Q-SERIES AND ORTHOGONAL POLYNOMIALS ASSOCIATED WITH BARNES' FIRST LEMMA*

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ABSTRACT. We exploit symmetry (recurrence relation) techniques for the derivation of properties associated with families of basic hypergeometric functions. Similar methods have been used by Nikiforov, Suslov, and Uvarov. Here we apply these ideas to find new proofs of Barnes' First Lemma and some of its q-analogues. We show that these integrals correspond to the weight functions determining the orthogonality relations for Hahn, q-Hahn, and big q-Jacobi polynomials. As another example of our method we introduce a biorthogonal system of rational functions whose weight function corresponds to the q-analogue of Kummer's Theorem.

1. Introduction. In papers Agarwal et al. (1987), Kalnins and Miller (1987), Miller (1988), the authors have advocated the exploitation of symmetry (recurrence relation) techniques for the derivation of properties associated with families of basic hypergeometric functions, in analogy with the local Lie theory techniques for ordinary hypergeometric functions. In particular, we have used these ideas to give simple derivations of the orthogonality relations for the Askey-Wilson and Wilson polynomials (Askey and Wilson (1985), Wilson (1980)), and, in particular, simple evaluations of the weight function integrals that determine the normalizations of these polynomials. Similar techniques have been employed by Nikiforov, Suslov, and Uvarov (1985) and Nikiforov and Suslov (1986), but they have apparently not applied them to the computation of contour integrals and summation formulas.

In §2 we use recurrence relations obeyed by a family of q-Hahn polynomials to derive the complex orthogonality of these polynomials and several q-analogues of Barnes' First Lemma, including those of Watson (1910) and Askey and Roy (1987, eq. 2.8). These integrals correspond to the square of the norm of the constant polynomial 1. Expanding one of these contour integrals by residues we obtain the real orthogonality relations for the big q-Jacobi polynomials.

In §3 we carry out the corresponding computations for the limiting case $q \to 1-$ and obtain the classical Barnes' Lemma, which we now see is associated with the orthogonality of the Hahn polynomials.

In §4 we work out a simple but nontrivial example of the use of these ideas to derive biorthogonality relations for rational basic hypergeometric functions. The associated summation formula is the q-analogue of Kummer's Theorem, originally due to Andrews (1973).

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This link between recurrence relations obeyed by families of special functions, orthogonality relations for the functions, associated contour integrals and summation formulas appears capable of extensive generalization.

Most of the computations in the following sections were checked with SMP.

2. q-analogues of Barnes' Lemma. We are concerned with the q-Hahn polynomials

$$\Phi_n^{a,b,c,d}(z) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, & q^{n-1}abcd, & az \\ ac, & ad & \end{matrix}\; ; q\right), \tag{2.1}$$

$$n = 0, 1, 2, \ldots$$

where the basic hypergeometric functions ${}_{p+1}\varphi_p$ are defined as usual by

$${}_{p+1}\varphi_p\left(\begin{matrix} a_1, \ldots & a_{p+1} \\ b_1, \ldots & b_p \end{matrix}\; ; x\right) = \sum_{m=0}^{\infty} \frac{(a_1;q)_m \ldots (a_{p+1};q)_m x^m}{(b_1;q)_m \ldots (b_p;q)_m (q;q)_m}$$

and

$$\begin{aligned} &(a;q)_0 = 1, \\ &(a;q)_m = (1-a)(1-aq)\ldots(1-aq^{m-1}), \quad m \geq 1. \end{aligned}$$

Initially we require $0 < |q| < 1$, $acd \neq 0$, and $|a|, |b|, |c|, |d| < 1$, but some of these conditions can be relaxed later. The polynomials obey the fundamental recurrence relations

$$\mu^{(a,b,c,d)}\Phi_n^{(a,b,c,d)} = \frac{q^{\frac{1}{2}}}{d}\left(1 - \frac{ad}{q}\right)\Phi_n^{(aq^{-\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{-\frac{1}{2}})}, \tag{2.2A}$$

$$\tau^{(a,b,c,d)}\Phi_n^{(a,b,c,d)} = \frac{a(1-q^{-n})(1-q^{n-1}abcd)}{q^{-\frac{1}{2}}(1-ad)(1-ac)}\Phi_{n-1}^{(aq^{\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}})} \tag{2.2B}$$

where

$$\begin{aligned} \mu^{(a,b,c,d)} &= \frac{1}{z}\left[(1 - azq^{-\frac{1}{2}})E_z^{\frac{1}{2}} - \left(1 - \frac{zq^{\frac{1}{2}}}{d}\right)E_z^{-\frac{1}{2}}\right], \\ \tau^{(a,b,c,d)} &= \frac{1}{z}[E_z^{\frac{1}{2}} - E_z^{-\frac{1}{2}}] \end{aligned}$$

and $E_z^{\alpha}f(z) = f(q^{\alpha}z)$. These relations follow from

$$\begin{aligned} \mu(za;q)_n &= \frac{q^{\frac{1}{2}}}{d}(1 - adq^{n-1})(zaq^{\frac{1}{2}};q)_n, \\ \tau(za;q)_n &= \frac{a}{q^{\frac{1}{2}}}(1-q^n)(zaq^{-\frac{1}{2}};q)_{n-1}. \end{aligned}$$

The existence of μ suggests the existence of a recurrence taking $\Phi_n^{(aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}})}$ to $\Phi_n^{(a,b,c,d)}$. Indeed, the appropriate operator is

$$\begin{aligned}
\mu^* &= -\frac{q^{\frac{1}{2}}}{\rho}\mu^{(bq^{\frac{1}{2}},aq^{-\frac{1}{2}},dq^{-\frac{1}{2}},cq^{\frac{1}{2}})},\\
&\mu^*\Phi_n^{(aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}})}\\
&= \frac{q^{\frac{1}{2}}}{\rho}\frac{(\frac{ad}{q}-q^{-n})(1-bcq^n)}{c(1-\frac{ad}{q})}\Phi_n^{(a,b,c,d)}, \quad \rho\neq 0,
\end{aligned} \tag{2.2C}$$

which follows from

$$\frac{\rho}{q^{\frac{1}{2}}}\mu^*(azq^{-\frac{1}{2}};q)_n = (az;q)_n(bcq^n-1)\frac{q^{-n}}{c} + (az;q)_{n-1}(acq^{n-1}-1)(q^n-1)\frac{q^{-n}}{c}.$$

Let $w_{a,b,c,d}(z)$ be a (complex-valued) weight function and $S_{a,b,c,d}$ be the indefinite inner product space of polynomials $f(z)$ with respect to the inner product

$$(f_1,f_2)_{a,b,c,d} = \frac{1}{2\pi i}\oint_C f_1(z)f_2(z)w_{a,b,c,d}(z)\frac{dz}{z} \tag{2.3}$$

where C is a deformation of the unit circle $|z|=1$. Now consider $\mu=\mu^{(a,b,c,d)}$ and $\mu^* = -\frac{q^{\frac{1}{2}}}{\rho}\mu^{(bq^{\frac{1}{2}},aq^{-\frac{1}{2}},dq^{-\frac{1}{2}},cq^{-\frac{1}{2}})}$ as mappings

$$\begin{aligned}
&\mu : S_{a,b,c,d} \to S_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}},\\
&\mu^* : S_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}} \to S_{a,b,c,d}
\end{aligned} \tag{2.4}$$

and determine $w_{a,b,c,d}$ so that

$$(\mu f, g)_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}} = (f,\mu^* g)_{a,b,c,d} \tag{2.5}$$

for all $f\in S_{a,b,c,d}$, $q\in S_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}}$. Condition (2.5) yields a q-difference equation for $w_{a,b,c,d}$ with solution

$$w_{a,b,c,d}(z) = \frac{(\frac{\rho z}{d};q)_\infty(\frac{qd}{z\rho};q)_\infty(\frac{\rho c}{z};q)_\infty(\frac{qz}{c\rho};q)_\infty}{(az;q)_\infty(bz;q)_\infty(\frac{c}{z};q)_\infty(\frac{d}{z};q)_\infty} \tag{2.6}$$

where

$$(x;q)_\infty = \lim_{n\to\infty}(x;q)_n.$$

This result is a consequence of the invariance of the contour integral under the changes of variable $z\to q^{\pm\frac{1}{2}}z$, and the property $h(qz) = -h(z)/z$ where $h(z) = (z;q)_\infty(q/z;q)_\infty$.

