Preface

This monograph is the result of an attempt to understand the role played by special function theory in the formalism of mathematical physics. It demonstrates explicitly that special functions which arise in the study of mathematical models of physical phenomena and the identities which these functions obey are in many cases dictated by symmetry groups admitted by the models. In particular it will be shown that the factorization method, a powerful tool for computing eigenvalues and recurrence relations for solutions of second order ordinary differential equations (Infeld and Hull [1]), is equivalent to the representation theory of four local Lie groups. A detailed study of these four groups and their Lie algebras leads to a unified treatment of a significant proportion of special function theory, especially that part of the theory which is most useful in mathematical physics.

Most of the identities for special functions derived in this book are known in one form or another. Our principal aim is not to derive new results but rather to provide insight into the structure of special function theory. Thus, all of the identities obtained here will be given an explicit group-theoretic interpretation instead of being considered merely as the result of some formal manipulation of infinite series.

The primary tools needed to deduce our results are multiplier representations of local Lie groups and representations of Lie algebras by generalized Lie derivatives. These concepts are introduced in Chapter 1 along with a brief survey of classical Lie theory. In Chapter 2 we state our main theme: Special functions occur as matrix elements and basis vectors corresponding to multiplier representations of local Lie groups. Chapters 3–6 are devoted to the explicit analysis of the special function theory of the complex local Lie groups with four-dimensional Lie algebras $\mathscr{G}(0, 0)$, $\mathscr{G}(0, 1)$, $\mathscr{G}(1, 0)$ and six-dimensional Lie algebra $\mathscr{T}_6$. Many fundamental properties of hypergeometric, confluent hypergeometric, and Bessel functions are obtained from this analysis. Furthermore, in Chapter 7 it is shown that the representation theory of these four Lie groups is completely equivalent to the factorization method of Infeld and Hull.

In Chapter 8 we determine the scope of our analysis by constructing the rudiments of a classification theory of generalized Lie derivatives. This classification theory enables us to decide in what sense the results of Chapters 3–6 are complete. Chapter 8 is more difficult than the rest of the book since it presupposes a knowledge of some rather deep results in local Lie theory and an acquaintance with the cohomology theory of Lie algebras. Hence, it may be omitted on a first reading.

Finally, in Chapter 9 we apply some of the results of the classification theory to obtain identities for special functions which are not related to the factorization method.

It should be noted that we will be primarily concerned with the representation theory of local Lie groups, a subject which was developed in the nineteenth century. The more recent and sophisticated theory of global Lie groups is by itself too narrow to obtain many fundamental identities for special functions. Also, unitary representations of Lie groups will occur only as special cases of our results; they will not be our primary concern.

The scope of this book is modest: we study no Lie algebras with dimension greater than 6. Furthermore, in the six-dimensional case, $\mathscr{T}_6$, we do not give complete results. (The so-called addition theorems of Gegenbauer type for Bessel functions would be obtained from such an analysis.) However, it should be clear to the reader that our methods can be generalized to higher dimensional Lie algebras.

We will almost exclusively be concerned with the derivation of recursion relations and addition theorems. The manifold applications of group theory in the derivation of orthogonality relations and integral transforms of special functions will rarely be considered. For these applications see the encyclopedic work of Vilenkin [3]. The overlap in subject matter between that book and this one is relatively small, except in the study of unitary representations of real Lie groups.

This monograph has been strongly influenced by the work of Bargmann [1–4], Infeld and Hull [1], Schwinger [1], Weisner [1–3], and Wigner [1, 2]. The author's primary contribution has been to weave these papers into a coherent theory relating Lie groups and special functions.

A knowledge of the basic elements of complex analysis, functional analysis, local Lie theory, and the theory of special functions is desirable for a proper understanding of the work presented here. This would correspond roughly with the background of an advanced graduate student in theoretical physics.

I wish to dedicate this book to the memory of Professor Bernard Friedman, in the hope that it will in a small way help to bridge the gap between the pure and applied sciences, a cause for which he was such an eloquent advocate.

January 1968 WILLARD MILLER, JR.