Appendix

For reference we list here some fundamental properties of special functions which occur frequently throughout the book. All of these properties can be found in Magnus *et al.* [1] and, of course, in Erdélyi *et al.* [1]. In the following expressions μ, ν and ξ are arbitrary complex numbers and n is an integer.

1 The Gamma Function

For Re $z > 0$ the function $\Gamma(z)$ is defined in terms of the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

However, by analytic continuation $\Gamma(z)$ can be extended to a function analytic in the whole complex plane, with the exception of simple poles at $z = -n$, $n = 0, 1, 2, \ldots$.

Functional equations:

$$\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{A.1}$$

Special values:

$$\begin{aligned} \Gamma(n+1) &= n! = n(n-1)\cdots(1), \qquad n = 0, 1, 2, \ldots . \\ \Gamma(\tfrac{1}{2}) &= \sqrt{\pi} \end{aligned} \tag{A.2}$$

Binomial coefficients:

$$\binom{\mu}{n} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)\, n!}. \tag{A.3}$$

2 The Hypergeometric Function

The hypergeometric series

$$\begin{aligned}F(a, b; c; z) &= {}_2F_1(a, b; c; z)\\ &= 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots\\ &\quad + \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)}\frac{z^n}{n!}\\ &\quad + \cdots \end{aligned} \tag{A.4}$$

defines a function which is analytic when $|z| < 1$. If a or b equals $-n$, $n = 0, 1, 2, \ldots$, the series is finite. $F(a, b; c; z)$ can be analytically continued to define a function analytic and single-valued throughout the cut z-plane, where the cut runs along the positive real axis from the branch point $+1$ to $+\infty$. For $\operatorname{Re} c > \operatorname{Re} b > 0$ this function has the integral representation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,(1-tz)^{-a}\,dt.$$

Limit relation:

$$\begin{aligned}\lim_{c\to -n}\frac{F(a, b; c; z)}{\Gamma(c)} &= \frac{a(a+1)\cdots(a+n)\,b(b+1)\cdots(b+n)}{(n+1)!}\\ &\quad\cdot z^{n+1}F(a+n+1, b+n+1; n+2; z),\\ &\quad n = 0, 1, 2, \ldots .\end{aligned} \tag{A.5}$$

$F(a, b; c; z)$ is a solution, regular at $z = 0$, of the hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)\,z]\frac{du}{dz} - abu = 0. \tag{A.6}$$

For $c \neq 0, -1, -2, \ldots$, this equation has a second solution $z^{1-c}F(a-c+1, b+1; c+1; z)$.

Differential recursion formulas:

$$\begin{aligned}\frac{d}{dz}F(a, b; c; z) &= \frac{ab}{c}F(a+1, b+1; c+1; z)\\ \left[a + z\frac{d}{dz}\right]F(a, b; c; z) &= aF(a+1, b; c; z)\\ \left[(a-c) + bz - z(1-z)\frac{d}{dz}\right]F(a, b; c; z) &= (a-c)F(a-1, b; c; z)\\ \left[(a+b-c) - (1-z)\frac{d}{dz}\right]F(a, b; c; z) &= \frac{(b-c)(c-a)}{c}F(a, b; c+1; z)\\ \left[(c-1) + z\frac{d}{dz}\right]F(a, b; c; z) &= (c-1)F(a, b; c-1; z)\end{aligned} \tag{A.7}$$

Transformation formulas:

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right)$$
$$= (1-z)^{c-a-b} F(c-a, c-b; c; z). \tag{A.8}$$

Some functions represented by hypergeometric functions:

(i) Legendre polynomials

$$P_n(\cos\theta) = F\left(n+1, -n; 1; \sin^2\frac{\theta}{2}\right), \quad n = 0, 1, 2, \ldots . \tag{A.9}$$

