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002

CANONICAL EQUATIONS AND SYMMETRY TECHNIQUES FOR q-SERIES*

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Abstract. The authors introduce symmetry techniques for the classification and derivation of generating functions for families of basic hypergeometric functions, in analogy with the Lie theory techniques for ordinary hypergeometric functions. To each family of basic hypergeometric functions there is associated a canonical system of partial q-difference equations and the symmetries of these equations are used to derive q-series identities and orthogonality relations for the special functions.

Key words. q-series, q-difference equations, basic hypergeometric functions

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1. Introduction. In [11], [13], [14] a Lie algebraic method was developed which associated with each family of multivariable hypergeometric functions a canonical system of partial differential equations constructed from the differential recurrence relations obeyed by the family. (The basic idea behind this method followed from the work of Weisner [16].) The hypergeometric functions arise by partial separation of variables in the canonical systems and any analytic solution of these equations can be considered as a generating function for this family. Furthermore the generating functions can be characterized in terms of symmetry operators for the canonical systems.

In this paper we present the foundations of an analogous theory for families of many-variable basic hypergeometric functions. To each family we associate a canonical system of partial q-difference equations constructed from the q-difference recurrence relations obeyed by the family. The basic hypergeometric functions arise by partial separation of variables in the canonical systems and any analytic solution of these equations is a generating function for the family. Symmetry operators for the canonical system can be used to characterize the generating functions. Thus a direct link is established between symmetries of the canonical system and identities obeyed by q-series.

In § 2 we show how to derive the canonical system of q-difference equations associated with a given family of q-series, using as examples the one-variable hypergeometric functions ${}_r\varphi_s$ and the two-variable function f_2, a q-analogue of the Appell function F_2. In § 3 we describe how to relate two different families of basic hypergeometric functions, that is, the procedures of *embedding* and *augmentation.* In the procedure of embedding the canonical system for one basic hypergeometric family restricts through a specialization of variables to the canonical system for a second hypergeometric family, so that the restricted family can be considered as a generating function for the second family. Augmentation is a process inverse to this. By augmentation we can write the defining equations for a generating function as the restriction of a canonical system of higher dimension.

In § 4 we apply our techniques to derive and characterize in terms of symmetries, a variety of generating functions for the families ${}_r\varphi_s$. In § 5 we treat the family ${}_2\varphi_1$ in somewhat more detail. (This family needs special treatment because its canonical

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system admits symmetries not shared by the systems for general ${}_r\varphi_s$.) Furthermore we show how orthogonality relations for q-series follow from symmetry ideas.

Our theory provides a simple uniform procedure for derivation and symmetry classification of a wide variety of q-identities, in analogy with the Lie theoretic procedures for ordinary hypergeometric series. The full power of the theory becomes evident in the study of many-variable q-series and in the study of Askey–Wilson polynomials, as we will show in future papers. However, in distinction to the case of differential equations we do not have the tools of local Lie transformation group theory or the relationship between Lie symmetries and separation of variables to help us obtain the generating functions in the most compact form. Our procedures enable us to classify and characterize generating functions in terms of symmetry operators; unaided, they do not enable us to write the generating functions in simplest form, i.e., factorized or in terms of a new choice of variables. It will be very interesting to see if (as is the case for differential equations) factorization and coordinates have symmetry operator interpretations.

The symmetry techniques presented here apply to formal power series and are essentially independent of convergence criteria. Hence, we shall ordinarily not specify the domains of validity for the identities derived in this paper. In most cases they can be determined easily for one-variable hypergeometric functions by the ratio test. For multivariable hypergeometric functions the full domain of convergence may be very difficult to determine (or even unknown). In those cases one can specialize some of the parameters in the functions (so that infinite series truncate to finite series for example) to guarantee convergence.

Finally we note that the symbolic method of Burchnall and Chaundy for ordinary hypergeometric series [4], [5], and some works of Hahn on q-series [7], [8] contain points of similarity with our method, although these authors did not use symmetry techniques.

2. The "basic" idea. We begin our study of canonical equations for q-series by deriving the canonical form associated with the q-hypergeometric functions ${}_r\varphi_s$:

$$(2.1)\qquad {}_r\varphi_s\left(\begin{matrix}a_1, \cdots, a_r\\ b_1, \cdots, b_s\end{matrix}; x\right)=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n x^n}{(b_1;q)_n\cdots(b_s;q)_n(q;q)_n}$$

where $a_i=q^{\alpha_i}$, $b_j=q^{\beta_j}$ and

$$(2.2)\qquad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},\qquad (a;q)_\infty=\prod_{m=0}^{\infty}(1-q^m a).$$

Here α_i, β_j, x are complex variables, ($\beta_j\neq 0, -1, -2, \cdots$) and we normally require that $0<q<1$. Note that for n a nonnegative integer we have

$$(2.3)\qquad (a;q)_n=(1-a)(1-qa)\cdots(1-q^{n-1}a).$$

As is well known [3], [5] ${}_r\varphi_s$ is a q-analogue of the hypergeometric series

$$(2.4)\qquad {}_rF_s\left(\begin{matrix}\alpha_1, \cdots, \alpha_r\\ \beta_1, \cdots, \beta_s\end{matrix}; x\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{x^n}{n!}$$

where

$$(2.5)\qquad (\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

and $\Gamma(\cdot)$ is the gamma function. For n a nonnegative integer

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1). \tag{2.6}$$

Here ${}_r\varphi_s$ and ${}_rF_s$ are related by

$$ {}_rF_s\left(\begin{matrix}\alpha_i\\ \beta_j\end{matrix}; x\right) = \lim_{q\to 1} {}_r\varphi_s\left(\begin{matrix}a^i\\ b_j\end{matrix}; \frac{x}{(1-q)^{r-s-1}}\right). \tag{2.7}$$

Let T_u be the q-dilation operator corresponding to the variable u, i.e., T_u maps a function f of the variables $u, v, w, \cdots$ to the function

$$T_u f(u, v, w, \cdots) = f(qu, v, w, \cdots). \tag{2.8}$$

From the q-series (2.1) one can easily verify the recurrence relations

$$\begin{aligned}
(1-a_kT_x){}_r\varphi_s\left(\begin{matrix}a_i\\ b_j\end{matrix}; x\right) &= (1-a_k){}_r\varphi_s\left(\begin{matrix}e^ka_i\\ b_j\end{matrix}; x\right), \qquad 1 \le k \le r,\\
(1-b_lq^{-1}T_x){}_r\varphi_s\left(\begin{matrix}a_i\\ b_j\end{matrix}; x\right) &= (1-b_lq^{-1}){}_r\varphi_s\left(\begin{matrix}a_i\\ e_lb_j\end{matrix}; x\right), \qquad 1 \le l \le s,\\
x^{-1}(1-T_x){}_r\varphi_s\left(\begin{matrix}a_i\\ b_j\end{matrix}; x\right) &= \frac{(1-a_1)\cdots(1-a_r)}{(1-b_1)\cdots(1-b_s)}{}_r\varphi_s\left(\begin{matrix}qa_i\\ qb_j\end{matrix}; x\right)
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
e^ka_i &= \begin{cases} a_i & \text{if } i \ne k,\\ qa_k & \text{if } i = k,\end{cases}\\
e_lb_j &= \begin{cases} b_j & \text{if } j \ne l,\\ q^{-1}b_l & \text{if } j = l.\end{cases}
\end{aligned} \tag{2.10}$$

Note that relations (2.9) imply the fundamental q-difference equation satisfied by the ${}_r\varphi_s$:

$$\{x(1-a_1T_x)\cdots(1-a_rT_x) - (1-T_x)(1-b_1q^{-1}T_x)\cdots(1-b_sq^{-1}T_x)\}{}_r\varphi_s\left(\begin{matrix}a_i\\ b_j\end{matrix}; x\right) = 0. \tag{2.11}$$

Indeed, for $\beta_j \ne 0, -1, -2, \cdots$ the only solution of this equation which is analytic in x at $x=0$ is ${}_r\varphi_s(^{a_i}_{b_j}; x)$.

Now we define the function ${}_r\Phi_s$ of $2(r+s)+1$ variables by

$$ {}_r\Phi_s(a_i, b_j; u_p) = {}_r\varphi_s\left(\begin{matrix}a_i\\ b_j\end{matrix}; \frac{u_{r+1}\cdots u_{r+s+1}}{u_1\cdots u_r}\right)u_1^{-\alpha_1}\cdots u_r^{-\alpha_r}u_{r+1}^{\beta_1-1}\cdots u_{r+s}^{\beta_s-1}. \tag{2.12}$$

Let $\Delta_p^{\pm}$ be the q-difference operators

$$\begin{aligned}
\Delta_p^+ &= u_p^{-1}(1-T_{u_p}),\\
\Delta_p^- &= u_p^{-1}(1-T_{u_p}^{-1}), \qquad 1 \le p \le r+s+1.
\end{aligned} \tag{2.13}$$

In terms of these operators, relations (2.9) take the simple form

$$\Delta_k^- \, {}_r\Phi_s = (1-a_k)\, {}_r\Phi_s\begin{pmatrix} e^k a_i \\ b_j \end{pmatrix}, \qquad 1 \leq k \leq r,$$

$$\Delta_{r+l}^+ \, {}_r\Phi_s = (1-b_l q^{-1})\, {}_r\Phi_s\begin{pmatrix} a_i \\ e_l b_j \end{pmatrix}, \qquad 1 \leq l \leq s, \tag{2.14}$$

$$\Delta_{r+s+1}^+ \, {}_r\Phi_s = \frac{(1-a_1)\cdots(1-a_r)}{(1-b_1)\cdots(1-b_s)}\, {}_r\Phi_s\begin{pmatrix} qa_i \\ qb_j \end{pmatrix}$$

and (2.11) becomes the (canonical) partial q-difference equation

$$\left(\prod_{k=1}^{r} \Delta_p^- - \prod_{p=r+1}^{r+s+1} \Delta_p^+\right) {}_r\Phi_s = 0. \tag{2.15}$$

Furthermore ${}_r\Phi_s$ satisfies the eigenvalue equations

$$\begin{aligned} T_{r+s+1}^{-1} T_k^{-1} \, {}_r\Phi_s &= q^{\alpha_k} \, {}_r\Phi_s, \qquad 1 \leq k \leq r, \\ T_{r+s+1}^{-1} T_{r+l} \, {}_r\Phi_s &= q^{\beta_l - 1} \, {}_r\Phi_s, \qquad 1 \leq l \leq s. \end{aligned} \tag{2.16}$$

Indeed, ${}_r\Phi_s$ is characterized by (2.15), (2.16): It is (to within a constant multiple) the only solution of these equations analytic in the u_p at $u_{r+s+1} = 0$.