It follows immediately that $\mu^*\mu$ is a self-adjoint operator

$$\mu^*\mu : S_{a,b,c,d} \to S_{a,b,c,d}$$

and from the recurrence relations (2.2) we have

$$\mu^*\mu\Phi_n^{(a,b,c,d)} = \lambda_n\Phi_n^{(a,b,c,d)}, \quad \lambda_n = \frac{q}{cd\rho}\left(\frac{ad}{q} - q^{-n}\right)(1-bcq^n). \tag{2.7}$$

Since eigenfunctions corresponding to distinct eigenvalues are orthogonal we have

$$(\Phi_n^{(a,b,c,d)}, \Phi_m^{(a,b,c,d)})_{a,b,c,d} = 0 \quad \text{for } m \neq n. \tag{2.8}$$

Relation (2.5) for $f = g \equiv 1$ yields

$$\|1\|^2_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}} = -\frac{d}{\rho c}\frac{1-bc}{\left(1-\frac{ad}{q}\right)}\|1\|^2_{a,b,c,d} \tag{2.9}$$

where

$$\|1\|^2_{a,b,c,d} = (1,1)_{a,b,c,d}.$$

The symmetry of the weight function in (a,b) yields an additional relation of the form (2.9). Furthermore the obvious relation

$$(\Phi_1^{(a,b,c,d)}, \Phi_0^{(a,b,c,d)})_{a,b,c,d} = 0,$$

the explicit expression (2.1), and the property $(1,p_n)_{a,b,c,d} = \|1\|^2_{aq^n,b,c,d}$ for $p_n(z) = (az;q)_n$, yield the relation

$$\|1\|^2_{aq,b,c,d} = \frac{(1-ad)(1-ac)}{(1-abcd)}\|1\|^2_{a,b,c,d}. \tag{2.10}$$

Again, the symmetry in (a,b) gives an additional relation. The solution of these q-difference equations is

$$\|1\|^2_{a,b,c,d} = \frac{(abcd;q)_\infty(\frac{c\rho}{d};q)_\infty(\frac{qd}{c\rho};q)_\infty}{(ad;q)_\infty(ac;q)_\infty(ac;q)_\infty(bc;q)_\infty(bd;q)_\infty}K(\rho,q) \tag{2.11}$$

where $K(\rho,q)$ is to be determined. In the special case $a = \rho/d$, $b = q/c\rho$ we can compute the (trivial) integral directly: $\|1\|^2_{\rho/d,q/c\rho,c,d} = 1$. Hence $K(\rho,q) = (\rho;q)_\infty(q/\rho;q)_\infty/(q;q)_\infty$ and we have

$$\begin{aligned}\|1\|^2_{a,b,c,d} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\frac{\rho}{d}e^{i\theta};q)_\infty(\frac{qd}{\rho}e^{-i\theta};q)_\infty(\rho c e^{i\theta};q)_\infty}{(ae^{i\theta};q)_\infty(be^{i\theta};q)_\infty(ce^{i\theta};q)_\infty} \\ &\quad\cdot\frac{(\frac{q}{c\rho}e^{i\theta};q)_\infty}{(de^{-i\theta};q)_\infty}d\theta \\ &= \frac{(abcd;q)_\infty(\frac{c\rho}{d};q)_\infty(\frac{qd}{c\rho};q)_\infty(\rho;q)_\infty(\frac{q}{\rho};q)_\infty}{(ad;q)_\infty(ac;q)_\infty(bc;q)_\infty(bd;q)_\infty(q;q)_\infty}\end{aligned} \tag{2.12}$$

in agreement with Askey and Roy (1986).

Now we consider the recurrence

$$\tau^{(a,b,c,d)} : S_{a,b,c,d} \to S_{aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}},$$

(2.2B) and compute the adjoint $\tau^* \equiv \tau^{*(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}}dq^{\frac{1}{2}})}$ such that

$$(\tau f, g)_{aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}} = (f, \tau^* g)_{a,b,c,d} \tag{2.13}$$

for all $f \in S_{a,b,c,d}$, $g \in S_{aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}}$. A straightforward computation yields

$$\tau^{*(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})} = \frac{1}{q^{\frac{1}{2}}z}\left[-(1-az)(1-bz)E_z^{\frac{1}{2}} + \left(1-\frac{z}{c}\right)\left(1-\frac{z}{d}\right)E_z^{-\frac{1}{2}}\right]. \tag{2.14}$$

It follows that $\tau^*\tau : S_{a,b,c,d} \to S_{a,b,c,d}$ is self-adjoint. Moreover, the action of τ^* on the polynomial basis is

$$\tau^*\Phi_{n-1}^{(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})} = \frac{(1-ac)(1-ad)}{q^{\frac{1}{2}}acd}\Phi_n^{(a,b,c,d)}. \tag{2.15}$$

This follows from

$$\tau^*(aq^{\frac{1}{2}}z;q)_k = \frac{q^{-k-\frac{1}{2}}}{acd}(1-acq^k)(1-adq^k)(az;q)_k - \frac{q^{-k-\frac{1}{2}}}{acd}(1-abcdq^k)(az;q)_{k+1}.$$

Thus

$$\tau^*\tau\Phi_n^{(a,b,c,d)} = \frac{1}{cd}(1-q^{-n})(1-q^{n-1}abcd)\Phi_n^{(a,b,c,d)}. \tag{2.16}$$

Setting $f = \Phi_n^{(a,b,c,d)}$, $g = \Phi_{n-1}^{(aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})}$ in (2.13) we obtain the recurrence

$$\|\Phi_n^{(a,b,c,d)}\|_{a,b,c,d}^2 = \frac{qa^2cd(1-q^{-n})(1-q^{n-1}abcd)}{(1-ad)^2(1-ac)^2}\|\Phi_{n-1}^{(aq^{\frac{1}{2}},\dots,dq^{\frac{1}{2}})}\|_{aq^{\frac{1}{2}},\dots,dq^{\frac{1}{2}}}^2, \tag{2.17}$$

which permits us to compute the norms $\|\Phi_n^{(a,\dots,d)}\|_{a,\dots,d}^2$ recursively from $\|1\|_{a,\dots,d}^2$. Note that the norms are all nonzero. We have shown that the q–Hahn polynomials $\{\Phi_n^{(a,b,c,d)}\}$ are uniquely characterized by their orthogonality with respect to the complex weight function $w_{a,b,c,d}$.

Since the weight function is symmetric in $\{a, b\}$ the orthogonal polynomials satisfies the transformation rule

$$ {}_3\varphi_2\left(\begin{matrix} q^{-n}, & q^{n-1}abcd, & bz \\ bc, & dc & \end{matrix};q\right) = \left(\frac{b}{a}\right)^n \frac{(ac;q)_n(ad;q)_n}{(bc;q)_n(bd;q)_n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, & q^{n-1}abcd, & az \\ ac, & ad & \end{matrix};q\right). \tag{2.18}$$

Also, the action of $\tau^{*(aq^{\frac{1}{2}},\cdots,dq^{\frac{1}{2}})}$ determines a Rodrigues formula.

