The Cheshire Cat effect in Lie theory. Lamé and Heun functions

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Abstract. There are many physically important linear PDEs that admit Lie symmetry groups, but for which the symmetry is destroyed by addition of potentials or consideration of non-symmetric solutions. For superintegrable systems on constant curvature spaces there is a Cheshire Cat effect: the symmetry disappears but its influence lingers on and can be used to analyze the solution spaces. We describe this approach to harmonic analysis and give examples that show how to obtain properties of Lamé and Heun functions, and insight into the classical Niven transform. Based on joint work with E. G. Kalnins, J. M. Kress and V. B. Kuznetsov.

1. Introduction

Most functions commonly called "special" obey symmetry properties best described via group theory. They arise as solutions of the PDEs of mathematical physics and can be characterized in terms of transformation properties under the Lie symmetries of the equations. There are, however, many important PDEs for which the initial symmetry is destroyed by addition of potentials or consideration of non-symmetric solutions. Fortunately, for superintegrable systems on constant curvature spaces there is a Cheshire Cat effect: the symmetry disappears but its influence lingers on and can be used to analyze the solution spaces. We consider solutions of Laplace-Beltrami eigenvalue problems via separation of variables. There is no general connection between symmetries of a manifold and the separable coordinate systems, but for constant curvature spaces (among others) a connection exists, at the level of 2nd-order elements in the universal enveloping algebra of the symmetry Lie algebra, and it persists even when the symmetry is completely broken by a potential. We describe these relations and give applications to Lamé and Heun functions, that directly have no Lie symmetry properties.

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2. Symmetries and variable separation

Let Δ_n be a Laplace-Beltrami operator for a Riemannian manifold V_n in n dimensions. The **Laplace-Beltrami eigenvalue equation** (with potential) for functions Ψ on V_n is $H\Psi(\mathbf{q}) = E\Psi(\mathbf{q})$ The **Laplace equation** is $H\Psi(\mathbf{q}) = 0$. The linear partial differential operator S is a **symmetry operator** for $(\Delta_n + V)\Phi = E\Phi$ if S maps local solutions Φ to local solutions $S\Phi$. Similarly, $\tilde{S}$ is a **conformal symmetry operator** for $(\Delta_n + V)\Phi = 0$ if $\tilde{S}$ maps local solutions Φ of this equation to local solutions $S\Phi$. The 1st-order symmetry operators for $(\Delta_n + V)\Phi = E\Phi$ form a Lie algebra, the **symmetry algebra** of this equation. The associated local **Lie symmetry group** maps solutions to solutions. There are similar definitions for conformal symmetries.

A set of orthogonal coordinates $\{x_\ell\}$ is ***R*-separable** for the Laplace-Beltrami equation if this equation admits solutions $\Psi = \exp(R(\mathbf{x}))\Pi_{i=1}^n \Psi_i(x^i) = e^R\Theta$, where $R(\mathbf{x})$ is a fixed function, independent of parameters, and the factors $\Psi_i(x^i)$ are the solutions of n ODEs (the separation equations) $\Psi_i'' + g_i(x^i)\Psi_i' - \left(f_i(x^i) + \sum_{j=1}^n \lambda_j s_{ij}(x^i)\right)\Psi_i = 0$, $i = 1, \cdots, n$ and $\lambda_1 = E$. The parameters λ_j are the **separation constants**. If $R \equiv 0$ we have *separation*, and if $R(x) = \sum_{i=1}^n R^{(i)}(x^i)$ we have *trivial* R-separation. There is a corresponding definition of R-separation for the Laplace equation with $E = 0$.

A basic result in the theory is [1] that every orthogonal R-separable coordinate system $\{x^i\}$ for $(\Delta_n + V)\Psi = E\Psi$ corresponds to a linearly independent set $\{S_1 = H = \Delta_n + V, S_2, \cdots, S_n\}$ of commuting 2nd-order partial differential symmetry operators. The R-separable solutions $\Psi_{\lambda_1,\cdots,\lambda_n}(\mathbf{x}) = \exp(R(\mathbf{x}))\Pi_{i=1}^n \Psi_i(x^i)$ are characterized as the simultaneous eigenfunctions of the commuting symmetry operators S_h: $S_h\Psi_{\lambda_1,\cdots,\lambda_n} = \lambda_h\Psi_{\lambda_1,\cdots,\lambda_n}$, $h = 1, \cdots, n$.

If $\mathscr{V}_n$ is a space of constant curvature and $V(\mathbf{q}) \equiv 0$ then all higher order symmetries of this space can be considered as elements of the universal enveloping algebra of the Lie symmetry algebra $\mathscr{G}_n$ of the manifold. In particular, the operators S_j describing variable separation are 2nd-order elements of the enveloping algebra of $\mathscr{G}_n$.