We can regard an analytic solution $\Psi(u_p)$ of the canonical equation

$$(\Delta_1^- \cdots \Delta_r^- - \Delta_{r+1}^+ \cdots \Delta_{r+s+1}^+)\Psi = 0 \tag{2.17}$$

as a generating function for basic hypergeometric functions. Indeed, expanding Ψ as a power series

$$\Psi(u_p) = \sum_{\alpha_i, \beta_j} f_{\alpha_i\beta_j}(x) u_1^{-\alpha_1} \cdots u_r^{-\alpha_r} u_{r+1}^{\beta_1 - 1} \cdots u_{r+s}^{\beta_s - 1} \tag{2.18}$$

where $x = u_{r+1} \cdots u_{r+s+1}/u_1 \cdots u_r$, we see that if Ψ is analytic at $x = 0$ and if no nonzero term occurs with some $\beta_j = 0, -1, -2, \cdots$ then always

$$f_{\alpha_i\beta_j}(x) = c_{\alpha_i\beta_j} \, {}_r\varphi_s\begin{pmatrix} a_i \\ b_j \end{pmatrix}; x \tag{2.19}$$

for some constants $c_{\alpha_i\beta_j}$. We shall typically compute such a generating function Ψ by characterizing it as a simultaneous eigenfunction of a set of $r+s$ commuting symmetry operators for (2.17). By a symmetry operator for the canonical equation we mean a linear operator L which maps any local analytic solution Ψ for (2.17) into another local analytic solution $L\Psi$. Clearly the dilation operators $T_{r+s+1}^{-1} T_k^{-1}$ $(1 \leq k \leq r)$, $T_{r+s+1}^{-1} T_{r+l}$ $(1 \leq l \leq s)$ are commuting symmetries, and the eigenvalue equations (2.16) characterize the basis solutions ${}_r\Phi_s$ in terms of these symmetries. Furthermore the q-difference operators Δ_i^- $(1 \leq i \leq r)$ and Δ_{r+h}^+ $(1 \leq h \leq s+1)$ are commuting symmetries. Note also that any permutation of the variables $\{u_i: 1 \leq i \leq r\}$ is a symmetry of (2.17) as is any permutation of the variables $\{u_{r+h}: 1 \leq h \leq s+1\}$. (For example, the transposition symmetry (u_{r+1}, u_{r+s+1}) implies that

$${}_r\varphi_s\begin{pmatrix} a_i b_1^{-1} q \\ q^2 b_1^{-1}, b_j b_1^{-1} q \end{pmatrix}; x \bigg) x^{1-\beta_1}$$

is another solution of (2.11).)

The canonical equation (2.17) for q-hypergeometric functions is a clear analogue of the canonical equation for the hypergeometric functions ${}_rF_s$ [13]. Indeed the basis functions

$$(2.20)\qquad {}_r\mathscr{F}_a\left(\begin{matrix}\alpha_i\\ \beta_j\end{matrix}; u_p\right) = {}_rF_s\left(\begin{matrix}\alpha_i\\ \beta_j\end{matrix}; \frac{u_{r+1}\cdots u_{r+s+1}}{u_1\cdots u_r}\right)u_1^{-\alpha_1}\cdots u_r^{-\alpha_r}u_{r+1}^{\beta_1-1}\cdots u_{r+s}^{\beta_s-1}$$

satisfy the canonical equation

$$(2.21)\qquad (\partial_{u_1}\cdots\partial_{u_r} - (-1)^r\partial_{u_{r+1}}\cdots\partial_{u_{r+s+1}})\,{}_r\mathscr{F}_s = 0$$

and the eigenvalue equations

$$(2.22)\qquad \begin{aligned}(-D_{r+s+1}-D_k)\,{}_r\mathscr{F}_s &= \alpha_k\,{}_r\mathscr{F}_s, & 1\le k\le r,\\ (-D_{r+s+1}+D_{r+l})\,{}_r\mathscr{F}_s &= (\beta_l-1)\,{}_r\mathscr{F}_s, & 1\le l\le s,\end{aligned}$$

where $D_p = u_p\partial_{u_p}$. Furthermore ${}_r\mathscr{F}_s$ is the only solution of (2.21), (2.22) that is analytic in the u_p at $u_{r+s+1}=0$.

Our procedure applies to a family of q-analogues for the ${}_rF_s$. Let δ be a function with domain $\{1, 2, \cdots, r+s+1\}$ and range contained in the set $\{+, -\}$. The canonical equation

$$(2.23)\qquad (\Delta_1^{\delta(1)}\cdots\Delta_r^{\delta(r)} - \Delta_{r+1}^{\delta(r+1)}\cdots\Delta_{r+s+1}^{\delta(r+s+1)})\Psi = 0$$

and eigenvalue equations

$$(2.24)\qquad \begin{aligned}T_{r+s+1}^{-1}T_k^{-1}\Psi &= q^{\alpha_k}\Psi, & 1\le k\le r,\\ T_{r+s+1}^{-1}T_{r+l}\Psi &= q^{\beta_l-1}\Psi, & 1\le l\le s,\end{aligned}$$

have the unique (to within a constant multiple) solution

$$(2.25)\qquad {}_r\Phi_s^\delta\left(\begin{matrix}a_i\\ b_i\end{matrix}; u_p\right) = {}_r\varphi_s^\delta\left(\begin{matrix}a_i\\ b_j\end{matrix}; \frac{u_{r+1}\cdots u_{r+s+1}}{u_1\cdots u_r}\right)u_1^{-\alpha_1}\cdots u_r^{-\alpha_r}u_{r+1}^{\beta_1-1}\cdots u_{r+s}^{\beta_s-1}$$

where

$$(2.26)\qquad \begin{aligned}{}_r\varphi_s^\delta\left(\begin{matrix}a_i\\ b_j\end{matrix}; x\right) = \sum_{n=0}^{\infty}&\frac{(q^{\delta'(1)\alpha_1}; q^{\delta'(1)1})_n}{(q^{\delta(r+1)\beta_1}; q^{\delta(r+1)1})_n}\cdots\frac{(q^{\delta'(r)\alpha_r}; q^{\delta'(r)1})_n}{(q^{\delta(r+s)\beta_s}; q^{\delta(r+s)1})_n}\\ &\cdot\frac{x^n}{(q^{\delta(r+s+1)}; q^{\delta(r+s+1)1})_n}\end{aligned}$$

and

$$(2.27)\qquad \delta'(p) = \begin{cases} + & \text{if } \delta(p) = -,\\ - & \text{if } \delta(p) = +.\end{cases}$$

Each of these q-analogues of ${}_rF_s$ can be further treated by the methods presented in this paper.

Canonical equations for many-variable hypergeometric q-series can be derived almost as easily as for the one-variable case. Consider for example the Appell function F_2:

$$(2.28)\qquad F_2\left(\begin{matrix}\alpha, \beta, \beta'\\ \gamma, \gamma'\end{matrix}; x, y\right) = \sum_{n,m=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n x^m y^n}{(\gamma)_m(\gamma')_n m!\,n!}.$$

As shown in [11] the canonical differential equations are

$$(2.29)\qquad (\partial_{u_1}\partial_{u_2} - \partial_{u_3}\partial_{u_4})\mathscr{F}_2 = 0, \qquad (\partial_{u_1}\partial_{u_5} - \partial_{u_6}\partial_{u_7})\mathscr{F}_2 = 0$$

with eigenvalue equations

$$(2.30)\qquad \begin{aligned} D_1+D_3+D_6 &\sim -\alpha, \qquad D_2+D_3\sim -\beta,\\ D_4-D_3\sim \gamma-1, \quad D_5+D_6 &\sim -\beta', \quad D_7-D_6\sim \gamma'-1. \end{aligned}$$

Here $A\sim\alpha$ stands for $A\mathcal{F}_2=\alpha\mathcal{F}_2$, and $D_i=u_i\partial_{u_i}$. Furthermore,

$$\mathcal{F}_2=F_2\left(\begin{matrix}\alpha,\beta,\beta'\\ \gamma,\gamma'\end{matrix};\frac{u_3u_4}{u_1u_2},\frac{u_6u_7}{u_1u_5}\right)u_1^{-\alpha}u_2^{-\beta}u_5^{-\beta'}u_4^{\gamma-1}u_7^{\gamma'-1}.$$