The complex orthogonality (2.8) for the q-Hahn polynomials leads to real discrete orthogonality for the big q-Jacobi polynomials (Andrews and Askey (1985)):

$$ {}_3\varphi_2\left(\begin{matrix} q^{-n}, & q^{n-1+\alpha+\beta+\gamma+\delta}, & q^{x+1} \\ q^{\alpha+\gamma}, & q^{\alpha+\delta} & \end{matrix};q\right). \tag{2.19}$$

The polynomials (2.19) are orthogonal with respect to the discrete measure with mass points and corresponding weights (Ismail and Wilson (1982)):

$$
\begin{aligned}
&x=\alpha+\delta+k-1 \qquad \frac{(q^{\delta-\gamma+k+1};q)_\infty (q^{k+1};q)_\infty}{(q^{\alpha+\delta+k};q)_\infty (q^{\beta+\delta+k};q)_\infty} q^k q^{\alpha+\delta-1}(1-q),\\
&x=\alpha+\gamma+k-1 \qquad \frac{(q^{\gamma-\delta+k+1};q)_\infty (q^{k+1};q)_\infty}{(q^{\alpha+\gamma+k};q)_\infty (q^{\beta+\gamma+k};q)_\infty} q^k q^{\alpha+\gamma-1}(1-q),
\end{aligned}
\tag{2.20}
$$

$k=0,1,2,\ldots$. To obtain this result from (2.6), (2.8) we first set $a=q^\alpha,\ldots,d=q^\delta$. If $\mathrm{Re}(\alpha,\beta,\gamma,\delta)>0$ and there are no double poles we can expand the integral $(\Phi_n,\Phi_m)_{a,b,c,d}$ by residues, using the simple poles of $w_{a,b,c,d}(z)$, (2.6), at $z=q^{\gamma+k}$, $z=q^{\delta+k}$. The result of this expansion is (2.20) with $z=q^x$. In particular, the dependence on ρ cancels out.

Our approach relating orthogonal polynomials to integrals of weight functions can be used to evaluate other important integrals. One class of such integrals can be conveniently studied through the change of variables given in the preceding paragraph:

$$
a=q^\alpha,\quad b=q^\beta,\quad c=q^\gamma,\quad d=q^\delta,\quad z=q^x. \tag{2.21}
$$

This change reduces q-difference equations for the weight function to ordinary difference equations. Indeed the operators (2.2) now take the form

$$
\begin{aligned}
\mu^{(\alpha,\beta,\gamma,\delta)} &= q^{-x}[(1-q^{\alpha+x-\frac12})E_x^{\frac12}-(1-q^{x-\delta+\frac12})E_x^{-\frac12}],\\
\tau^{(\alpha,\beta,\gamma,\delta)} &= q^{-x}[E_x^{\frac12}-E_x^{-\frac12}],\\
\mu* &= -q^{-x}[(1-q^{\beta+x})E_x^{\frac12}-(1-q^{x-\gamma})E_x^{-\frac12}]
\end{aligned}
\tag{2.22}
$$

where $E_x^s g(x)=g(x+s)$.

The orthogonal functions are now polynomials in q^x. We require that the inner product take the form

$$
(f_1,f_2)_{\alpha,\beta,\gamma,\delta}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f_1(q^x)f_2(q^x)w_{\alpha,\beta,\gamma,\delta}(x)dx \tag{2.23}
$$

where the contour in the complex x-plane will run from $-i\infty$ to $+i\infty$ so that decreasing sequences of poles for w lie on the left and increasing sequences of poles lie on the right. The condition that μ^* be the adjoint of μ now becomes

$$
(\mu f,g)_{\alpha-\frac12,\beta+\frac12,\gamma+\frac12,\delta-\frac12}=(f,\mu^* g)_{\alpha,\beta,\gamma,\delta} \tag{2.24}
$$

with $f\in S_{\alpha,\beta,\gamma,\delta}$, $g\in S_{\alpha-\frac12,\beta+\frac12,\gamma+\frac12,\delta-\frac12}$, where $S_{\alpha,\beta,\gamma,\delta}$ is the space of polynomials in q^x with inner product (2.23). The nonunique solution for w is

$$
w_{\alpha,\beta,\gamma,\delta}(x)=\frac{(q^{x-\gamma+1};q)_\infty (q^{x-\delta+1};q)_\infty}{(q^{x+\alpha};q)_\infty (q^{x+\beta};q)_\infty} q^x H(\alpha,\beta,\gamma,\delta,x) \tag{2.25}
$$

where H is an analytic function of its variables satisfying the periodicity properties

$$\begin{aligned}
&H(\alpha,\beta,\gamma,\delta,x) = H(\alpha,\beta,\gamma,\delta,x+1),\\
&H\left(\alpha-\frac{1}{2},\beta-\frac{1}{2},\gamma\pm\frac{1}{2},\delta\mp\frac{1}{2},x+\frac{1}{2}\right)\\
&= H\left(\alpha-\frac{1}{2},\beta-\frac{1}{2},\gamma\pm\frac{1}{2},\delta\mp\frac{1}{2},x+\frac{1}{2}\right)\\
&= H\left(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2},x+\frac{1}{2}\right).
\end{aligned} \tag{2.26}$$

One solution of (2.26) is

$$H(\alpha,\beta,\gamma,\delta,x) = \frac{\sin\pi(\gamma-\delta)}{\sin\pi(\gamma-x)\sin\pi(\delta-x)}$$

so that the weight function becomes

$$w_{\alpha,\beta,\gamma,\delta}(x) = \frac{(q^{x-\gamma+1};q)_\infty (q^{x-\delta+1};q)_\infty q^x \sin\pi(\gamma-\delta)}{(q^{x+\alpha};q)_\infty (q^{x+\beta};q)_\infty \sin\pi(\gamma-x)\sin\pi(\delta-x)}. \tag{2.27}$$

Then the polynomials are
(2.28)

$$\Phi_n^{(\alpha,\beta,\gamma,\delta)}(q^x) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+\gamma+\delta-1}, & q^{\alpha+x} \\ q^{\alpha+\gamma}, & q^{\alpha+\delta} & \end{matrix} ;q\right), \qquad n=0,1,2,\ldots$$

and the eigenvalue equation is

$$\mu^*\mu\Phi_n^{\alpha,\beta,\gamma,\delta} = q^{\frac{1}{2}-\gamma-\delta}(q^{\alpha+\delta-1}-q^{-n})(1-q^{\beta+\gamma+n})\Phi_n^{(\alpha,\beta,\gamma,\delta)}.$$

We have immediately

$$(\Phi_n^{(\alpha,\ldots,\delta)},\Phi_m^{(\alpha,\ldots,\delta)})_{\alpha,\ldots,\delta} = 0, \qquad m\neq n$$

and

$$\|1\|^2_{\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}} = q^{-\frac{1}{2}+\delta-\gamma}\frac{(1-q^{\beta+\gamma})}{(1-q^{\alpha+\delta-1})}\|1\|^2_{\alpha,\beta,\gamma,\delta}. \tag{2.29}$$

The symmetry of w in (α,β) leads to a similar recurrence. Also, the skew-symmetry of w in γ,δ yields a new recurrence. Finally, the orthogonality

$$(\Phi_1^{(a,\ldots,\delta)},\Phi_0^{(\alpha,\ldots,\delta)}) = 0$$

leads to the recurrences

$$\|1\|^2_{\alpha+1,\beta,\gamma,\delta} = \frac{(1-q^{\alpha+\gamma})(1-q^{\alpha+\delta})}{(1-q^{\alpha+\beta+\gamma+\delta})}\|1\|^2_{\alpha,\beta,\gamma,\delta}$$

and

$$\|1\|^2_{\alpha,\beta,\gamma+1,\delta} = -\frac{q^{\delta-\gamma}(1-q^{\alpha+\gamma})(1-q^{\beta+\gamma})}{(1-q^{\alpha+\beta+\gamma+\delta})}\|1\|^2_{\alpha,\beta,\gamma,\delta}$$

with the solution

$$\|1\|^2_{\alpha,\beta,\gamma,\delta} = \frac{(q^{\alpha+\beta+\gamma+\delta};q)_\infty (q^{\delta-\gamma};q)_\infty (q^{1+\gamma-\delta};q)_\infty q^\gamma}{(q^{\alpha+\delta};q)_\infty (q^{\alpha+\gamma};q)_\infty (q^{\beta+\gamma};q)_\infty (q^{\beta+\delta};q)_\infty} M(\alpha,\beta,\gamma,\delta) \tag{2.30}$$

where M is an analytic function of its arguments, symmetric in (α, β) and in (γ, δ), satisfying the periodicity relations

$$\begin{aligned} M(\alpha,\beta,\gamma,\delta) &= M(\alpha+1,\beta,\gamma,\delta) = M(\alpha,\beta,\gamma+1,\delta) \\ &= M\left(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}\right). \end{aligned} \tag{2.31}$$