(ii) General spherical harmonics

$$\mathfrak{P}_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right)$$
$$\left(\arg\frac{z+1}{z-1} = 0 \text{ for } z \text{ real and } > 1\right).$$

$$\mathfrak{Q}_\nu^\mu(z) = \frac{e^{\mu\pi i}}{2^{\nu+1}}\frac{\Gamma(\nu+\mu+1)\,\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{3}{2})}(z^2-1)^{\mu/2} z^{-\nu-\mu-1}$$
$$\cdot F\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$
$$(\arg(z^2-1) = 0, \text{ for } z \text{ real and } > 1,$$
$$\arg z = 0, \text{ for } z \text{ real and } > 0).$$

(iii) $$\mathfrak{P}_\nu^{\mu,\xi}(z) = \frac{1}{\Gamma(\xi+\mu+1)}\left(\frac{z+1}{2}\right)^{(2\nu-\xi-\mu)/2}\left(\frac{z-1}{2}\right)^{(\mu+\xi)/2}$$
$$\cdot F\left(-\nu+\mu, -\nu+\xi; \mu+\xi+1; \frac{z-1}{z+1}\right)$$
$$= \frac{1}{\Gamma(\xi+\mu+1)}\left(\frac{z+1}{2}\right)^{(\xi-\mu)/2}\left(\frac{z-1}{2}\right)^{(\xi+\mu)/2}$$
$$\cdot F\left(\nu+\xi+1, -\nu+\xi; \mu+\xi+1; \frac{1-z}{2}\right)$$
$$(\arg(z+1) = \arg(z-1) = 0 \text{ for } z \text{ real and } > 1).$$

(iv) Gegenbauer polynomials

$$C_n^\nu(t) = \frac{\Gamma(n+2\nu)}{\Gamma(n+1)\,\Gamma(2\nu)} F\left(n+2\nu, -n; \nu+\tfrac{1}{2}; \frac{1-t}{2}\right).$$

(v) Jacobi polynomials

$$\mathscr{F}_n(\alpha, \gamma, x) = F(-n, \alpha + n; \gamma; x), \qquad n = 0, 1, 2, \ldots .$$

3 The Confluent Hypergeometric Function

The confluent hypergeometric series

$$ {}_1F_1(a; c; z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \cdots + \frac{a(a+1)\cdots(a+n-1)}{c(c+1)\cdots(c+n-1)}\frac{z^n}{n!} + \cdots, \tag{A.10}$$

defines an entire function of z. This function is a solution of the differential equation

$$z\frac{d^2v}{dz^2} + (c - z)\frac{dv}{dz} - av = 0. \tag{A.11}$$

For c not an integer, a second solution of (A.11) is

$$z^{1-c}\,{}_1F_1(a - c + 1; 2 - c; z).$$

Limit relation:

$$\lim_{c\to -n}\frac{{}_1F_1(a; c; z)}{\Gamma(c)} = \frac{a(a+1)\cdots(a+n)}{(n+1)!}\,z^{n+1}\,{}_1F_1(a + n + 1; n + 2; z), \qquad n = 0, 1, 2, \ldots . \tag{A.12}$$

Recursion formulas:

$$\begin{aligned}
\frac{d}{dz}\,{}_1F_1(a; c; z) &= \frac{a}{c}\,{}_1F_1(a+1; c+1; z)\\
\left[-1 + \frac{d}{dz}\right]{}_1F_1(a; c; z) &= \frac{a-c}{c}\,{}_1F_1(a; c+1; z),\\
\left[(c-1) + z\frac{d}{dz}\right]{}_1F_1(a; c; z) &= (c-1)\,{}_1F_1(a; c-1; z), \qquad \text{(A.13)}\\
\left[a + z\frac{d}{dz}\right]{}_1F_1(a; c; z) &= a\,{}_1F_1(a+1; c; z),\\
\left[-z + (c-a) + z\frac{d}{dz}\right]{}_1F_1(a; c; z) &= (c-a)\,{}_1F_1(a-1; c; z).
\end{aligned}$$