Now consider the q-analogue

$$f_2\left(\begin{matrix}a,b,b'\\ c,c'\end{matrix};x,y\right)=\sum_{n,m=0}^{\infty}\frac{(a;q)_{m+n}(b;q)_m(b';q)_nx^my^n}{(c;q)_m(c';q)_n(q;q)_m(q;q)_n}$$

where $a=q^\alpha$, $b=q^\beta$, etc. The function

$$(2.31)\qquad f_2=f_2\left(\begin{matrix}a,b,b'\\ c,c'\end{matrix};\frac{u_3u_4}{u_1u_2},\frac{u_6u_7}{u_1u_5}\right)u_1^{-\alpha}u_2^{-\beta}u_5^{-\beta'}u_4^{\gamma-1}u_7^{\gamma'-1}$$

satisfies the recurrence relations

$$(2.32)\qquad \begin{gathered} \Delta_1^-f_2=(1-a)f_2(aq),\qquad \Delta_2^-f_2=(1-b)f_2(bq),\\ \Delta_3^+f_2=\frac{(1-a)(1-b)}{(1-c)}f_2\left(\begin{matrix}aq,bq\\ cq\end{matrix}\right),\\ \Delta_4^+f_2=(1-cq^{-1})f_2(cq^{-1}),\qquad \Delta_5^-f_2=(1-b')f_2(b'q),\\ \Delta_6^+f_2=\frac{(1-a)(1-b')}{(1-c')}f_2\left(\begin{matrix}aq,b'q\\ c'q\end{matrix}\right),\\ \Delta_7^+f_2=(1-c'q^{-1})f_2(c'q^{-1}), \end{gathered}$$

hence the canonical equations

$$(2.33)\qquad \Delta_1^-\Delta_2^--\Delta_3^+\Delta_4^+\sim 0,\qquad \Delta_1^-\Delta_5^--\Delta_6^+\Delta_7^+\sim 0.$$

(Here again $A\sim\chi$ signifies that f_2 is an eigenfunction of the operator A with eigenvalue χ.) Furthermore f_2 satisfies the dilation eigenvalue equations

$$(2.34)\qquad \begin{gathered} T_1T_3T_6\sim a^{-1},\quad T_2T_3\sim b^{-1},\quad T_4T_3^{-1}\sim cq^{-1},\\ T_5T_6\sim b'^{-1},\quad T_7T_6^{-1}\sim c'q^{-1}, \end{gathered}$$

and for $\gamma,\gamma'\neq 0,-1,-2,\cdots$ the only solution of equations (2.33), (2.34) analytic in the u_i at $u_3=u_6=0$ is (2.31). The standard pair of q-difference equations for the function f_2

$$\begin{gathered} [(1-aT_xT_y)(1-bT_x)-x^{-1}(1-T_x)(1-cq^{-1}T_x)]f_2=0,\\ [(1-aT_xT_y)(1-b'T_y)-y^{-1}(1-T_y)(1-c'q^{-1}T_y)]f_2=0, \end{gathered}$$

is obtained directly from the canonical equations by setting $x=u_3u_4/u_1u_2$, $y=u_6u_7/u_1u_5$ and factoring out the remaining "ignorable" variables. Note the perfect correspondence between the differential equations (2.29), (2.30) for the Appell function and the q-difference equations (2.33), (2.34) for the q-analogue.

Just as in the single-variable case we can study a family of q-analogues for F_2, one for each function δ with domain $\{1, 2, \cdots, 7\}$ and range contained in $\{+, -\}$. The canonical equations are

$$\begin{aligned} &\Delta_1^{\delta(1)}\Delta_2^{\delta(2)} - \Delta_3^{\delta(3)}\Delta_4^{\delta(4)} \sim 0, \\ &\Delta_1^{\delta(1)}\Delta_5^{\delta(5)} - \Delta_6^{\delta(6)}\Delta_7^{\delta(7)} \sim 0 \end{aligned} \tag{2.35}$$

and the corresponding q-series is

$$\begin{aligned} f_2^{\delta}\left(\begin{matrix} a, b, b' \\ c, c' \end{matrix}; x, y\right) = \sum_{m,n=0}^{\infty} &\frac{(q^{\delta'(1)\alpha}; q^{\delta'(1)1})_{m+n}(q^{\delta'(2)\beta}; q^{\delta'(2)1})_m}{(q^{\delta(r)\gamma}; q^{\delta(r)1})_m (q^{\delta(7)\gamma'}; q^{\delta(7)1})_n} \\ &\cdot \frac{(q^{\delta'(5)\beta'}; q^{\delta'(5)1})_n x^m y^n}{(q^{\delta(3)}; q^{\delta(3)1})_m (q^{\delta(6)}; {}^{\delta(6)1})_n}. \end{aligned} \tag{2.36}$$

Similar computations can be performed for any two-variable (or many-variable) q-hypergeometric series g. Corresponding to each parameter $a = q^{\alpha}$ such that the symbol $(a; q)_{Am+Bn}$ appears in the numerator of the expansion

$$g = \sum_{m,n} \frac{(a; q)_{Am+Bn} \cdots}{(c; q)_{Cm+Dn} \cdots} \frac{x^m y^m}{(q; q)_m (q; q)_n}$$

there is a "raising operator" $E^{\alpha} = \Delta^-$ associated with the recurrence relation

$$(1 - aT_x^A T_y^B)g = (1 - a)g(aq).$$

Similarly, for each denominator parameter c we can construct a "lowering operator" $E_{\gamma} = \Delta^+$ associated with

$$(1 - cq^{-1}T_x^C T_y^D)g = (1 - cq^{-1})g(cq^{-1}).$$

Application of $x^{-1}(1 - T_x)$, corresponding to Δ^+, takes each numerator parameter a to aq^A and each denominator parameter c to cq^C, whereas application of $y^{-1}(1 - T_y)$, corresponding to Δ^+, takes each numerator parameter a to aq^B and each denominator parameter c to cq^D.

Hrabowski [9] has discussed the general procedures for associating a system of canonical differential equations and eigenvalue equations with a given hypergeometric series and, conversely, for associating one or more hypergeometric series with a given system of canonical differential equations and eigenvalue equations. His analysis applies, with minor modifications, to q-hypergeometric series as we will discuss in a subsequent paper. The principal distinction is that there are two types of q-difference operators $\Delta_u^{\pm}$ and only one type of differential operator ∂_u.

3. Embeddings and augmentations. If the canonical equations of a q-hypergeometric series can be identified with a subset of the canonical equations of a second q-hypergeometric series, then by a suitable restriction of coordinates we can regard the second series to be a generating function for the first series. As an illustrative example we consider the canonical equation

$$\Delta_1^- \Delta_2^- - \Delta_3^+ \Delta_4^+ \sim 0 \tag{3.1}$$

for the basic hypergeometric function

$$\begin{gathered} {}_2\Phi_1 = {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; \frac{u_3 u_4}{u_1 u_2}\right) u_1^{-\alpha} u_2^{-\beta} u_3^{\gamma-1}, \\ T_4^{-1} T_1^{-1} \sim q^{\alpha}, \qquad T_4^{-1} T_2^{-1} \sim q^{\beta}, \qquad T_4^{-1} T_3 \sim q^{\gamma-1}, \end{gathered} \tag{3.2}$$

and the canonical equations (2.33) and eigenvalue equations (2.34) for the q-Appell function f_2. Identifying the variables $u_1, \cdots, u_4$ for ${}_2\Phi_1$ with the variables $u_1, \cdots, u_4$ for f_2 we see from (2.33) and (3.1) that for any choice of u_5, u_6, u_7 the function $f_2(u_p)$ can be regarded as a solution of the canonical equation (3.1) for ${}_2\Phi_1$. For uniqueness we require $u_5 = u_6 = u_7 = 1$ and obtain the solution

$$f_2 = f_2\left(\begin{matrix} a, b, b' \\ c, c' \end{matrix}; \frac{u_3 u_4}{u_1 u_2}, \frac{1}{u_1}\right) u_1^{-\alpha} u_2^{-\beta} u_4^{\gamma-1} \tag{3.3}$$

of (3.1). Our approach is to characterize the generating function (3.3) in terms of symmetry operators for (3.1). However the remaining canonical equation for f_2 and the eigenvalue equations (2.34) involve the variables u_5, u_6, u_7. We need to evaluate the operators $\Delta_5^-, \Delta_6^+, \Delta_7^+$ applied to f_2 for $u_5 = u_6 = u_7 = 1$, in terms of operators acting only on functions of the variables $u_1, \cdots, u_4$. From (2.34) we find

$$T_6 \sim a^{-1} T_1^{-1} T_3^{-1}, \qquad T_5^{-1} \sim b' a^{-1} T_1^{-1} T_3^{-1}, \qquad T_7 \sim c' q^{-1} a^{-1} T_1^{-1} T_3^{-1}.$$

Thus the solution (3.3) is characterized by the equations

$$\begin{gathered} T_2 T_3 \sim b^{-1}, \qquad T_4 T_3^{-1} \sim c q^{-1}, \\ \Delta_1^-\left(1 - \frac{b'}{a} T_1^{-1} T_3^{-1}\right) - \left(1 - \frac{1}{a} T_1^{-1} T_3^{-1}\right)\left(1 - \frac{c'}{qa} T_1^{-1} T_3^{-1}\right) \sim 0. \end{gathered} \tag{3.4}$$

Note that the operators $T_2 T_3$, $T_4 T_3^{-1}$, Δ_1^- and $T_1^{-1} T_3^{-1}$ are all symmetries of $\Delta_1^- \Delta_2^- - \Delta_3^+ \Delta_4^+$, so that we have characterized the solution (3.3) of (3.1) in terms of a set of (mixed) eigenvalue equations for symmetries of (3.1). (It is not always the case that the generating function of the restricted canonical system obtained through this process is characterized in terms of symmetries of the restricted system. An example is the restriction of the canonical system for the Appell function F_1 to the wave equation [11]. However, in this case and in all other such examples known to the authors appropriate functional linear combinations of the mixed eigenvalue equations can be expressed in terms of symmetry operators and the resulting system still uniquely characterizes the generating function.) This is the process of *embedding*.

The process inverse to embedding is augmentation. Here we are given a canonical system of q-difference equations and a characterization of a generating function for this system by a set of mixed eigenvalue equations expressed in terms of symmetry operators for the canonical system. Our aim is to establish simple rules for determination of the generating function as an explicit q-hypergeometric series by recasting the defining equations as a canonical system with dilation eigenvalue equations in a greater number of variables than the original problem.