To evaluate M we replace α by $\alpha+k$, k a positive integer, and rewrite the integral for $\|1\|^2_{\alpha+k,\beta,\gamma,\delta}$ in the form $(x = iy)$

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{(q^{\alpha+k+\delta};q)_\infty (q^{\alpha+k+\delta};q)_\infty}{(q^{\alpha+\beta+\gamma+\delta+k};q)_\infty (q^{iy+\alpha+k};q)_\infty}\right] \frac{(q^{iy-\gamma+1};q)_\infty}{(q^{iy+\beta};q)_\infty} \\ &\qquad \cdot \frac{(q^{iy-\delta+1};q)_\infty q^{iy}\,dy}{\sin\pi(\gamma-iy)\sin\pi(\delta-iy)} \\ &= \frac{q^\gamma (q^{\delta-\gamma};q)_\infty (q^{1+\gamma-\delta};q)_\infty}{(q^{\beta+\gamma};q)_\infty (q^{\beta+\delta};q)_\infty} M(\alpha,\beta,\gamma,\delta). \end{aligned} \tag{2.32}$$

Notice that the right-hand side of (2.32) is independent of k and that the bracketed quantity on the left goes to 1 as $k \to +\infty$. From the Lebesgue dominated convergence theorem we conclude that

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{q^{iy-\gamma+1};q)_\infty (q^{iy-\delta+1};q)_\infty q^{iy}}{(q^{iy+\beta};q)_\infty \sin\pi(\gamma-iy)\sin\pi(\delta-iy)} \\ &= \frac{q^\delta (q^{\delta-\gamma};q)_\infty (q^{1+\gamma-\delta};q)_\infty}{(q^{\beta+\gamma};q)_\infty (q^{\beta+\delta};q)_\infty} M(\alpha,\beta,\gamma,\delta). \end{aligned}$$

It follows immediately that M is independent of α and β. Now in (2.32) set $k = 0$, $\alpha = 1-\gamma$, $\beta = 1-\delta$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{q^{iy}\sin\pi(\gamma-\delta)\,dy}{\sin\pi(\gamma-iy)\sin\pi(\delta-iy)} = \frac{M(\gamma,\delta)q^\gamma(1-q^{\delta-\gamma})}{(1-q)(q;q)_\infty}. \tag{2.33}$$

The rather elementary integral on the left-hand side of (2.33) can be easily evaluated by residues and the resulting geometric series summed to yield $M(\gamma,\delta) = (q;q)_\infty/\pi$. Thus

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(q^{iy-\gamma+1};q)_\infty (q^{iy-\delta+1};q)_\infty q^{iy}\sin\pi(\gamma-\delta)}{(q^{iy+\alpha};q)_\infty (q^{iy+\beta};q)_\infty \sin\pi(\gamma-iy)\sin\pi(\delta-iy)} dy \\ &= \frac{(q;q)_\infty q^\gamma (q^{\alpha+\beta+\gamma+\delta};q)_\infty (q^{\delta-\gamma};q)_\infty (q^{1+\gamma-\delta};q)_\infty}{\pi (q^{\alpha+\delta};q)_\infty (q^{\alpha+\gamma};q)_\infty (q^{\beta+\gamma};q)_\infty (q^{\beta+\delta};q)_\infty}, \end{aligned} \tag{2.34}$$

which is Watson's q-analogue of Barnes' First Lemma (Watson (1910)).

By choosing other solutions H of relations (2.26) we can evaluate other integrals in the form (2.30), (2.31). For example, if $H = \cos^{-2}\pi x$ then $M = (q;q)_\infty q^{\frac{1}{4}}(1-q^{\frac{1}{2}})/2\pi(q^\gamma - q^\delta)$. However, in general one cannot evaluate M by the simple method we used for (2.34).

3. The classical Barnes' Lemma. To see the relationship between our results and Barnes' Lemma we could, with care, let $q \to 1-$ in expression (2.12) (i.e., at the end of our construction). However, it is more instructive to take the limit $q \to 1-$ immediately and then proceed step-by-step through the argument of the preceding section. From this point of view the functions to be considered are the Hahn polynomials

$$\Phi_n^{(\alpha,\beta,\gamma,\delta)}(x) = {}_3F_2\left(\begin{matrix} -n, & n+\alpha+\beta+\gamma+\delta-1, & \alpha+x \\ \alpha+\gamma, & \alpha+\delta & \end{matrix}; 1\right) \tag{3.1}$$

$$n = 0, 1, 2, \ldots, \qquad \alpha, \beta, \gamma, \delta > 0$$

where ${}_3F_2$ is a generalized hypergeometric function:

$${}_{p+1}F_p\left(\begin{matrix} \alpha, \ldots, \alpha_{p+1} \\ \beta_1, \ldots, \beta_p \end{matrix}, z\right) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_{p+1})_m z^m}{(\beta_1)_m \ldots (\beta_p)_m m!},$$

$$(\alpha)_m = \begin{cases} 1 & \text{if} \quad m = 0, \\ \alpha(\alpha+1)\ldots(\alpha+m-1) & \text{if} \quad m \geq 1. \end{cases}$$

The recurrence relations are

$$\mu^{(\alpha,\beta,\gamma,\delta)}\Phi_n^{(\alpha,\beta,\gamma,\delta)} = (\alpha+\delta-1)\Phi_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2})}, \tag{3.2A}$$

(3.2B)
$$\mu^{*(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2})}\Phi_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2})} = \frac{(n+\alpha+\delta-1)(n+\beta+\gamma)}{(\alpha+\delta-1)}\Phi_n^{(\alpha,\beta,\gamma,\delta)},$$

$$\tau^{(\alpha,\beta,\gamma,\delta)}\Phi_n^{(\alpha,\beta,\gamma,\delta)} = \frac{-n(n+\alpha+\beta+\gamma+\delta-1)}{(\alpha+\delta)(\alpha+\gamma)}\Phi_{n-1}^{(\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2})}, \tag{3.2C}$$

$$\tau^{*(\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2})}\Phi_{n-1}^{(\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2})} = (\alpha+\delta)(\alpha+\gamma)\Phi_n^{(\alpha,\beta,\gamma,\delta)}, \tag{3.2D}$$

where

$$\begin{aligned}
\mu^{(\alpha,\beta,\gamma,\delta)} &= \left(\alpha+x-\frac{1}{2}\right)E_x^{\frac{1}{2}} + \left(\delta-x-\frac{1}{2}\right)E_x^{-\frac{1}{2}}, \\
\mu^{*(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2})} &= (\beta+x)E_x^{\frac{1}{2}} + (\gamma-x)E_x^{-\frac{1}{2}}, \\
\tau^{(\alpha,\beta,\gamma,\delta)} &= E_x^{\frac{1}{2}} - E_x^{-\frac{1}{2}}, \\
\tau^{*(\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2})} &= (\gamma-x)(\delta-x)E_x^{-\frac{1}{2}} - (\alpha+x)(\beta+x)E_x^{\frac{1}{2}},
\end{aligned}$$

and $E_x^s g(x) = g(x+s)$.