Transformation formula:

$${}_1F_1(a; c; z) = e^z\,{}_1F_1(c - a; c; -z). \tag{A.14}$$

Some functions represented by confluent hypergeometric functions:

(i) Generalized Laguerre functions

$$L_\nu^{(\alpha)}(z) = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\,\Gamma(\nu+1)}\,{}_1F_1(-\nu;\alpha+1;z).$$

(ii) Bessel functions

$$J_\nu(z) = \frac{e^{-iz}(z/2)^\nu}{\Gamma(\nu+1)}\,{}_1F_1(\nu+\tfrac{1}{2};2\nu+1;2iz). \tag{A.15}$$

(iii) Parabolic cylinder functions

$$D_\nu(z) = 2^{\nu/2}\exp\left(-\frac{z^2}{4}\right)\left[\frac{\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-\nu}{2}\right)}\,{}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right)+\frac{z}{\sqrt{2}}\right.$$
$$\left.\cdot\frac{\Gamma(-\frac{1}{2})}{\Gamma\left(-\frac{\nu}{2}\right)}\,{}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right)\right].$$

4 Parabolic Cylinder Functions

The parabolic cylinder function $D_\nu(z)$, (A.15), (iii), is a solution of the differential equation

$$\frac{d^2u}{dz^2}+\left(\nu+\frac{1}{2}-\frac{z^2}{4}\right)u=0 \tag{A.16}$$

as are the functions $D_\nu(-z)$, $D_{-\nu-1}(iz)$, $D_{-\nu-1}(-iz)$. Here $D_\nu(z)$ and $D_{-\nu-1}(iz)$ are linearly independent for all ν, and if ν is not an integer $D_\nu(z)$ and $D_\nu(-z)$ are linearly independent. If $\nu = n$ is a nonnegative integer then

$$D_n(z) = 2^{-n/2}\exp(-z^2/4)\,H_n(2^{-1/2}z), \tag{A.17}$$

where

$$H_n(z) = (-1)^n\exp(z^2)\frac{d^n}{dz^n}\exp(-z^2) \tag{A.18}$$

is the Hermite polynomial of order n.

Recursion formulas:

$$\left[\frac{z}{2}+\frac{d}{dz}\right]D_\nu(z)=\nu D_{\nu-1}(z),\qquad \left[\frac{z}{2}-\frac{d}{dz}\right]D_\nu(z)=D_{\nu+1}(z). \tag{A.19}$$

5 Bessel Functions

The Bessel function $J_\nu(z)$ can be defined by the series expansion

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1; -z^2/4), \qquad |\arg z| < \pi, \tag{A.20}$$

where

$${}_0F_1(c; z) = 1 + \frac{1}{c}\frac{z}{1!} + \frac{1}{c(c+1)}\frac{z^2}{2!} + \cdots + \frac{1}{c(c+1)\cdots(c+n-1)}\frac{z^n}{n!} + \cdots. \tag{A.21}$$

Clearly, $z^{-\nu}J_\nu(z)$ is an entire function of z. The $J_\nu(z)$ is a solution of Bessel's equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) u = 0, \tag{A.22}$$

as is $J_{-\nu}(z)$. For ν not an integer, $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent. However, if $\nu = n$ is an integer then $J_{-n}(z) = (-1)^n J_n(z)$ and $J_n(z)$ is the only solution of (A.22) which is regular at $z = 0$.

Recursion formulas:

$$\left[\frac{\nu}{z} - \frac{d}{dz}\right] J_\nu(z) = J_{\nu+1}(z), \qquad \left[\frac{\nu}{z} + \frac{d}{dz}\right] J_\nu(z) = J_{\nu-1}(z). \tag{A.23}$$

Spherical Bessel functions:

$$j_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{n+\frac{1}{2}}(z), \qquad n \geqslant 0. \tag{A.24}$$