To see how this procedure works, consider the generating function, characterized as the function analytic in $u_1, \cdots, u_4$ at $u_3 = 0$ and satisfying the canonical equation (3.1) and the mixed eigenvalue equations (3.4). Since the last equation in (3.4) is neither a canonical equation or a dilation eigenvalue equation for (3.1), we cannot determine the power series expression for the generating function by inspection. However, we can replace the expressions

$$1 - b' a^{-1} T_1^{-1} T_3^{-1}, \qquad 1 - a^{-1} T_1^{-1} T_3^{-1}, \qquad 1 - c' q^{-1} a^{-1} T_1^{-1} T_3^{-1}$$

by Δ_5^-, Δ_6^+, Δ_7^+, respectively, for $u_5 = u_6 = u_7 = 1$ where $T_5^{-1} \sim b' a^{-1} T_1^{-1} T_3^{-1}$, $T_6 \sim a^{-1} T_1^{-1} T_3^{-1}$ and $T_7 = c' q^{-1} a^{-1} T_1^{-1} T_3^{-1}$. Then for general u_p the defining equations of the generating function take the canonical form (2.33), (2.34) with the unique solution (2.31), analytic at $u_3 = u_6 = 0$. Setting $u_5 = u_6 = u_7 = 1$ we obtain the generating function.

(Note that the choice Δ_5^-, Δ_6^+, Δ_7^+ is unique. If we had taken Δ_5^+ for example, we would have obtained the condition $T_5T_1T_3 \sim b'a^{-1}$, but $T_5T_1T_3$ is *not* a symmetry of the canonical equations (2.33).) We also note that q-analogues of the Appell function F_3 and the Horn function $\mathscr{H}_2$ correspond to these same canonical equations but have different analyticity properties [11].

The following sections contain several more examples of augmentation.

4. Generating functions for ${}_r\varphi_s$. Here we will present several examples showing how generating functions for the ${}_r\varphi_s$, (2.1) are associated with the canonical equation

$$(\Delta_1^- \cdots \Delta_r^- - \Delta_{r+1}^+ \cdots \Delta_{r+s+1}^+)\Psi = 0. \tag{4.1}$$

Recall that

$${}_r\Phi_s = {}_r\varphi_s\left(\begin{matrix} a_i \\ b_j \end{matrix}; \frac{u_{r+1}\cdots u_{r+s+1}}{u_1\cdots u_r}\right) u_1^{-\alpha_1}\cdots u_r^{-\alpha_r} u_{r+1}^{\beta_1-1}\cdots u_{r+s}^{\beta_s-1}$$

is a solution of (4.1) and that Δ_i^-, $(1 \leqq i \leqq r)$, Δ_{r+k}^+, $(1 \leqq k \leqq s)$, $T_{r+s+1}T_i$ and $T_{r+s+1}^{-1}T_k$ are symmetries of this equation.

There appears to be no convenient general q-analogue of the local Lie theory which permits us to compute Lie group symmetries of differential equations from Lie algebra symmetries through the process of exponentiation. However, in particular cases the analogy is successful. Consider the q-exponential

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \qquad |x| < 1 \tag{4.2}$$

satisfying $\Delta_x^+ e_q = e_q$ [15, p. 92]. In a formal sense at least, the operator $e_q(\lambda\Delta_1^-)$, $\lambda \in \mathbb{C}$, is a symmetry of (4.1). Applying this operator to a basis solution ${}_r\Phi_s$ and making use of (2.14), (4.2) we obtain

$$e_q(\lambda\Delta_1^-)\,{}_r\Phi_s(a_1) = \sum_{n=0}^{\infty} \frac{\lambda^n (a_1;q)_n}{(q;q)_n}\,{}_r\varphi_s(a_1q^n). \tag{4.3}$$

To compute the left-hand side of (4.3) we utilize Heines's (q-binomial) theorem [15, p. 92], [2],

$$\sum_{m=0}^{\infty} \frac{(a;q)_m}{(q;q)_m} t^m = \frac{(at;q)_\infty}{(t;q)_\infty}$$

to derive

$$e_q(\lambda\Delta_x^-)x^n = x^n \frac{(\lambda q^{-n}/x;q)_\infty}{(\lambda/x;q)_\infty} = \frac{x^n}{(\lambda/x,q)_{-n}}. \tag{4.4}$$

From (4.4), (2.1) and (2.12) we find

$$\begin{aligned} e_q(\lambda\Delta_1^-)\,{}_r\Phi_s(a_1) = \frac{(\lambda/a_1u_1;q)_\infty}{(\lambda/u_1;q)_\infty}\,{}_r\varphi_{s+1}\left(\begin{matrix} a_i \\ b_j \end{matrix}; \lambda/a_1u_1; \frac{u_{r+1}\cdots u_{r+s+1}}{u_1\cdots u_r}\right) \\ \cdot u_1^{-\alpha_1}\cdots u_r^{-\alpha_r}u_{r+1}^{\beta_1-1}\cdots u_{r+s}^{\beta_s-1} \end{aligned} \tag{4.5}$$

so that

$$\frac{(x/a_1;q)_\infty}{(x;q)_\infty}\,{}_r\varphi_{s+1}\left(\begin{matrix} a_i \\ b_j \end{matrix}; x/a_1; y\right) = \sum_{n=0}^{\infty} \frac{(a_1^{-1};q)_n}{(q;q)_n} x^n\,{}_r\varphi_s\left(\begin{matrix} a_1q^n, a_2, \cdots, a_r \\ b_j \end{matrix}; y\right). \tag{4.6}$$

Another way to understand this result is to note that the right-hand side of (4.3) is an eigenfunction of the operator $T_{r+s+1}T_1 - \lambda q^{-1}\Delta_1^- T_{r+s+1}T_1$ with eigenvalue a_1^{-1}. The method of augmentation can then be employed to derive (4.5). Still another point of view is that since ${}_r\Phi_s(a_1)$ is an eigenfunction of the symmetry operator $T_{r+s+1}T_1$ with eigenvalue a_1^{-1} then $e_q(\lambda\Delta_1^-){}_r\Phi_s(a_1)$ must be an eigenfunction of the formal symmetry operator

$$e_q(\lambda\Delta_1^-)T_{r+s+1}T_1E_q(-\lambda\Delta_1^-) \tag{4.7}$$

with the same eigenvalue. Here [15, p. 93]

$$E_q(x)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}x^n=(-x;q)_\infty,$$
$$\Delta_x^- E_q=-q^{-1}E_q,\qquad e_q(x)E_q(-x)=1. \tag{4.8}$$

Let X, Y be linear operators.

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$$f(\lambda)\equiv e_q(\lambda X)YE_q(-\lambda X)=\sum_{n=0}^{\infty}\frac{\lambda^n}{(q;q)_n}[X,Y]_n \tag{4.9}$$

where

$$[X,Y]_0=Y,\qquad [X,Y]_1=XY-YX,$$
$$[X,Y]_{n+1}=X[X,Y]_n-q^n[X,Y]_nX,\qquad n=1,2,\cdots. \tag{4.10}$$

Proof. This result is equivalent to the identity $\Delta_\lambda^+ f(\lambda)=Xf(\lambda)-T_\lambda f(\lambda)X$ and the identity can be verified by formal power series expansion in the variable λ. (The authors learned of this result from Mourad Ismail and Dennis Stanton.)

For $X=\Delta_1^-$, $Y=T_{r+s+1}T_1$ it is easily verified that $[X,Y]_1=(1-q^{-1})\Delta_1^- T_{r+s+1}T_1$, $[X,Y]_2=0$ so, by the lemma:

$$e_q(\lambda\Delta_1^-)T_{r+s+1}T_1E_q(-\lambda\Delta_1^-)=T_{r+s+1}T_1-q^{-1}\lambda\Delta_1^- T_{r+s+1}T_1.$$

Using the q-binomial theorem we can verify that

$$E_q(-\lambda\Delta_x^+)x^n=x^n\frac{(\lambda/x;q)_\infty}{(\lambda q^n/x;q)_\infty}. \tag{4.11}$$

From (2.14) we have

$$E_q(-\lambda\Delta_{r+1}^+){}_r\Phi_s(b_1)=\sum_{n=0}^{\infty}\frac{(b\lambda/q)^n(b^{-1}q;q)_n}{(q;q)_n}{}_r\Phi_s(b_1q^{-n}),$$

$$E_q(-\lambda\Delta_{r+s+1}^+){}_r\Phi_s\begin{pmatrix}a_i\\ b_j\end{pmatrix}=\sum_{n=0}^{\infty}\frac{(-\lambda)^nq^{n(n-1)/2}}{(q;q)_n}\cdot\frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n}{}_r\Phi_s\begin{pmatrix}a_iq^n\\ b_jq^n\end{pmatrix}.$$

Applying (4.11) we obtain the generating functions

$$\frac{(qx/b;q)_\infty}{(x;q)_\infty}{}_{r+1}\varphi_s\begin{pmatrix}a_i,x\\ b_j\end{pmatrix};y\Big)=\sum_{n=0}^{\infty}\frac{(q/b_1;q)_n}{(q;q)_n}{}_r\varphi_s\begin{pmatrix}a_i\\ b_1q^{-n},b_2\cdots b_s\end{pmatrix};y\Big)x^n,$$
$${}_{r+1}\varphi_s\begin{pmatrix}a_i,x\\ b_j\end{pmatrix};y\Big)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n}{}_r\varphi_s\begin{pmatrix}a_iq^n\\ b_jq^n\end{pmatrix};y\Big)(-xy)^n. \tag{4.12}$$