We define a complex inner product by

$$(g_1, g_2)_{\alpha,\beta,\gamma,\delta} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} g_1(x)g_2(x)w_{\alpha,\beta,\gamma,\delta}(x)dx \tag{3.3}$$

for polynomials g_1, g_2 where the integration path is the imaginary axis in the complex x-plane. Let $S_{\alpha,\beta,\gamma,\delta}$ be the space of polynomials with this inner product. We require that

$$(\mu f, g)_{\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}} = (f, \mu^* g)_{\alpha,\beta,\gamma,\delta} \tag{3.4}$$

for all $f \in S_{\alpha,\beta,\gamma,\delta}$, $g \in S_{\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}}$. In order that (3.4) hold, the weight function must satisfy a difference equation whose solution is (essentially)

$$w_{\alpha,\beta,\gamma,\delta}(x) = \Gamma(\alpha+x)\Gamma(\beta+x)\Gamma(\gamma-x)\Gamma(\delta-x) \tag{3.5}$$

where Γ is the gamma function (Whittaker and Watson (1958, Chap. XII)). Here we are using the fundamental recurrence for the gamma function

$$\Gamma(z+1) = z\Gamma(z).$$

It now follows that $\mu^*\mu : S_{\alpha,\beta,\gamma,\delta} \to S_{\alpha,\beta,\gamma,\delta}$ is self-adjoint with respect to this inner product and has eigenfunctions $\Phi_n^{\alpha,\beta,\gamma,\delta}$:

$$\mu^*\mu\Phi_n^{\alpha,\beta,\gamma,\delta} = (n+\alpha+\delta-1)(n+\beta+\gamma)\Phi_n^{\alpha,\beta,\gamma,\delta}. \tag{3.6}$$

Since eigenfunctions corresponding to distinct eigenvalues are orthogonal we have

$$(\Phi_n^{(\alpha,\beta,\gamma,\delta)}, \Phi_m^{(\alpha,\beta,\gamma,\delta)})_{\alpha,\beta,\gamma,\delta} = 0, \qquad n \neq m. \tag{3.7}$$

A similar computation gives

$$(\tau f, g)_{\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2}} = (f, \tau^* g)_{\alpha,\beta,\gamma,\delta} \tag{3.8}$$

for the same weight function and all

$$g \in S_{\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2}}, \qquad f \in S_{\alpha,\beta,\gamma,\delta}.$$

Thus $\tau^*\tau$ is self-adjoint on $S_{\alpha,\beta,\gamma,\delta}$ and

$$\tau^*\tau\Phi_n^{(\alpha,\beta,\gamma,\delta)} = -n(n+\alpha+\beta+\gamma+\delta-1)\Phi_n^{(\alpha,\beta,\gamma,\delta)}. \tag{3.9}$$

Setting $f = g = 1$ in (3.4) we find

$$||1||^2_{\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}} = \frac{(\beta+\gamma)}{(\alpha+\delta-1)}||1||^2_{\alpha,\beta,\gamma,\delta}. \tag{3.10}$$

Symmetry of the weight function in (α,β) and in (γ,δ) gives three more such relations. Furthermore

$$(\Phi_1^{(\alpha,\beta,\gamma,\delta)}, \Phi_0^{(\alpha,\beta,\gamma,\delta)})_{\alpha,\beta,\gamma,\delta} = 0,$$

which, from (3.1) and (3.5), implies

$$||1||^2_{\alpha+1,\beta,\gamma,\delta} = \frac{(\alpha+\gamma)(\alpha+\delta)}{(\alpha+\beta+\gamma+\delta)}||1||^2_{\alpha,\beta,\gamma,\delta} \tag{3.11}$$

and also

$$\|1\|^2_{\alpha,\beta,\gamma+1,\delta} = \frac{(\gamma+\alpha)(\gamma+\beta)}{(\alpha+\beta+\gamma+\delta)}\|1\|^2_{\alpha,\beta,\gamma,\delta}. \tag{3.12}$$

The symmetry of the weight function in (α, β) and in (γ, δ) gives two more such relations. It follows that

$$\|1\|^2_{\alpha,\beta,\gamma,\delta} = \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} M(\alpha,\beta,\gamma,\delta) \tag{3.13}$$

where M is symmetric in (α, β) and in (γ, δ), and satisfies the peridocity properties

$$\begin{aligned} M(\alpha+1,\beta,\gamma,\delta) &= M\left(\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}\right) \\ &= M(\alpha,\beta,\gamma+1,\delta) = M(\alpha,\beta,\gamma,\delta). \end{aligned} \tag{3.14}$$

To evaluate M we replace α by $\alpha+k$ and γ by $\gamma+k$, k a positive integer, and write the expression (3.13) in the form

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\Gamma(\alpha+\beta+\gamma+\delta+2k)\Gamma(\alpha+k+iy)\Gamma(\gamma+k-iy)}{\Gamma(\alpha+\gamma+2k)\Gamma(\alpha+\delta+k)\Gamma(\beta+\gamma+k)}\right) \\ &\quad\cdot\Gamma(\beta+iy)\Gamma(\delta-iy)dy = \Gamma(\beta+\delta)M(\alpha,\beta,\gamma,\delta). \end{aligned} \tag{3.15}$$

From Stirling's formula (Whittaker and Watson (1958, Chap. XII)), we have

$$\lim_{k\to+\infty}\frac{\Gamma(\alpha+\beta+\gamma+\delta+2k)\Gamma(\alpha+k+iy)\Gamma(\gamma+k-iy)}{\Gamma(\alpha+\gamma+2k)\Gamma(\alpha+\delta+k)\Gamma(\beta+\gamma+k)} = 2^{\beta+\delta}$$

and it follows easily that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\beta+iy)\Gamma(\delta-iy)dy = 2^{-(\beta+\delta)}\Gamma(\beta+\delta)M.$$

Hence, M is a constant, independent of $\alpha, \beta, \gamma, \delta$. To evaluate the constant we set $\alpha = \beta+\gamma = \delta = \frac{1}{2}$ in (3.13) and use the reflection formula for gamma functions:

$$\|1\|^2_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}} = M = 2\pi\int_0^{\infty}\frac{dy}{\cosh \pi y} = 1.$$

Thus,

$$\begin{aligned} \|1\|^2_{\alpha,\beta,\gamma,\delta} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\alpha+iy)\Gamma(\beta+iy)\Gamma(\gamma-iy)\Gamma(\delta-iy)dy \\ &= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}. \end{aligned} \tag{3.16}$$

This is Barnes' First Lemma (Bailey (1935, p.6), Slater (1966, p. 109)).

In relation (3.8) we set $f=\Phi_n^{(\alpha,\beta,\gamma,\delta)}$, $g=\Phi_{n-1}^{(\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12)}$ to obtain the recurrence

$$\begin{aligned}||\Phi_n^{(\alpha,\beta,\gamma,\delta)}||^2_{\alpha,\beta,\gamma,\delta} &= \frac{-n(n+\alpha+\beta+\gamma+\delta-1)}{(\alpha+\gamma)^2(\alpha+\delta)^2}\cdot \\ &\cdot ||\Phi_{n-1}^{(\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12)}||^2_{\alpha+\frac12,\beta+\frac12,\gamma+\frac12,\delta+\frac12}.\end{aligned} \tag{3.17}$$

It follows that the norms of the orthogonal polynomials are nonzero and can be computed recursively from $||1||^2_{\alpha,\beta,\gamma,\delta}$. Thus these polynomials are defined uniquely by their orthogonality with respect to the weight function w.