If $\mathscr{V}_n$ is a space of constant curvature and $V(\mathbf{q}) \neq 0$ then the symmetry operators (or conformal symmetry operators) describing variable separation take the form $S_j = \mathscr{S}_j + W_j$ where $\mathscr{S}_j$ is a 2nd-order element in the enveloping algebra of $\mathscr{G}_n$ and W_j is a scalar (potential) function. Thus even though the 1st order symmetry may be destroyed by the potential, the $\mathscr{S}_j$ retain their connection with the original symmetry algebra, e.g. [2]. This is the grin of the Cheshire Cat and is discussed by Kalnins [3] in his proceedings paper on superintegrable systems.

Finding all orthogonal separable coordinate systems $\mathbf{q}$ for a given space $\mathscr{V}_n$ is difficult. However, for real n-dimensional Euclidean space, the n-sphere, and the n-hyperboloid of two sheets, we have a graphical procedure to classify and construct all possibilities, [4, 5].

Here I discuss examples of special functions that arise through variable separation, but that have no simple transformation properties under the Lie symmetry algebra. (The Cheshire Cat is fading, but has not yet disappeared.) For the Euclidean space Laplace equation

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2)\Psi = 0$$

orthogonal separation is possible in the 11 Helmholtz separable systems [6] and nontrivial R-separation in 6 additional systems [7]. Each system is characterized by a pair of commuting

2nd-order conformal symmetry operators for the Laplacian. The conformal symmetry algebra of this equation is 10-dimensional, with basis

$$P_X = \partial_X,\ M_{YX} = -M_{XY} = Y\partial_X - X\partial_Y,\ D = -(\frac{1}{2} + X\partial_X + Y\partial_Y + Z\partial_Z),\ K_X = -2XD - R^2\partial_X$$

etc., where $R^2 = X^2 + Y^2 + Z^2$. The 55 2nd-order operators formed from this Lie algebra of differential operators satisfy 20 relations on the solution space, among which are

$$\mathbf{P}\cdot\mathbf{P} \equiv P_X^2 + P_Y^2 + P_Z^2 = 0,\ \mathbf{M}\cdot\mathbf{M} \equiv M_{YX}^2 + M_{XZ}^2 + M_{ZY}^2 = \frac{1}{4} - D^2.$$

Every R-separable solution set is characterized by a pair of 2nd-order commuting conformal symmetries. For ellipsoidal coordinates ($R = 0$) the operators can be chosen as $\mathbf{M}\cdot\mathbf{M} + (a-1)P_Y^2 + aP_Z^2$, $M_{XZ}^2 + aM_{YZ}^2 - aP_Z^2$, whereas for conical coordinates ($R = 0$) they are $\mathbf{M}\cdot\mathbf{M}$, $M_{XZ}^2 + aM_{YZ}^2$. The complicated characterizations suggest what is true, that the Lamé functions associated with these separable systems have no simple tranformation properties under the symmetry algebra, [8].

Computations involving separable solutions of the Laplace equation are simplified by making use of a 2-variable model for the solution space: we represent solutions $\Psi(X,Y,Z)$ in an integral form

$$\Psi(X,Y,Z) = \int_{C_1} d\beta \int_{\tilde{C}_2} d\varphi h[\beta,\varphi] \exp\left[\beta(iX\cos\varphi + iY\sin\varphi - Z)\right] \equiv I(h),$$

where h is analytic on a complex domain that contains the integration contours $C_1 \times C_2$ and is chosen such that $I(h)$ converges absolutely, and arbitrary differentiation with respect to X,Y,Z is permitted under the integral sign. For each h, $\Psi = I(h)$ is a solution of the Laplace equation and the action of the conformal symmetries on the solution space corresponds to the operators

$$\mathscr{P}_X = i\beta w_1,\ \mathscr{P}_Y = i\beta w_2,\ \mathscr{P}_Z = -\beta,\ D = \beta\partial_\beta + \frac{1}{2},\ \mathscr{M}_{XY} = -w_2\partial_{w_1},$$

$$\mathscr{M}_{ZX} = iw_1\beta\partial_\beta + iw_2^2\partial_{w_1},\ \mathscr{M}_{ZY} = iw_2\beta\partial_\beta - iw_1w_2\partial_{w_1}$$

where $w_1^2 + w_2^2 = 1$. (Indeed, $w_1 = \cos\varphi,\ w_2 = \sin\varphi$.) We shall not make use of the symmetries K.

