Appendix B: Completely Continuous Symmetric Operators

Let $T$ be a bounded operator on the Hilbert space $\mathcal{H}$. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of $T$ in case there exists a nonzero $u \in \mathcal{H}$ such that $Tu = \lambda u$. Each such nonzero $u$ is an eigenvector of $T$ corresponding to eigenvalue $\lambda$. If $\mathcal{H}$ is $n$-dimensional and $T$ is self-adjoint it is well-known that there exists an ON basis $\{e_j\}$ for $\mathcal{H}$ consisting of eigenvectors of $T$:

$$Tu_j = \lambda_j u_j, \quad j = 1, \ldots, n.$$  

The matrix of $T$ with respect to this basis is diagonal:

$$T = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$  

However, if $\mathcal{H}$ is infinite-dimensional and $T$ is self-adjoint it is usually not possible to find an ON basis for $\mathcal{H}$ consisting of eigenvectors. There is a sense in which $T$ can be diagonalized (the spectral theorem for self-adjoint operators) but this is not a straightforward extension of the procedure for diagonalizing self-adjoint operators on finite-dimensional spaces.

Nevertheless, there is a class of operators $T$ of great importance in mathematical physics for which the eigenvectors do form an ON basis in $\mathcal{H}$: the completely continuous self-adjoint operators.

A subset $\mathcal{B}$ of $\mathcal{H}$ is bounded if there exists a constant $C > 0$ such that $\|v\| < C$ for all $v \in \mathcal{B}$.  

**Definition:** An operator $T$ on $\mathcal{H}$ is completely continuous if for every bounded sequence $\{v_j\}$ in $\mathcal{H}$, there is a subsequence $\{v_{j_k}\}$, $j_k < j_{k+1} < \cdots < j_k < \cdots$, such that $\{Tv_{j_k}\}$ is convergent.
Note: It is easy to show that a completely continuous operator is bounded.

**Example 1:** Every linear operator on a finite-dimensional space is completely continuous.

**Example 2:** The identity operator $\mathbf{E}$ on an infinite dimensional space is not completely continuous. (Hint: Look at the action of $\mathbf{E}$ on an ON basis of $\mathcal{H}$.)

**Example 3:** Let $\mathcal{H}=L_2(\mathbb{R}_m)$, (appendix A), and let $h_\gamma(x), g_\gamma(x) \in C(\mathbb{R}_m)$, $\gamma = 1, \ldots, m$. Let $K$ be the operator on $L_2(\mathbb{R}_m)$ defined by

$$Kf(x) = \sum_{\gamma=1}^{m} K(x,\gamma) f(\gamma) w(\gamma) \, \, \, \, f \in L_2(\mathbb{R}_m)$$

where

$$K(x,\gamma) = \sum_{\gamma=1}^{m} h_\gamma(x) g_\gamma(\gamma)$$

is the kernel of the integral operator $K$ and $w(\gamma)$ is the weight function on $L_2(\mathbb{R}_m)$. Then $K$ is completely continuous. This follows from the fact that $R_K$ is finite-dimensional.

**Example 4:** Let $\mathcal{H}=L_2(\mathbb{R}_m)$ and let $K$ be an integral operator (B.1) where now we require only that the kernel $K(x,\gamma)$ be continuous in $x$ and $\gamma$. Then $K$ is completely continuous. Moreover, if $K(x,\gamma)=\overline{K(\gamma,x)}$ then $K$ is self-adjoint.

Note: We give no proofs in this appendix. For detailed proofs the reader can consult [Helwig, 1] or [Stakgold, 1]. However, the reader should be able to supply the elementary proof of

**Lemma B.1:** Let $\mathcal{T}$ be a bounded self-adjoint operator on $\mathcal{H}$. Then the eigenvalues of $\mathcal{T}$ are real and eigenvectors corresponding to distinct
eigenvalues are orthogonal.

**Theorem 2.14:** Let $\mathbf{T}$ be a nonzero completely continuous self-adjoint operator on the separable Hilbert space $\mathcal{H}$. Let $C_\lambda = \{ u \in \mathcal{H} : \mathbf{T}u = \lambda u \}$ be the eigenspace corresponding to the eigenvalue $\lambda$. Then

a) $\mathbf{T}$ has at least one nonzero eigenvalue $\lambda_1$ and at most countably many, $\lambda_1 \geq \lambda_2 \geq \cdots$. Each eigenspace $C_{\lambda_i}$ for $\lambda_i \neq 0$ is finite-dimensional. If there are an infinite number of eigenvalues then $\lim_{i \to \infty} \lambda_i = 0$.

b) Let $\lambda_1, \lambda_2, \cdots$ be the eigenvalues of $\mathbf{T}$, possibly including $\lambda = 0$, and let $\{ u_{ij}^\dagger, j = 1, 2, \cdots, \dim C_{\lambda_i} \}$ be an ON basis for $C_{\lambda_i}$. Then $\{ u_{ij}^\dagger, j = 1, 2, \cdots, \dim C_{\lambda_i}, i=1,2,\cdots \}$ is an ON basis for $\mathcal{H}$.

c) If $u \in R_T$, $u = \sum u_i^\dagger \psi_i$, then

$$u = \sum_{i,j} (T \psi_i, u_i^\dagger) u_j^\dagger = \sum_{i,j} (\psi_i, T u_i^\dagger) u_j^\dagger = \sum_{i,j} \lambda_i (\psi_i, u_i^\dagger) u_j^\dagger.$$

Note: Part c) follows immediately from a) and b). The sum in the expansion of $u$ goes only over those eigenvectors corresponding to nonzero eigenvalues.

Consider the completely continuous self-adjoint integral operator $\mathbf{K}$ on $L_2(\mathcal{M})$, (example 4). The kernel $K(x,y)$ of $\mathbf{K}$ is continuous in all its arguments and satisfies $K(x,y) = K(y,x)$. The preceding theorem clearly applies to $\mathbf{K}$. Moreover, by making use of the special structure of $\mathbf{K}$ we can obtain more information about the expansion a). The eigenvectors $u_i^\dagger(x)$ are now functions in $L_2(\mathcal{M})$.

**Theorem 2.15:** 1) Let $\lambda$ be a nonzero eigenvalue of $\mathbf{K}$ and $z(x)$ a corresponding eigenfunction. Then $z(x) \in C(\mathcal{M})$. 2) More generally,
if \( u(x) \in R_\lambda \) then \( u(x) \in C(\partial M) \). 3) If \( u(x) \in R_\lambda \), \( u(x) = \sum u(x) \)
then
\[
    u(x) = \sum_{l,j} (u_l, u_j) u_l^j(x) = \sum_{l,j} \gamma_l^j (\tau, u_l^j) u_l^j(x)
\]
where the series converges uniformly to \( u(x) \) (pointwise) for all \( x \in \partial M \).

The point of statement 3) is that the expansion of \( u \in R_\lambda \) in terms of the eigenfunctions \( u_l^j(x) \), converges not only in the norm but also pointwise uniformly.