The symmetry of the weight function in (α,β) implies the identity

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}-n, & n+\alpha+\beta+\gamma+\delta-1, & x+\beta \\ \beta+\gamma, & \beta+\delta & \end{matrix};1\right)\\ &=\frac{(\alpha+\gamma)_n(\alpha+\delta)_n}{(\beta+\gamma)_n(\beta+\delta)_n}{}_3F_2\left(\begin{matrix}-n, & n+\alpha+\beta+\gamma+\delta-1, & x+\alpha \\ \alpha+\gamma, & \alpha+\delta & \end{matrix};1\right).\end{aligned}$$

Furthermore, the symmetry of the weight function with respect to the interchanges $x\leftrightarrow -x$, $\alpha\leftrightarrow\delta$, $\beta\leftrightarrow\gamma$ implies

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}-n, & n+\alpha+\beta+\gamma+\delta-1, & -x+\alpha \\ \alpha+\gamma, & \alpha+\delta & \end{matrix};1\right)\\ &=(-1)^n\frac{(\beta+\delta)_n}{(\alpha+\gamma)_n}{}_3F_2\left(\begin{matrix}-n, & n+\alpha+\beta+\gamma+\delta-1, & x+\delta \\ \alpha+\delta, & \beta+\delta & \end{matrix};1\right).\end{aligned}$$

4. Biorthogonality relations. We will extend the ideas of the previous sections by considering rational functions rather than polynomials. Thus the basic object of study will be the rational function of z

$$p_n^{a,b}(z)=\frac{(az;q)_n}{(bz;q)_n} \tag{4.1}$$

rather than the polynomial $(az;q)_n$ of §2. Two fundamental recurrences are

$$\mu_1 p_{n-1}^{a,bq}=\frac{(a-\frac{b\rho}{a}q^{-n+1})}{a-b}p_{n-2}^{a,b}+\frac{b}{a}\frac{(-b+\rho q^{-n+1})}{a-b}p_n^{a,b}, \tag{4.2A}$$

$$\mu_2 p_n^{a,b}=(a-b)(1-q^n)p_{n-1}^{a,bq}, \tag{4.2B}$$

where

$$\mu_1=\frac{(1-\frac{b\rho z}{aq})}{(1-\frac{az}{q})}E_z^{-1}+\frac{b}{a}\frac{1-\rho z}{1-bz}I_z,$$

$$\mu_2=\frac{q}{z}(1-bq^{-1}z)(1-bz)[I_z-E_z^{-1}],$$

and

$$E_z^s f(z)=f(q^s z),\quad I_z f(z)=f(z).$$

It is not difficult to verify that the eigenvalue equation

$$\mu_1\mu_2\Psi(z) = \lambda\Psi(z) \tag{4.3}$$

has the solutions

$$\Psi_\ell^{a,b,\rho}(z) = \sum_{k=0}^{\ell} \frac{(q^{-2\ell};q^2)_k(\frac{b}{\rho}q^{2\ell-1};q^2)_k}{(q^2;q^2)_k(\frac{a^2q}{b\rho};q^2)_k}\frac{(az;q)_{2k}}{(bz;q)_{2k}}q^{2k} \tag{4.4}$$

corresponding to the eigenvalues

$$\lambda_\ell = \frac{b}{a}(1-q^{-2\ell})(bq^{2\ell}-\rho q), \quad \ell = 0,1,2,\dots. \tag{4.5}$$

Let $S_e^{a,b,\rho}$ be the complex vector space of all finite linear combinations of the functions $\{\Psi_\ell^{a,b,\rho}\}$. Consider the bilinear form

$$< f,g >_{a,b,\rho}^e = \oint_C f(z)g(z)w_{a,b,\rho}^e(z)\frac{dz}{z} \tag{4.6}$$

where $g \in S_e^{a,b,\rho}$, w is a weight function, C is a positively oriented closed curve in the complex z-plane and $f \in \hat{S}_e^{a,b,\rho}$ (a space to be determined). We interpret $\mu_1\mu_2$ as the map

$$\mu_1\mu_2 : S_e^{a,b,\rho} \to S_e^{a,b,\rho}$$

and try to determine w, C, and $\hat{S}_e^{a,b,\rho}$ such that the adjoint eigenvalue equation

$$(\mu_1\mu_2)^*\hat{\Psi}_\ell^{a,b,\rho} = \lambda_\ell\hat{\Psi}_\ell^{a,b,\rho}, \quad \ell = 0,1,2,\dots \tag{4.7}$$

has hypergeometric solutions $\hat{\Psi}_\ell^{a,b,\rho}$ where

$$(\mu_1\mu_2)^* : \hat{S}_e^{a,b,\rho} \to \hat{S}_e^{a,b,\rho}.$$

An evident solution is

$$\begin{aligned}\hat{\Psi}_\ell^{a,b,\rho}(z) &= \Psi_\ell^{a,b,\rho}\left(\frac{q}{b\rho z}\right)\\ &= \sum_{k=0}^{\ell} \frac{(q^{-2\ell};q^2)_k(\frac{b}{\rho}q^{2\ell-1};q^2)_k}{(q^2;q^2)_k(\frac{a^2q}{b\rho};q^2)_k}\frac{(\frac{aq}{b\rho z};q)_{2k}}{(\frac{q}{\rho z};q)_{2k}}q^{2k}\end{aligned} \tag{4.8}$$

with $\hat{S}_e^{a,b,\rho}$ as the space of all finite linear combinations of the $\{\hat{\Psi}_\ell^{a,b,\rho}\}$. The weight function must satisfy the recurrence

$$\frac{w_{a,b,\rho}^e(qz)}{w_{a,b,\rho}^e(z)} = -\frac{(1-az)(1-\frac{1}{\rho z})}{(1-\frac{a}{b\rho z})(1-bz)}. \tag{4.9}$$

This recurrence has many solutions, depending on our choice of the zeros and the poles of w in the z-plane. One of the solutions with the simplest pole structure is

$$w_{a,b,\rho}^e(z) = \frac{(bz;q)_\infty(-\rho z;q)_\infty(-\frac{q}{\rho z};q)_\infty}{(az;q)_\infty(\frac{qa}{\rho bz};q)_\infty(\rho z;q)_\infty} \tag{4.10}$$

where we assume

$$|q| < |\rho| < 1, \qquad |qa| < |\rho b|. \tag{4.11}$$

For C we take the unit circle: $|z| = 1$. We will adopt solution (4.10) in the computations to follow.

Note that

$$\mu_2 \Psi_\ell^{a,b,\rho} = \frac{q^2(a-b)(1-q^{-2\ell})(1-\frac{b}{\rho}q^{2\ell-1})}{(1-\frac{a^2q}{b\rho})}\Theta_{\ell-1}^{a,b,\rho} \tag{4.12}$$

where

$$\begin{aligned} &\Theta_{-1}^{a,b,\rho} = 0, \\ &\Theta_\ell^{a,b,\rho} = \sum_{k=0}^{\ell} \frac{(q^{-2\ell};q^2)_k(\frac{b}{\rho}q^{2\ell+3};q^2)_k}{(q^2;q^2)_k(\frac{a^2q^3}{b\rho};q^2)_k}\frac{(az;q)_{2k+1}q^{2k}}{(bzq;q)_{2k+1}}, \\ &\qquad \ell = 0,1,\ldots . \end{aligned} \tag{4.13}$$

It follows from (4.3)-(4.5) that

$$\mu_1 \Theta_{\ell-1}^{a,b,\rho} = \frac{(a^2q-\rho b)}{qa(a-b)}\Psi_\ell^{a,b,\rho}, \quad \ell = 1,2,\ldots . \tag{4.14}$$

Let $S_o^{a,b,\rho}$ be the space of all finite linear combinations of the $\{\Theta_\ell^{a,b,\rho}\}$. We have the interpretation

$$\mu_1 : S_o^{a,b,\rho} \to S_e^{a,b,\rho}, \ \mu_2 : S_e^{a,b,\rho} \to S_o^{a,b,\rho}.$$