Let us find an integral representation for solutions Ψ that are eigenfunctions of the dilation operator D with eigenvalue $-\ell - \frac{1}{2}$. We choose C_1 and C_2 as unit circles in the β and $t = e^{i\varphi}$ complex planes, respectively, and require ℓ to be a non-negative integer. Setting $\mathscr{D}h = (-\ell - \frac{1}{2})h$ we find $h(\beta,t) = \beta^{-\ell-1}j(t),\ j(t) = \sum_{m=-\ell}^{\ell} a_m t^m$. Then we evaluate the β integral by residues to obtain

$$\Psi(X,Y,Z) = I(h) = \int_0^{2\pi} [X\cos\varphi + Y\sin\varphi + iZ]^\ell\, j(e^{i\varphi})d\varphi.$$

Since $\mathbf{M}\cdot\mathbf{M} = \frac{1}{4} - D^2$ we have $\mathbf{M}\cdot\mathbf{M}\Psi = -\ell(\ell+1)\Psi$. For $j(t) = t^m$, $-\ell \leq m \leq \ell$ we have $M^0\Psi = m\Psi$ so Ψ must be a multiple of the solid harmonic $R^\ell Y_\ell^m(\theta,\phi)$, expressed in spherical coordinates. This model has an obvious extension to the Laplace equation in n dimensions.

3. Niven operators

Niven constructed an operator that maps harmonic functions, i.e., solutions of the $n = 3$ Laplace equation that are homogeneous of degree ℓ in X,Y,Z, into ellipsoidal solutions. Indeed, it maps a conical coordinate solution to an ellipsoidal solution, and is an infinite-order differential operator. A detailed technical construction is given in [9]. Here we give a much simpler treatment. Our theory extends to cover Niven operators in n dimensions and for many new coordinate systems.

Let H_ℓ be the space of solutions of the Laplace equation, homogeneous of degree ℓ. There is an operator F_ℓ, the Niven operator, such that relations

$$\begin{aligned} \left(\mathbf{M}\cdot\mathbf{M}+(a-1)P_Y^2+aP_Z^2\right)F_\ell &= F_\ell(\mathbf{M}\cdot\mathbf{M}) \\ \left(M_{XZ}^2+aM_{YZ}^2+aP_Z^2\right)F_\ell &= F_\ell(M_{XZ}^2+aM_{YZ}^2), \end{aligned} \tag{1}$$

$$F_\ell = {}_0F_1\left(\begin{matrix} - \\ -\ell-1/2 \end{matrix}; \frac{1}{4}((a-1)P_Y^2+aP_Z^2)\right)$$

hold on H_ℓ. Thus F_ℓ is an intertwining operator on H_ℓ beween the spaces of separated conical solutions and of separated ellipsoidal solutions, each expressible in terms of Lamé functions [9, 8]. We verify (1) using the model, on the space $\mathscr{H}_\ell$ of functions $h(\beta,t) = \beta^{-\ell-1}j(t)$. Setting $t = e^{i\varphi}$ we have

$$(a-1)\mathscr{P}_Y^2+a\mathscr{P}_Z^2 = \beta^2(a\sin^2\varphi+\cos^2\varphi).$$

Set $\mathscr{F}_\ell = \mathscr{F}_\ell(x)$, where $x = \beta^2(a\sin^2\varphi+\cos^2\varphi)$. Thus on $\mathscr{H}_\ell$ the Niven operator is just multiplication by an ordinary analytic function of x. The first equation (1) on $\mathscr{H}_\ell$ then reduces to a second-order ODE for $\mathscr{F}_\ell$:

$$4x\mathscr{F}_\ell''+(-4\ell+2)\mathscr{F}_\ell'-\mathscr{F}_\ell = 0.$$

The solution bounded at 0 is $\mathscr{F}_\ell = {}_0F_1(-\ell-1/2;x/4)$. Transferring this operator over to the solution space via $F_\ell\Psi = I(\mathscr{F}_\ell h)$ we obtain the required result. Note that a solution of the intertwining equations is given by $\Psi_E = I(h)$ where $h = \mathscr{F}_\ell\beta^{-\ell-1}j(t)$. Here, $j(t)$ satisfies the eigenvalue equation for Lamé functions, [8], $(\mathscr{M}_{XZ}^2+a\mathscr{M}_{YZ}^2)j = \lambda_1 j$ for operators $\mathscr{M}$ on the space of functions corresponding to homogeneity of degree ℓ, exactly the same equation as satisfied by the conical coordinate eigenfunctions. Thus we obtain the classical result [9] that if $L_\ell^m(\alpha)$ are Lamé polynomials then

$$L_\ell^m(\alpha)L_\ell^m(\beta)L_\ell^m(\gamma) = c\int_{-2K}^{2K}P_\ell(\mu)L_\ell^m(\delta)d\delta,$$

$$\mu = k^2\mathrm{sn}\alpha\ \mathrm{sn}\beta\ \mathrm{sn}\gamma\,\mathrm{sn}\delta-(k^2/k'^2)\mathrm{cn}\alpha\ \mathrm{cn}\beta\ \mathrm{cn}\gamma\,\mathrm{cn}\delta-(1/k'^2)\mathrm{dn}\alpha\ \mathrm{dn}\beta\ \mathrm{dn}\gamma\,\mathrm{dn}\delta$$

where $P_\ell(z)$ is a Legendre polynomial.