Now we consider some examples where the generating functions are directly characterized in terms of symmetry operators for the canonical equation (4.1). Let A, B, C be nonnegative integers with $B \geqq 1$, $A+B=r$, $C+B=s$. We look for an eigenfunction of (4.1) characterized by the following conditions:

$$(4.13)\quad \begin{array}{ll} \Delta_1^-\Delta_2^- \cdots \Delta_A^- - \Delta_{r+1}^+\Delta_{r+2}^+ \cdots \Delta_{r+C}^+\Delta_{r+s+1}^+ \sim 0, & \\ \Delta_{A+1}^-\Delta_{A+2}^- \cdots \Delta_r^- - \Delta_{r+C+1}^+\Delta_{r+C+2}^+ \cdots \Delta_{r+s}^+ \sim 0, & \\ T_1^{-1}T_{r+s+1}^{-1} \sim a_1, & T_{r+1}T_{r+s+1}^{-1} \sim c_1 q^{-1}, \\ \vdots & \vdots \\ T_A^{-1}T_{r+s+1}^{-1} \sim a_A, & T_{r+C}T_{r+s+1}^{-1} \sim c_C q^{-1}, \\ T_{A+1}T_r^{-1} \sim b_1 q^{-1}, & T_{r+C+1}^{-1}T_r^{-1} \sim d_1, \\ \vdots & \vdots \\ T_{A+B-1}T_r^{-1} \sim b_{B-1}q^{-1}, & T_{r+C+B}^{-1}T_r^{-1} \sim d_B. \end{array}$$

Choosing u_r and u_{r+s+1} as the distinguished variables such that the generating function is analytic at $u_r = u_{r+s+1} = 0$, we see that equations (4.13) are in canonical form with solution

$$\begin{aligned} &{}_A\varphi_C\left(\begin{matrix} a_1, \cdots, a_A \\ c_1, \cdots, c_C \end{matrix}; \frac{u_{r+1}\cdots u_{r+C}}{u_1 \cdots u_A} u_{r+s+1}\right) u_1^{-\alpha_1} \cdots u_A^{-\alpha_A} u_{r+1}^{\gamma_1 - 1} \cdots u_{r+C}^{\gamma_C - 1} \\ &\quad \cdot {}_B\varphi_{B-1}\left(\begin{matrix} d_1, \cdots, d_B \\ b_1, \cdots, b_{B-1} \end{matrix}; \frac{qb_1 \cdots b_{B-1}u_{A+1} \cdots u_{A+B-1}}{d_1 \cdots d_B u_{r+C+1} \cdots u_{r+s}} u_r\right) \\ &\quad \cdot u_{A+1}^{\beta_1 - 1} \cdots u_{A+B-1}^{\beta_{B-1}} u_{r+C+1}^{-\delta_1} \cdots u_{r+s}^{-\delta_B} \\ &= \sum_{n=0}^{\infty} X_n\, {}_{A+B}\varphi_{C+B}\left(\begin{matrix} a_1, \cdots, a_A; q^{-n}, q^{1-n}/b_1, \cdots, q^{1-n}/b_{B-1} \\ c_1, \cdots, c_C; q^{1-n}/d_1, \cdots, q^{1-n}/d_B \end{matrix}; \frac{u_{r+1}\cdots u_{r+s+1}}{u_1 \cdots u_r}\right) \\ &\quad \cdot u_1^{-\alpha_1} \cdots u_A^{-\alpha_A} u_{A+1}^{\beta_1 + n - 1} \cdots u_{A+B-1}^{\beta_{B-1}+n-1} u_r^n u_{r+1}^{\gamma_1 - 1} \cdots u_{r+C}^{\gamma_C - 1} u_{r+C+1}^{-\delta_1 - n} \cdots u_{r+s}^{-\delta_B - n}. \end{aligned}$$

Setting $u_{r+s+1} = 0$, we find that

$$X_n = \left(\frac{qb_1 \cdots b_{B-1}}{d_1 \cdots d_B}\right)^n \frac{(d_1; q)_n \cdots (d_B; q)_n}{(b_1; q)_n \cdots (b_{B-1}; q)_n (q; q)_n}$$

and the generating function simplifies to

$$\begin{aligned} &{}_A\varphi_C\left(\begin{matrix} a_1, \cdots, a_A \\ c_1, \cdots, c_C \end{matrix}; \frac{d_1 \cdots d_B}{b_1 \cdots b_{B-1}}; tz\right) {}_B\varphi_{B-1}\left(\begin{matrix} d_1, \cdots, d_B \\ b_1, \cdots, b_{B-1} \end{matrix}; t\right) \\ (4.14)\qquad &= \sum_{n=0}^{\infty} \frac{(d_1; q)_n \cdots (d_B; q)_n}{(b_1; q)_n \cdots (b_{B-1}; q)_n} \\ &\quad \cdot {}_{A+B}\varphi_{C+B}\left(\begin{matrix} a_1, \cdots, a_A; q^{-n}, q^{1-n}/b_1, \cdots, q^{1-n}/b_{B-1} \\ c_1, \cdots, c_C; q^{1-n}/d_1, \cdots, q^{1-n}/d_B \end{matrix}; qz\right) \frac{t^n}{(q; q)_n}. \end{aligned}$$

Finally we derive a generating function for the q-series

$$
\begin{aligned}
&{}_{p+1}\hat{\varphi}_{p+r}\left(\begin{matrix} a_1, \cdots, a_{p+1} \\ b_1, \cdots, b_{p+r} \end{matrix}; q^{-1}, x\right) \\
(4.15)\qquad &= \sum_{n=0}^{\infty} \frac{(a_1; q^{-1})_n \cdots (a_{p+1}; q^{-1})_n}{(b_1; q^{-1})_n \cdots (b_{p+r}; q^{-1})_n} \frac{x^n}{(q^{-1}; q^{-1})_n} \\
&= \sum_{n=0}^{\infty} \frac{(a_1^{-1}; q)_n \cdots (a_{p+1}^{-1}; q)_n (-1)^{rn} q^{rn(n-1)/2}}{(b_1^{-1}; q)_n \cdots (b_{p+r}^{-1}; q)_n (q; q)_n} \left(\frac{x a_1 \cdots a_{p+1}}{b_1 \cdots b_{p+r} q}\right)^n.
\end{aligned}
$$

Consider the equations

$$(4.16)\qquad \Delta_1^+ \cdots \Delta_{p+1}^+ - \Delta_{p+2}^- \cdots \Delta_{2p+r+2}^- \sim 0,$$

$$
(4.17)\qquad \begin{aligned}
&T_1^{-1} T_{2p+r+2}^{-1} \sim q^{\alpha_i} = a_i^{-1}, \qquad 1 \leqq i \leqq p+1, \\
&T_j T_{2p+r+2}^{-1} \sim q^{\beta_j - 1} = b_j^{-1} q^{-1}, \qquad p+2 \leqq j \leqq 2p+r+1,
\end{aligned}
$$

in canonical form. The solution of these equations, analytic at $u_{2p+r+2} = 0$ is:

$$
(4.18)\qquad \begin{aligned}
&{}_{p+1}\hat{\varphi}_{p+r}\left(\begin{matrix} a_1, \cdots, a_{p+1} \\ b_1, \cdots, b_{p+r} \end{matrix}; q^{-1}, \frac{u_{p+2} \cdots u_{2p+r+2}}{u_1 \cdots u_{p+1}}\right) \\
&\cdot u_1^{-\alpha_1} \cdots u_{p+1}^{-\alpha_{p+1}} u_{p+2}^{\beta_1 - 1} \cdots u_{2p+r+1}^{\beta_{p+r}-1}.
\end{aligned}
$$

We search for a generating function satisfying (4.16) and the following conditions:

$$
(4.19)\qquad \begin{aligned}
&\text{(a)}\qquad \Delta_1^+ + c T_1 T_{2p+r+2} \sim 1, \\
&\text{(b)}\qquad T_1^{-1} T_{2p+r+2}^{-1} \sim a_i^{-1}, \qquad 2 \leqq i \leqq p+1, \\
&\qquad\qquad T_j T_{2p+r+2}^{-1} \sim b_j^{-1} q^{-1}, \qquad p+2 \leqq j \leqq 2p+r+1.
\end{aligned}
$$

Introducing a new variable u_{2p+r+3} and conditions

$$(4.20)\qquad T_1 T_{2p+r+2} T_{2p+r+3} \sim c^{-1}, \qquad \Delta_1^+ - \Delta_{2p+r+3}^- \sim 0$$

we see that (4.20) reduces to (4.19) (a) when $u_{2p+r+3} = 1$ and conditions (4.16), (4.19) (b), (4.20) are in canonical form. It is straightforward to write down the generating function analytic at $u_1 = u_{2p+r+3} = 0$ and, with simplification, to obtain the identity

$$
(4.21)\qquad \begin{aligned}
&\frac{(ct; q)_\infty}{(t; q)_\infty} {}_{p+1}\Phi_{p+r+1}\left[\begin{matrix} c, a_2^{-1}, \cdots, a_{p+1}^{-1} \\ ct, b_1^{-1}, \cdots, b_{p+r}^{-1} \end{matrix}; q, xt\right] \\
&= \sum_{k=0}^{\infty} \frac{(c; q)_k}{(q; q)_k} {}_{p+1}\Phi_{p+r}\left[\begin{matrix} q^{-n}, a_2^{-1}, \cdots, a_{p+1}^{-1} \\ b_1^{-1}, \cdots, b_{p+r}^{-1} \end{matrix}; q, xq^n\right] t^k
\end{aligned}
$$

where

$$
(4.22)\qquad \begin{aligned}
&{}_{p+1}\Phi_{p+r}\left[\begin{matrix} a_1, \cdots, a_{p+1} \\ b_1, \cdots, b_{p+r} \end{matrix}; q, x\right] \\
&= \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_{p+1}; q)_n (-1)^r q^{rn(n-1)/2}}{(b_1; q)_n \cdots (b_{p+r}; q)_n} \frac{x^n}{(q; q)_n}.
\end{aligned}
$$

The generating functions derived above are not "deep." Indeed each can be proven by equating coefficients of powers of appropriate variables and using the q-binomial theorem. Furthermore, more general generating functions hold when some of the q-shifted factorials are replaced by arbitrary sequences; see [6]. Our point is that generating functions in their totality can be classified and derived using symmetry methods. In the following section we consider cases where the q-series have a richer symmetry structure and the generating functions are more interesting.