Furthermore, $(\mu_1\mu_2)^*$ factors as $(\mu_1\mu_2)^* = \mu_2^*\mu_1^*$ where

$$\begin{aligned} \mu_1^* &= \frac{q^2}{\rho^2 z}(1-\rho z)\left(1-\frac{\rho z}{q}\right)(I_z - E_z^1), \\ \mu_2^* &= \frac{b^2q^2}{qa^2}\frac{(1-az)}{(1-\frac{b\rho z}{a})}E_z^1 + \frac{b\rho^2}{qa}\frac{(1-\frac{bz}{q})}{(1-\frac{\rho z}{q})}I_z \end{aligned}$$

and

$$\begin{aligned} \mu_1^*\hat{\Psi}_\ell^{a,b,\rho} &= \frac{\frac{q^3}{\rho}(\frac{a}{b}-1)}{(1-\frac{a^2q}{b\rho})}(1-q^{-2\ell})\left(1-\frac{b}{\rho}q^{2\ell-1}\right)\hat{\Theta}_{\ell-1}^{a,b,\rho}, \\ \mu_2^*\hat{\Theta}_{\ell-1}^{a,b,\rho} &= -\frac{b^2\rho^2(1-\frac{a^2q}{b\rho})}{aq^2(a-b)}\hat{\Psi}_\ell^{a,b,\rho}, \end{aligned} \tag{4.15}$$

with

$$\begin{aligned} &\hat{\Theta}_{-1}^{a,b,\rho}(z) = 0, \\ &\hat{\Theta}_\ell^{a,b,\rho}(z) = \sum_{k=0}^{\ell} \frac{(q^{-2\ell};q^2)_k(\frac{b}{\rho}q^{2\ell+3};q^2)_k}{(q^2;q^2)_k(\frac{a^2q^3}{b\rho};q^2)_k}\frac{(\frac{aq}{b\rho z};q)_{2k+1}}{(\frac{q^2}{\rho z};q)_{2k+1}}q^{2k}, \\ &\qquad \ell = 0,1,\ldots . \end{aligned} \tag{4.16}$$

Let $\hat{S}_o^{a,b,\rho}$ be the space of all finite linear combinations of the functions $\{\hat{\Theta}_\ell^{a,b,\rho}\}$. Then we have the interpretations

$$\mu_1^*: \hat{S}_e^{a,b,\ell} \to \hat{S}_o^{a,b,\rho}, \quad \mu_2^*: \hat{S}_o^{a,b,\rho} \to \hat{S}_e^{a,b,\rho}.$$

We now try to determine a weight function $w^o_{a,b,\rho}(z)$ such that the adjoint relation

$$< \mu_1^* f, g >^o_{a,b,\rho} = < f, \mu_1 g >^e_{a,b,\rho} \tag{4.17}$$

holds for all $f \in \hat{S}_e^{a,b,\rho}$, $g \in S_o^{a,b,\rho}$, where

$$< g_1, g_2 >^o_{a,b,\rho} = \oint_C g_1(z) g_2(z) w^o_{a,b,\rho}(z) \frac{dz}{z}. \tag{4.18}$$

A straightforward computation yields

$$\begin{aligned} w^o_{a,b,\rho}(z) &= \frac{\rho^2 b z w^e_{a,b,\rho}(z)}{aq^2(1-bz)(1-\frac{\rho z}{q})} \\ &= \frac{\rho^2 bz(bqz;q)_\infty(-\rho z;q)_\infty(-\frac{q}{\rho z};q)_\infty}{aq^2(az;q)_\infty(\frac{qa}{\rho bz};q)_\infty(\frac{\rho z}{q};q)_\infty}. \end{aligned} \tag{4.19}$$

We can similarly verify that the adjoint relation

$$< \mu_2^* g, f >^e_{a,b,\rho} = < g, \mu_2 f >^o_{a,b,\rho} \tag{4.20}$$

holds for all $f \in S_e^{a,b,\rho}$, $g \in \hat{S}_o^{a,b,\rho}$.

It follows immediately from these adjoint relations and the eigenvalue equations (4.3) and (4.7) that the biorthogonality relations

$$< \hat{\Psi}_\ell^{a,b,\rho}, \Psi_{\ell'}^{a,b,\rho} >^e_{a,b,\rho} = 0, \tag{4.21A}$$

$$< \hat{\Theta}_\ell^{a,b,\rho}, \Theta_{\ell'}^{a,b,\rho} >^o_{a,b,\rho} = 0, \tag{4.21B}$$

hold for all $\ell \neq \ell'$. Our remaining problem is to compute the left-hand sides of expressions (4.21A), (4.21B) for $\ell = \ell'$.

One appropriate operator for this problem is

$$\xi = \frac{q}{z}(1-bqz)\left(1-\frac{b^2\rho z}{a^2}\right) I_z - \frac{q}{z}\frac{(1-bz)(1-bqz)(1-\frac{b\rho z}{aq})}{(1-\frac{az}{q})} E_z^{-1} \tag{4.22}$$

which satisfies

$$\begin{aligned} \xi p^{a,bq}_{2k+1} &= (a-bq)\frac{b\rho}{a^2}\left(1-\frac{q^{2k+1}a^2}{b\rho}\right) p^{a,bq^2}_{2k}, \\ & \quad k = 0,1,2,\ldots \end{aligned} \tag{4.23}$$

where the basis functions $p_n^{a,b}$ are defined by (4.1). It follows that

$$\begin{aligned} \xi \Theta_\ell^{a,b,\rho} &= -q(a-bq)\left(1-\frac{b\rho}{a^2 q}\right) \Psi_\ell^{a,bq^2,\rho q^{-2}}, \\ & \quad \ell = 0,1,\ldots \end{aligned} \tag{4.24}$$

and that ξ has the interpretation

$$\xi : S_o^{a,b,\rho} \to S_e^{a,bq^2,\rho q^{-2}}.$$

We define the adjoint operator

$$\xi^* : \hat{S}_e^{a,bq^2,\rho q^{-2}} \to \hat{S}_o^{a,b,\rho}$$

by

$$< g, \xi f >^e_{a,bq^2,\rho q^{-2}} = < \xi^* g, f >^o_{a,b,\rho} \tag{4.25}$$

where $g \in \hat{S}_e^{a,bq^2,\rho q^{-2}}$, $f \in S_o^{a,b,\rho}$. A straightforward computation yields

$$\xi^* = qE_z^1 + \frac{a}{b}\frac{(1-\frac{b^2\rho z}{a^2})}{(1-\frac{\rho z}{q^2})} I_z \tag{4.26}$$

and

$$\frac{(a-bq)}{q}\xi^* \hat{\Psi}_\ell^{a,bq^2,\rho q^{-2}} = -\frac{q^2}{a}\frac{(1-\frac{a^2q^{2\ell+1}}{b\rho})}{(1-\frac{a^2q}{b\rho})}(b^2 - a^2q^{-2\ell-2})\hat{\Theta}_\ell^{a,b,\rho}. \tag{4.27}$$

Set

(4.28)
$$||\Psi_\ell^{a,b,\rho}||_e^2 = < \hat{\Psi}_\ell^{a,b,\rho}, \Psi_\ell^{a,b,\rho} >^e_{a,b,\rho}, \quad ||\Theta_\ell^{a,b,\rho}||_o^2 = < \hat{\Theta}_\ell^{a,b,\rho}, \Theta_\ell^{a,b,\rho} >^o_{a,b,\rho},$$
$$\ell = 0, 1, 2, \ldots.$$

¿From (4.14), (4.15), and (4.17) with $f = \hat{\Psi}_\ell^{a,b,\rho}$, $g = \Theta_{\ell-1}^{a,b,\rho}$ we have

$$||\Psi_\ell^{a,b,\rho}||_e^2 = -\frac{a\rho q^2(a-b)^2(1-q^{-2\ell})(1-\frac{b}{\rho}q^{2\ell-1})}{(b\rho - a^2q)^2}||\Psi_{\ell-1}^{a,b,\rho}||_o^2. \tag{4.29}$$

Relation (4.20) yields the same recurrence. Expressions (4.24), (4.25), and (4.27) with $f = \Theta_\ell^{a,b,\rho}$, $g = \hat{\Psi}_\ell^{a,bq^2,\rho q^{-2}}$ produce

$$||\Psi_\ell^{a,bq^2,\rho q^{-2}}||_e^2 = -\frac{a^2q^3(1-\frac{a^2q^{2\ell+1}}{b\rho})(b^2 - a^2q^{-2\ell-2})}{b\rho(a-bq)^2(1-\frac{a^2q}{b\rho})^2}||\Theta_\ell^{a,b,\rho}||_o^2. \tag{4.30}$$

It follows from these results that if we know $||1^{a,b,\rho}||_e^2$ for all a, b, ρ then we can compute recursively expressions (4.28) for all ℓ. In general these norms will be nonzero.

