3 \leq S_1.$$

This proof yields an estimate for the error made in approximating S by S_M.

Theorem 2 *(Alternating Series Estimation Theorem) If the alternating series* $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *satisfies the conditions of the Alternating Series Convergence Theorem, then the error* $E_M = S - S_M$ *made by approximating* S *by the partial sum* S_M *is bounded in absolute value by* a_{M+1}*, i.e.,*

$$|S - S_M| = |E_M| \leq a_{M+1}.$$

Note: If the conditions for the Alternating Series Convergence Theorem hold only for $n \geq n_0$ then the Alternating Series Estimation Theorem still holds, but only for partial sums S_M with $M > n_0$

Example 1

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

It is easy to verify that this series satisfies the conditions of the Alternating Series Convergence Theorem. Thus we have

$$|S - S_M| \leq a_{M+1} = \frac{1}{(M+1)^2}.$$

If we want to approximate S *to an accuracy of* 10^{-4} *then we need to choose* M *such that* $\frac{1}{(M+1)^2} \leq 10^{-4}$*, or* $(M+1)^2 \geq 10^4$*. Thus* $M + 1 \geq 100$ *and the smallest* M *that will suffice is* $M = 99$*. If we sum the first 99 terms of this series we will obtain an estimate of the sum that is accurate to within 4 decimal places.*

Example 2

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

This is the same convergence alternating series as our first example. What is the maximum error in approximating S *by* S_{1000}*? According to the Estimation Theorem*

$$|S - S_{1000}| \leq a_{1000+1} = \frac{1}{(1001)^2} \approx 10^{-6}.$$

Example 3

$$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n(n+2)}$$

This is an alternating series for $x \geq 0$ and it satisfies the conditions of the Alternating Series Convergence Theorem Convergence Theorem for $0 \leq x \leq 1$. (Note that for $x > 1$ the series diverges.) For any x in the interval $[0, 1]$ the error in approximating $S(x)$ by $S_M(x)$ has the bound

$$|S(x) - S_M(x)| \leq a_{M+1}(x) = \frac{x^{M+1}}{(M+1)(M+3)} \leq \frac{1}{(M+1)(M+3)}.$$

Thus, if we want an approximation of $S(x)$ by $S_M(x)$ with maximum error 10^{-6} for all $x \in [0, 1]$ then we must require

$$\frac{1}{(M+1)(M+3)} \leq 10^{-6}$$

or $1000000 \leq (M+1)(M+3)$. Using a calculator, one can verify that the smallest integer M that will work is $M = 999$.

Example 4

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = e^{-x}.$$

For any positive x and for $n + 1 \geq x$ this satisfies the conditions of the Alternating Series Convergence Theorem. In particular, for $0 \leq x \leq 1$ the conditions are satisfied for all n. For any x in the interval $[0, 1]$ the error in approximating $S(x)$ by $S_M(x)$ has the bound

$$|S(x) - S_M(x)| \leq a_{M+1}(x) = \frac{x^{M+1}}{(M+1)!} \leq \frac{1}{(M+1)!}.$$

Thus, if we want an approximation of $S(x)$ by $S_M(x)$ with maximum error 10^{-6} for all $x \in [0, 1]$ then we must require

$$\frac{1}{(M+1)!} \leq 10^{-6}$$

or $10^6 \leq (M+1)!$. Using a calculator, one can verify that the smallest integer M that will work is $M = 9$. Note that we could also have solved this problem using Taylor's Theorem with remainder.

Example 5

$$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = \ln(1+x), \quad -1 < x \le 1.$$

For any x such that $0 \le x \le 1$ this satisfies the conditions of the Alternating Series Convergence Theorem. Thus for any x in the interval $[0,1]$ the error in approximating $S(x)$ by $S_M(x)$ has the bound

$$|S(x) - S_M(x)| \le a_{M+1}(x) = \frac{x^{M+1}}{M+1} \le \frac{1}{M+1}.$$

Thus, if we want an approximation of $S(x)$ by $S_M(x)$ with maximum error 10^{-6} for all $x \in [0,1]$ then we must require

$$\frac{1}{M+1} \le 10^{-6}$$

or $10^6 \le M+1$. Thus the smallest integer M that will work is $M = 999,999$. Note that we could also have solved this problem using Taylor's Theorem with intergral or derivative remainder, but the computation would have been more involved because of the need to compute the $M+1$st derivative of $\ln(1+x)$. On the other hand, Taylor's Theorem applies to series that may not be alternating.