5. Generating functions for ${}_2\varphi_1$. The canonical equations for the q-hypergeometric functions ${}_2\varphi_1$ and ${}_1\varphi_1$ admit certain simple symmetry operators which do not extend to symmetries of the equations for general ${}_r\varphi_s$. This is closely related to the fact that ${}_2\varphi_1$ and ${}_1\varphi_1$ obey q-difference recurrence relations not shared by general ${}_r\varphi_s$. We shall examine the case ${}_2\varphi_1$ in some detail. Here the canonical equation is

$$Q \equiv \Delta_1^-\Delta_2^- - \Delta_3^+\Delta_4^+ \sim 0 \tag{5.1}$$

and the eigenvalue equations for the basis

$${}_2\Phi_1 = {}_2\varphi_1\begin{pmatrix} a, b \\ c \end{pmatrix}; \frac{u_3u_4}{u_1u_2}\Big) u_1^{-\alpha}u_2^{-\beta}u_3^{\gamma-1}$$

are

$$T_4^{-1}T_1^{-1} \sim a, \qquad T_4^{-1}T_2^{-1} \sim b, \qquad T_4^{-1}T_3 \sim cq^{-1}. \tag{5.2}$$

In addition to the dilation operators T_4T_1, T_4T_2, $T_4T_3^{-1}$, their products and inverses, we have as symmetries of (5.1) the operators

$$E^\alpha = \Delta_1^-, \quad E^\beta = \Delta_2^-, \quad E_\gamma = \Delta_3^+, \quad E^{\alpha\beta\gamma} = \Delta_4^+,$$

$$E_\alpha = -q^{-1}u_3u_4T_1^{-2}T_4^{-2}\Delta_2^- + u_1u_3T_1^{-1}T_4^{-1}\Delta_3^+ - qu_1T_3T_4^{-1} + u_1T_1^{-1}T_3T_4^{-2},$$

$$E_\beta = -u_3u_4q^{-1}T_2^{-2}T_4^{-2}\Delta_1^- + u_2u_3T_2^{-1}T_4^{-1}\Delta_3^+ - qu_2T_3T_4^{-1} + u_2T_2^{-1}T_3T_4^{-2},$$

$$E^\gamma = u_3\left[qT_3T_4^{-2} - T_1^{-1}T_2^{-1}T_4^{-2} + q^2\frac{u_1u_2}{u_3}T_3\Delta_4^-T_4^{-1}\right], \tag{5.3}$$

$$E_{\alpha\beta\gamma} = -u_1u_2T_4^{-1}\Delta_3^+ + q^{-1}u_4T_4^{-1} - q^{-2}u_4T_1^{-1}T_2^{-1}T_4^{-2},$$

$$E^{\alpha\gamma} = u_3T_4^{-1}\Delta_1^- + qu_2\Delta_4^-, \qquad E^{\beta\gamma} = u_3T_4^{-1}\Delta_2^- + qu_1\Delta_4^-,$$

$$E_{\alpha\gamma} = -u_1\Delta_3^+ + q^{-1}u_4T_1^{-1}T_4^{-1}\Delta_2^-, \qquad E_{\beta\gamma} = -u_2\Delta_3^+ + q^{-1}u_4T_2^{-1}T_4^{-1}\Delta_1^-.$$

These symmetry operators correspond to recurrence relations for the ${}_2\varphi_1$ since, when applied to the standard basis ${}_2\Phi_1\binom{a,b}{c}$, they yield

$$E^\alpha\,{}_2\Phi_1 = (1-a)\,{}_2\Phi_1(aq), \qquad E_\alpha\,{}_2\Phi_1 = (a-c)\,{}_2\Phi_1(aq^{-1}),$$

$$E^\beta\,{}_2\Phi_1 = (1-b)\,{}_2\Phi_1(bq), \qquad E_\beta\,{}_2\Phi_1 = (b-c)\,{}_2\Phi_1(bq^{-1}),$$

$$E^\gamma\,{}_2\Phi_1 = \frac{(c-a)(c-b)}{c-1}\,{}_2\Phi_1(cq), \qquad E_\gamma\,{}_2\Phi_1 = \left(1-\frac{c}{q}\right){}_2\Phi_1(cq^{-1}),$$

$$E^{\alpha\beta\gamma}\,{}_2\Phi_1 = \frac{(1-a)(1-b)}{1-c}\,{}_2\Phi_1\begin{pmatrix} aq, bq \\ cq \end{pmatrix}, \qquad E_{\alpha\beta\gamma}\,{}_2\Phi_1 = \left(\frac{c}{q}-1\right){}_2\Phi_1\begin{pmatrix} aq^{-1}, bq^{-1} \\ cq^{-1} \end{pmatrix}, \tag{5.4}$$

$$E^{\alpha\gamma}\,{}_2\Phi_1 = \frac{(b-c)(1-a)}{1-c}\,{}_2\Phi_1\begin{pmatrix} aq \\ cq \end{pmatrix}, \qquad E_{\alpha\gamma}\,{}_2\Phi_1 = \left(1-\frac{c}{q}\right){}_2\Phi_1\begin{pmatrix} aq^{-1} \\ cq^{-1} \end{pmatrix},$$

$$E^{\beta\gamma}\,{}_2\Phi_1 = \frac{(a-c)(1-b)}{1-c}\,{}_2\Phi_1\begin{pmatrix} bq \\ cq \end{pmatrix}, \qquad E_{\beta\gamma}\,{}_2\Phi_1 = \left(1-\frac{c}{q}\right){}_2\Phi_1\begin{pmatrix} bq^{-1} \\ cq^{-1} \end{pmatrix}.$$

There appear to be no simple standardized expressions for these symmetries as q-difference operators applied to the null space of Q. Indeed one can always multiply each such symmetry by a dilation symmetry and obtain a recurrence relation (5.4) equivalent to the original relation. Second, given any q-difference operator D we can add DQ to any symmetry operator, since DQ acts as the zero operator on the null space of Q. One can use the above modifications to simplify considerably some of the expressions for the operators (5.3), but at the expense of complicating relations (5.4).

From the raising and lowering operators E we can form the following equations, each equivalent to the canonical equation (5.1):

$$
\begin{aligned}
&E^{\alpha}E_{\alpha}+T_4^{-1}(qT_3-T_1^{-1})(1-q^{-1}T_4^{-1}T_1^{-1})\sim 0,\\
&E^{\beta}E_{\beta}+T_4^{-1}(qT_3-T_2^{-1})(1-q^{-1}T_4^{-1}T_2^{-1})\sim 0,\\
&E^{\gamma}E_{\gamma}+T_4^{-2}(T_3-T_1^{-1})(T_3-T_2^{-1})\sim 0,\\
&E^{\alpha\beta\gamma}E_{\alpha\beta\gamma}+(1-q^{-1}T_4^{-1}T_1^{-1})(1-q^{-1}T_4^{-1}T_2^{-1})\sim 0,\\
&E^{\alpha\gamma}E_{\alpha\gamma}-T_4^{-1}(T_2^{-1}-T_3^{-1})(1-q^{-1}T_4^{-1}T_1^{-1})\sim 0,\\
&E^{\beta\gamma}E_{\beta\gamma}-T_4^{-1}(T_1^{-1}-T_3^{-1})(1-q^{-1}T_4^{-1}T_2^{-1})\sim 0.
\end{aligned}
\tag{5.5}
$$

The operators E and the dilation symmetries T_4T_1, T_4T_2, $T_4T_3^{-1}$ form a q-analogue of the 15-dimensional conformal symmetry algebra of the wave equation in four-dimensional space-time. Although these q-symmetry operators do not generate a finite-dimensional Lie algebra under operator commutation they still permit us to construct the invariants (5.5).

The method of augmentation can be used to obtain explicit expressions for many generating functions characterized by E symmetry operators. For example, while the conditions

$$E_{\alpha}\sim -c,\quad T_2T_4\sim b^{-1},\quad T_3T_4^{-1}\sim cq^{-1} \tag{5.6}$$

would be difficult to solve directly, due to the complicated expression for E_{α}, we note that the first of these conditions implies $E^{\alpha}E_{\alpha}\sim -cE^{\alpha}$ so from the first expression (5.5) for $E^{\alpha}E_{\alpha}$

$$\Delta_1^{-}-(1-c^{-1}T_4^{-1}T_1^{-1})(1-q^{-1}T_4^{-1}T_1^{-1})\sim 0.$$

Setting $T_5\sim c^{-1}T_4^{-1}T_1^{-1}$, $T_6\sim q^{-1}T_4^{-1}T_1^{-1}$, we see that the desired generating functions are the restrictions to $u_5=u_6=1$ of certain solutions of the canonical system

$$
\begin{aligned}
&\Delta_1^{-}\Delta_2^{-}-\Delta_3^{+}\Delta_4^{+}\sim 0, &&\Delta_1^{-}-\Delta_5^{+}\Delta_6^{+}\sim 0,\\
&T_2T_4\sim b^{-1}, &&T_3T_4^{-1}\sim cq^{-1},\\
&T_1T_4T_5\sim c^{-1}, &&T_1T_4T_6\sim q^{-1}.
\end{aligned}
\tag{5.7}
$$

We can immediately write down a series solution:

$$\Psi=f_2\left(\begin{matrix}c,\,b,\,0\\ c,\,c\end{matrix};\frac{u_3u_4}{u_1u_2},\frac{u_5u_6}{u_1}\right)u_1^{-\gamma}u_2^{-\beta}u_3^{\gamma-1}u_6^{\gamma-1} \tag{5.8}$$

where f_2 is defined by (2.31). Setting $u_5=u_6=1$ we find the (not very interesting) generating function

$$f_2\left(\begin{matrix}c,\,b,\,0\\ c,\,c\end{matrix};z,t\right)=\sum_{n=0}^{\infty}\frac{1}{(q;q)_n}\,{}_2\varphi_1\left(\begin{matrix}cq^n,\,b\\ c\end{matrix};z\right)t^n. \tag{5.9}$$

We shall return to this example after introducing the q-Kummer transformation symmetry.