Replacing ρz by z in (4.6), (4.10) we have

$$||1^{a,b,\rho}||_e^2 = G(u,v,q) = \frac{1}{2\pi}\oint_C \frac{(\frac{uz}{v};q)_\infty(-z;q)_\infty(-\frac{q}{z};q)_\infty}{(uz;q)_\infty(\frac{qv}{z};q)_\infty(z;q)_\infty}\frac{dz}{z} \tag{4.31}$$

where $u = a/\rho$, $v = a/b$. From the relation

$$< \hat{\Psi}_0^{a,b,\rho}, \Psi_1^{a,b,\rho} >_e = 0$$

it is straightforward to compute the recurrence

$$\|1^{aq^2,bq^2,\rho}\|_e^2 = \frac{(1-\frac{a^2q}{b\rho})}{(1-\frac{bq}{\rho})}\|1^{a,b,\rho}\|_e^2$$

or $G(q^2u,v,q) = (1-uvq)G(u,v,q)/(1-uq/v)$, which implies

$$G(u,v,q) = \frac{(\frac{uq}{v};q^2)_\infty}{(uvq;q^2)_\infty}G(v,q). \tag{4.32}$$

To finish the computation of (4.31) we utilize the recurrence operator

$$\eta = \frac{(aq-b)}{(1-az)}\left[\left(1-\frac{bz}{q}\right)E_z^{-1} + \frac{b}{aq}\left(1-\frac{a^2qz}{b}\right)I_z\right]. \tag{4.33}$$

Here,

$$\eta p_n^{a,b} = aq(1-q^{-n})p_{n-2}^{aq,b} + \frac{(-b^2+a^2q^{2-n})}{aq}p_n^{aq,b},$$

hence

$$\eta\Psi_\ell^{a,b,\rho} = -\frac{q^{-2\ell}(b-\frac{a^2}{\rho}q^{2\ell+1})}{abq(1-\frac{a^2q}{b\rho})}(-a^2q^2+b^2q^{2\ell})\Psi_\ell^{aq,b,\rho}. \tag{4.34}$$

Interpreting $\eta : S_e^{a,b,\rho} \to S_e^{aq,b,\rho}$ we compute the adjoint $\eta^* : \hat{S}_e^{aq,b,\rho} \to \hat{S}_e^{a,b,\rho}$ with the result

$$\eta^* = (aq-b)\left[\frac{(1-\rho z)}{\rho z}(1-az)E_z^1 + \frac{b}{aq}\left(1-\frac{qa}{\rho bz}\right)\left(1-\frac{a^2qz}{b}\right)I_z\right]. \tag{4.35}$$

This leads to the recurrence

$$\eta^*\hat{\Psi}_\ell^{aq,b,\rho} = -\frac{(aq-b)^2}{aq}\left(1-\frac{a^2q}{b\rho}\right)\hat{\Psi}_\ell^{a,b,\rho}. \tag{4.36}$$

Thus the relation

$$< \eta^*1, 1 >_{a,b,\rho}^e = < 1, \eta 1 >_{aq,b,\rho}^e$$

implies

$$\|1^{aq,b,\rho}\|_e^2 = \left(1-\frac{a^2q}{bm}\right)\left(\frac{b-aq}{b+aq}\right)\|1^{a,b,\rho}\|_e^2$$

or $G(qv,q) = (1-qv)G(v,q)/(1+qv)$. We conclude that $G(v,q) = (-vq;q)_\infty K(q)/(vq;q)_\infty$ where $K(q)$ is to be determined. Setting $u=0$, $v=1$ in (4.31) we find $G(0,-1,q) = 1 = (q;q)_\infty K(q)/(-q;q)_\infty$, so

$$\|1^{a,b,\rho}\|_e^2 = \frac{(-q;q)_\infty(\frac{bq}{\rho};q^2)_\infty(-\frac{aq}{b};q)_\infty}{(q;q)_\infty(\frac{a^2q}{b\rho};q^2)_\infty(\frac{aq}{b};q)_\infty}. \tag{4.37}$$

Expanding the original integral $\|1^{a,b,\rho}\|_e^2$ by residues inside the circle C, we see that our result is equivalent to the summation formula

$${}_2\varphi_1\left(\begin{matrix} vq, & uvq \\ uq & \end{matrix}; -\frac{1}{v}\right) = \frac{(-q;q)_\infty(\frac{uq}{v};q^2)_\infty(uvq^2;q^2)_\infty}{(uq;q)_\infty(-\frac{1}{v};q)_\infty}. \tag{4.38}$$

This is a q-analogue of Kummer's Theorem, first proved by Andrews (1973); see also Andrews (1977, p. 20).

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Note added in proof. A referee points out that the system of biorthogonal functions in §4 is a special case of a system found by J.A. Wilson in 1977 but not yet published.

References

1. A.K. Agarwal, E.G. Kalnins, and W. Miller (1987), *Canonical equations and symmetry techniques for q-series*, SIAM J. Math. Anal. **18**, 1519-1538.
2. G.E. Andrews (1973), *On the q-analogue of Kummer's theorem and applications*, Duke Math. J. **40**, 525–528.
3. G.E. Andrews (1977), *The Theory of Partitions*, Addison-Wesley, Reading, MA.
4. G.E. Andrews and R. Askey (1985), *Classical Orthogonal Polynomials*, Lecture Notes # 1171, Springer-Verlag, New York, Berlin, 36–62.
5. R. Askey and R. Roy (1986), *More q-beta integrals*, Rocky Mtn. J. Math. **26**, 365–372.
6. R. Askey and J. Wilson (1985), *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs of the AMS **no. 319**.
7. W.N. Bailey (1964), *Generalized Hypergeometric Series*, Cambridge University Press, 1935, Reprinted by Stechert-Hafner, New York.
8. M.E.-H. Ismail and J.A. Wilson (1982), *Asymptotic and generating relations for the q-Jacobi and ${}_4\varphi_3$ polynomials*, J. Approx. Theory **36**, 43–54.
9. E.G. Kalnins and W. Miller, *Symmetry techniques for q-series: Askey-Wilson polynomials*, Rocky Mtn. J. Math., to appear.
10. W. Miller (1987), *A note on Wilson polynomials*, SIAM J. Math. Anal. **18**, 1221–1226.
11. A.F. Nikiforov and S.K. Suslov (1986), *Classical orthogonal polynomials of a discrete variable on nonuniform lattices*, Lett. Math. Phys. **11**, 27–34.
12. A. F. Nikiforov, S.K. Suslov, and V.B. Uvarov (1985), *Classical Orthogonal Polynomials of a Discrete Variable*, Nauka, Moscow (in Russian).
13. L.J. Slater (1966), *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge.
14. G.N. Watson (1910), *The continuation of functions defined by generalized hypergeometric series*, Trans Cambridge Phil. Soc. **21**, 281–299.
15. E.T. Whittaker and G.N. Watson (1958), *A Course in Modern Analysis*, Cambridge University Press, Cambridge.
16. J. Wilson (1980), *Some hypergeometric orthogonal polynomials*, SIAM J. Math. Anal. **11**, 690–701.