The q-Kummer transformation

$$(5.10)\qquad {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{(ax;q)_\infty}{(x;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n (c/b;q)_n q^{n(n-1)/2}}{(c;q)_n (q;q)_n} \frac{(-bn)^n}{(ax;q)_n}$$

[1] and the q-Euler transform

$$(5.11)\qquad {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{(abx/c;q)_\infty}{(x;q)_\infty} {}_2\varphi_1\left(\begin{matrix} c/a, c/b \\ c \end{matrix}; \frac{abx}{c}\right)$$

[15, p. 97], [2] can be related to symmetries of (5.1). For this we consider the restriction of the operator Q, (5.1), to the space of convergent Laurent series in the monomials

$$(5.12)\qquad f_{k,\alpha,\beta,\gamma} = \left(\frac{u_3 u_4}{u_1 u_2}\right)^k u_1^{-\alpha} u_2^{-\beta} u_3^{\gamma-1}$$

where k is a nonnegative integer and α, β, γ are complex numbers such that $\gamma \neq 0, -1, -2, \cdots$. (That is, we do not consider the complication of logarithmic solutions.) We define the operator R_1 on this space as the unique linear operator such that

$$R_1(f_{k,\alpha,\beta,\gamma}) = \frac{(azq^k;q)_\infty}{(z;q)_\infty} q^{k(k-1)/2} \left(\frac{-cz}{b}\right)^k u_1^{-\alpha} u_2^{\beta-\gamma} u_3^{\gamma-1}$$

$$(5.13)\qquad = \left(\frac{-c}{b}\right)^k q^{k(k-1)/2} \sum_{n=0}^{\infty} \frac{(aq^k;q)_n}{(q;q)_n} z^{n+k} u_1^{-\alpha} u_2^{\beta-\gamma} u_3^{\gamma-1},$$

$$z = \frac{u_3 u_4}{u_1 u_2}, \quad a = q^\alpha, \quad b = q^\beta, \quad c = q^\gamma.$$

(This is a q-analogy of the inversion in a cone conformal symmetry of $\partial_{12} - \partial_{34} \sim 0$.) Similarly we define R_2 by

$$(5.14)\qquad R_2(f_{k,\alpha,\beta,\gamma}) = \left(\frac{-cz}{a}\right)^k q^{k(k-1)/2} \frac{(bzq^k;q)_\infty}{(z;q)_\infty} u_1^{\alpha-\gamma} u_2^{-\beta} u_3^{\gamma-1}$$

and linearity, and S by linearity and

$$(5.15)\qquad S(f_{k,\alpha,\beta,\gamma}) = \frac{(cz/ab;q)_\infty}{(z;q)_\infty} \left(\frac{cz}{ab}\right)^k u_1^{\beta-\gamma} u_2^{\alpha-\gamma} u_3^{\gamma-1}.$$

It is not difficult to show that (5.10), (5.11) are equivalent to the assertion that R_1, R_2 and S are symmetries of (5.1). (The direct proofs of (5.10), (5.11) involve nothing more complicated than the q-Vandermonde theorem [3].)

Note that a basis for the solution space of $Q \sim 0$ and the eigenvalue equations $T_1^{-1}T_4^{-1} \sim a$, $T_2^{-1}T_4^{-1} \sim b$, $T_3 T_4^{-1} \sim cq^{-1}$ consists of the functions

$${}_2\Phi_1\left(\begin{matrix} a, b \\ c \end{matrix}\right) = {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) u_1^{-\alpha} u_2^{-\beta} u_3^{\gamma-1} \quad \text{and}$$

$${}_2\Phi_1'\left(\begin{matrix} a, b \\ c \end{matrix}\right) = {}_2\varphi_1\left(\begin{matrix} qa/c, qb/c \\ q^2/c \end{matrix}; z\right) z^{1-\gamma} u_1^{-\alpha} u_2^{-\beta} u_3^{\gamma-1}.$$

Furthermore the operators R_1, R_2, S satisfy

$$R_1\,{}_2\Phi_1\begin{pmatrix} a,\, b \\ c \end{pmatrix} = {}_2\Phi_1\begin{pmatrix} a,\, c/b \\ c \end{pmatrix},$$

$$R_1\,{}_2\Phi_1'\begin{pmatrix} a,\, b \\ c \end{pmatrix} = (-c^{1/2}/b)^{1-\gamma}\,{}_2\Phi_1'\begin{pmatrix} a,\, c/b \\ c \end{pmatrix},$$

(5.16)
$$R_2\,{}_2\Phi_1\begin{pmatrix} a,\, b \\ c \end{pmatrix} = {}_2\Phi_1\begin{pmatrix} c/a,\, b \\ c \end{pmatrix},$$

$$R_2\,{}_2\Phi_1'\begin{pmatrix} a,\, b \\ c \end{pmatrix} = (-c^{1/2}/a)^{1-\gamma}\,{}_2\Phi_1'\begin{pmatrix} c/a,\, b \\ c \end{pmatrix},$$

$$R_1^2 = R_2^2 = S^2 = I, \qquad R_1R_2 = R_2R_1 = S,$$

where I is the identity operator. Other easily derived properties of R_1 are as follows:

$$R_1E^\alpha R_1^{-1} = E^\alpha, \qquad R_1E^\beta R_1^{-1} = E_\beta T_2T_4,$$

$$R_1E_\alpha R_1^{-1} = E_\alpha, \qquad R_1E_{\alpha\gamma}R_1^{-1} = -E_{\alpha\beta\gamma},$$

$$R_1E_\beta R_1^{-1} = qE^\beta T_3T_2, \qquad R_1E^\gamma R_1^{-1} = -qE^{\beta\gamma}T_3T_2,$$

(5.17)
$$R_1E_\gamma R_1^{-1} = E_{\beta\gamma}, \qquad R_1E^{\alpha\beta\gamma}R_1^{-1} = E^{\alpha\gamma}T_2T_4,$$

$$R_1T_1^{-1}T_4^{-1}R_1^{-1} = T_1^{-1}T_4^{-1}, \qquad R_1T_2^{-1}T_4^{-1}R_1^{-1} = qT_2T_3,$$

$$R_1T_3T_4^{-1}R_1^{-1} = T_3T_4^{-1}.$$

Similar results for R_2 follow from (5.17) and the interchanges $1\leftrightarrow 2$, $\alpha\leftrightarrow\beta$, and the corresponding results for S follow from $S = R_1R_2 = R_2R_1$.

As an example of the use of these symmetries we consider a generating function Ψ characterized by

(5.18)
$$E^\alpha E_\beta \sim \lambda, \qquad T_4^{-1}T_3 \sim \lambda, \qquad T_4^2T_1T_2 \sim 1/\lambda\mu q.$$

Due to the occurrence of the operator E_β, it is not easy to find a simple form for the generating function by direct computation from Ψ. However, we can transform this problem into a simpler one. Indeed, $\Psi' = R_1\psi$ satisfies

(5.19)
$$qE^\alpha E^\beta T_3T_2 \sim \lambda, \qquad T_4^{-1}T_3 \sim \lambda, \qquad T_1^{-1}T_2 \sim \mu.$$

Although these equations are not in canonical form they are easy to solve by substituting a formal power series for Ψ'. The solution analytic at $u_1 = u_4 = 0$ is

$$\Psi' = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2 - lk - l(l+1)/2}\, u_1^k u_2^{k+\mu} u_3^{l+\lambda}(q^{\mu+1}u_4)^l}{(q^{\mu+1};q)_k(q^{\lambda+1};q)_l(q;q)_k(q;q)_l}.$$

From (5.13) we then find easily that

(5.20)
$$\Psi = R_1\Psi' = \sum_{k=0}^{\infty} \frac{(z^{-1};q)_k(-zt/q)^k}{(q^{\mu+1};q)_k(q;q)_k} \sum_{l=0}^{\infty} \frac{(-q^{\lambda-1}zt)^l q^{l^2}}{(q^{\lambda+1};q)_l(q;q)_l}\, u_2^{-\mu-\lambda-1}u_3^\lambda$$

where $z = u_3u_4/u_1u_2$, $t = u_1/u_2$. (The factorization in this expression is not surprising, based on variable separation for the corresponding differential equation problem and the fact that the operators E^α, E_β commute.) The generating relation is

(5.21)
$$\Psi = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n(q^{\mu+1};q)_n}\,{}_2\varphi_1\begin{pmatrix} q^{-n},\, q^{\mu+\lambda+n+1} \\ q^{\lambda+1} \end{pmatrix};z\Big)\, t^n u_2^{-\mu-\lambda-1}u_3^\lambda$$

where the coefficient of ${}_2\varphi_1$ has been determined by setting $z=0$ in (5.20). This is the generating function for little q-Jacobi polynomials [10]; set $z=qx$.

For our final example we return to (5.6) and note that the function $\Psi'=R_2\Psi$ satisfies the conditions

$$E^{\alpha}T_1T_4\sim -1,\quad T_2T_4\sim b^{-1},\quad T_3T_4^{-1}\sim cq^{-1}$$

which are easily solved by power series substitution to yield

$$\Psi'=\sum_{k,l=0}^{\infty}\frac{q^{-k(k+1)/2-lk}(q^{\beta};q)_k(-z)^k u_1^{l+k}u_2^{-\beta}u_3^{\gamma-1}}{(q^{\gamma};q)_k(q;q)_k(q;q)_l}.$$

Then from (5.14) we find

$$\Psi=R_2\Psi'=\frac{(q^{\beta}z;q)_{\infty}}{(z;q)_{\infty}(t;q)_{\infty}}\sum_{k=0}^{\infty}\frac{q^{k^2}(q^{\gamma}zt)^k t^{\gamma}u_2^{-\beta}u_3^{\gamma-1}}{(q^{\beta}z;q)_k(q^{\gamma};q)_k(q;q)_k}$$
$$=\sum_{n=0}^{\infty}\frac{1}{(q;q)_n}\,{}_2\varphi_1\left(\begin{matrix}cq^n,b\\c\end{matrix};z\right)t^{n+\gamma}u_2^{-\beta}u_3^{\gamma-1}$$

where $z=u_3u_4/u_1u_2$, $t=u_1^{-1}$.

Our symmetry approach has profound relationships with the theory of orthogonal polynomials. We shall illustrate these relationships by presenting a new derivation of the orthogonality for little q-Jacobi polynomials which we normalize in the form

$$\Phi_n^{(a,b)}(x)={}_2\varphi_1\left(\begin{matrix}q^{-n},q^{n+1}ab\\aq\end{matrix};qx\right),\qquad n=0,1,2,\cdots \tag{5.22}$$

with $-1<q<1$, $0<aq<1$, $bq<1$ [10]. The symmetries $E^{\alpha\beta\gamma}$ and $E_{\alpha\beta\gamma}$, (5.3), induce recurrences for these polynomials:

$$\tau^{(a,b)}\Phi_n^{(a,b)}(x)=\frac{q(1-q^{-n})(1-q^{n+1}ab)}{(1-aq)}\Phi_{n-1}^{(aq,bq)}(x),$$
$$\tau^{*(aq,bq)}\Phi_{n-1}^{(aq,bq)}(x)=-(1-q)\Phi_n^{(a,b)}(x) \tag{5.23}$$

where

$$\tau^{(a,b)}=\Delta_x^+,\qquad \tau^{*(aq,bq)}=(x-1)T_x^{-1}+(aq-abq^2x).$$

The existence of this pair of "raising" and "lowering" operators suggests that there might exist a Hilbert space structure with respect to which τ^* and τ are mutually adjoint, so that $\tau^*\tau$ would be selfadjoint.

To be more explicit, let $w_{a,b}(x)$ be a (complex-valued) weight function and $S_{a,b}$ the indefinite inner-product space of polynomials $f(x)$ with respect to the inner product

$$(f_1,f_2)_{a,b}=\frac{1}{2\pi i}\int_C f_1(x)f_2(x)w_{a,b}(x)\frac{dx}{x} \tag{5.24}$$

where the contour C is a deformation of the circle $|x|=1+\varepsilon$, $\varepsilon>0$ in the complex x-plane. Consider $\tau^{(a,b)}$ and $\tau^{*(aq,bq)}$ as mappings:

$$\tau^{(a,b)}:S_{a,b}\to S_{aq,bq},\qquad \tau^{*(aq,bq)}:S_{aq,bq}\to S_{a,b}$$

and determine $w_{a,b}(x)$ so that

$$(\tau f,g)_{aq,bq}=(f,\tau^*g)_{a,b} \tag{5.25}$$

for all $f \in S_{a,b}$, $g \in S_{aq,bq}$. A straightforward "integration by parts" yields the following conditions:

$$\frac{-q}{x}\frac{w_{aq,bq}(x/q)}{w_{a,b}(x)} = x-1, \qquad \frac{1}{x}\frac{w_{aq,bq}(x)}{w_{a,b}(x)} = aq(1-bqx)$$

with the solution

$$w_{a,b}(x) = \frac{(x/a;\, q)_\infty (qa/x;\, q)_\infty}{(xbq;\, q)_\infty (1/x;\, q)_\infty s(a, q)},$$

(5.26)

$$s(a, q) = (aq;\, q)_\infty (1/a;\, q)_\infty (-aq;\, q)_\infty (-1/a;\, q)_\infty.$$

It follows immediately that $\tau^*\tau$ is selfadjoint on $S_{a,b}$ and from the recurrence relations (5.23) we have

(5.27) $$\tau^*\tau\Phi_n^{(a,b)} = \lambda_n \Phi_n^{(a,b)}, \qquad \lambda_n = -q(1-q^{-n})(1-q^{n+1}ab).$$

Clearly $\lambda_n \neq \lambda_m$ if $n \neq m$ and since eigenfunctions corresponding to distinct eigenvalues are orthogonal we have

(5.28) $$(\Phi_n^{(a,b)}, \Phi_m^{(a,b)})_{a,b} = 0 \quad \text{for } n \neq m.$$

From (5.23) and (5.25) with $f = \Phi_n^{(a,b)}$, $g = \Phi_{n-1}^{(aq,bq)}$ we obtain the following recurrence:

(5.29) $$\|\Phi_n^{(a,b)}\|_{a,b}^2 = \frac{-q(1-q^{-n})(1-q^{n+1}ab)}{(1-aq)^2}\|\Phi_{n-1}^{(aq,bq)}\|_{aq,bq}^2.$$

From (5.29) we can compute $\|\Phi_n^{(a,b)}\|_{a,b}^2$ once we know $\|1\|_{a,b}^2 = (\Phi_0^{(a,b)}, \Phi_0^{(a,b)})_{a,b}$ for all admissible a, b.

We now turn to the task of computing $\|1\|_{a,b}^2$. We know that $(\Phi_1^{(a,b)}, \Phi_0^{(a,b)})_{a,b} = 0$ and, substituting the explicit expression (5.22) for the orthogonal polynomial $\Phi_1^{(a,b)}(x)$, we can write this relation in the form

(5.30) $$\|1\|_{a,bq}^2 = \frac{(1-bq)}{(1-abq^2)}\|1\|_{a,b}^2.$$

(Here we have used the evident fact that $(1-xbq, 1)_{a,b} = \|1\|_{a,bq}^2$.) To obtain an additional condition on the norm we consider the symmetries E^γ, E_γ in the form:

$$\mu^{(a,b)}\Phi_n^{(a,b)} = (1-a)\Phi_n^{(aq^{-1},bq)},$$

(5.31)

$$\mu^{*(aq^{-1},bq)}\Phi_n^{(aq^{-1},bq)} = \frac{q^{-n}(1-aq^n)(1-bq^{n+1})}{a(1-a)}\Phi_n^{(a,b)}$$

where

$$\mu^{(a,b)} = 1 - aT_x^1, \qquad \mu^{(a,b)}: S_{a,b} \to S_{aq^{-1},bq},$$

$$\mu^{*(aq^{-1},bq)} = \frac{x-1}{ax}T_x^{-1} + \frac{1-bqx}{ax}, \qquad \mu^{*(aq^{-1},bq)}: S_{aq^{-1},bq} \to S_{a,b}.$$

It is easily verified that

(5.32) $$(\mu f, g)_{aq^{-1},bq} = (f, \mu^* g)_{a,b}$$

for all $f \in S_{a,b}$, $g \in S_{aq^{-1},bq}$ so that μ and μ^* are mutually adjoint. Setting $f = \Phi_0^{(a,b)}$, $g = \Phi_0^{(aq^{-1},bq)}$ in this relation, we see immediately that

(5.33) $$\|1\|_{aq^{-1},bq}^2 = \frac{(1-bq)}{a(1-a)}\|1\|_{a,b}^2.$$

The recurrences (5.30), (5.33) have the solution

$$\|1\|^2_{a,b} = \frac{(abq^2; q)_\infty K(q)}{(bq; q)_\infty (aq; q)_\infty (-1/a; q)_\infty (-aq; q)_\infty}.$$

Thus

$$\frac{1}{2\pi i}\int_C \frac{(x/a; q)_\infty (qa/x; q)_\infty dx}{(1/a; q)_\infty (1/x; q)_\infty (xbq; q)_\infty x} = \frac{(abq^2; q)_\infty K(q)}{(bq; q)_\infty (aq; q)_\infty}. \tag{5.34}$$

(Here we are assuming $a > 1+\varepsilon > 1 > qa$.) With the choice $bq = 1/a$ the integral becomes trivial to evaluate and we find that $K(q) = 1/(q; q)_\infty$. The complex orthogonality relations just determined can be recast as real discrete orthogonality through evaluation of the contour integral by residues at the poles $x = q^k$, $k = 0, 1, 2, \cdots$. The final result is

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(aq)^k (q^{k+1}; q)_\infty}{(bq^{k+1}; q)_\infty} \Phi_n^{(a,b)}(q^k) \Phi_m^{(a,b)}(q^k) \\ &\qquad = \frac{(aq)^n (abq^{n+1}; q)_\infty (q; q)_n \delta_{m,n}}{(bq^{n+1}; q)_\infty (aq; q)_\infty (aq; q)_n (1 - abq^{2n+1})}. \end{aligned} \tag{5.35}$$

Note that the proof of this result follows entirely from the symmetries; no special function identities are needed.

The ideas behind this derivation of orthogonality relations can be generalized substantially. In particular in [12] it is shown how to derive the orthogonality relations for the Askey–Wilson polynomials (the most general extension of the classical orthogonal polynomials known) using this symmetry method. A simple corollary of the derivation is an identity for ${}_4\varphi_3$ polynomials (Sears' transformation) that includes the q-Kummer and q-Euler transforms as special cases.

The fundamental symmetry concepts introduced in this paper extend to the very important q-series of the form

$${}_{r+1}\varphi_r\left(\begin{matrix} a_i, x \\ b_j \end{matrix}; q\right) \quad \text{and} \quad {}_{r+1}\varphi_r\left(\begin{matrix} a_i, ax, a/x \\ b_j \end{matrix}; q\right).$$

They will be the subject of future papers by the authors.

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