To clarify the mechanism behind the Niven operator construction we consider the case of general ellipsoidal coordinates for the $N = 4$ Laplace equation. General ellipsoidal separable solutions for the Laplace equaton are characterized by the commuting operators

$$\begin{aligned} \Gamma_1' &= \mathbf{M}\cdot\mathbf{M}+\left((a+b+ab)P_X^2+(a+b)P_Y^2+(1+b)P_Z^2+(1+a)P_T^2\right), \\ \Gamma_2' &= (a+b)M_{XY}^2+(1+b)M_{XZ}^2+(1+a)M_{XT}^2+bM_{YZ}^2+aM_{YT}^2+M_{ZT}^2 \\ &+ \left((1+a+b)P_X^2+abP_Y^2+bP_Z^2+aP_T^2\right), \\ \Gamma_3' &= \frac{1}{2}\left(abM_{XY}^2+bM_{XZ}^2+aM_{XT}^2\right)+\frac{1}{6}\left(abP_X^2\right). \end{aligned} \tag{2}$$

Here, we have chosen the parameters for the ellipsoidal coordinates as

$$(e_1, e_2, e_3, e_4) = (0, 1, a, b),$$

where $1 < a < b$. The associated conical cordinates are characterized by the commuting operators

$$\begin{aligned}
\Gamma_1 &= \mathbf{M} \cdot \mathbf{M} \\
\Gamma_2 &= (a+b)M_{XY}^2 + (1+b)M_{XZ}^2 + (1+a)M_{XT}^2 + bM_{YZ}^2 + aM_{YT}^2 + M_{ZT}^2, \\
\Gamma_3 &= \frac{1}{2}\left(abM_{XY}^2 + bM_{XZ}^2 + aM_{XT}^2\right).
\end{aligned} \tag{3}$$

Note that $\Gamma'_j = \Gamma_j + \Phi_j$, $j = 1, 2, 3$ where the Φ_j are linear combinations of squares of linear momentum operators. Thus,

$$[\Gamma'_i, \Gamma'_j] = [\Gamma_i, \Gamma_j] = [\Phi_i, \Phi_j] = 0, \quad i, j = 1, 2, 3.$$

It follows that the commutivity of the Γ'_j implies the important commutation relations

$$[\Gamma_i, \Phi_j] + [\Phi_i, \Gamma_j] = 0, \quad 1 \leq i, j \leq 3. \tag{4}$$

Theorem: Let H_ℓ be the space of solutions of the 4D Laplace equation that are homogeneous of degree ℓ. There exists a Niven operator F_ℓ, such that the identities

$$\Gamma'_j F_\ell = F_\ell \Gamma_j, \quad j = 1, 2, 3.$$

hold on H_ℓ. The operator can be chosen in the form

$$F_\ell = {}_0F_1\left(-\ell - 1; \frac{1}{4}((1+a+b)P_X^2 + (a+b)P_Y^2 + (1+b)P_Z^2 + (1+a)P_T^2)\right).$$

The theorem is proved by employing the 3 variable model. These results extend to all dimensions and to other separable systems.

4. Product formulas

Product formulas for Lamé and Heun functions are obtained by comparing the expression of an ellipsoidal or conical coordinate separable solution as a product of Lamé (or Heun) functions to the integral transform of a reduced product of such functions obtained from a lower variable model. Here is a typical example. Consider solutions of the 4D Laplace equation in the variables X, Y, Z, and T. We choose new coordinates

$$X + iY = R\sqrt{\frac{uv}{a}} e^{i\varphi}, \quad Z = R\sqrt{\frac{(u-1)(v-1)}{1-a}}, \quad T = R\sqrt{\frac{(u-a)(v-a)}{a(a-1)}}.$$

In these coordinates the Laplace equation becomes

$$(\partial_R^2 + \frac{3}{R}\partial_R + \frac{1}{R^2}\{\frac{-4}{u-v}[\sqrt{\frac{P(u)}{u}}\partial_u(\sqrt{uP(u)}\partial_u) - \sqrt{\frac{P(v)}{v}}\partial_v(\sqrt{vP(v)}\partial_v)) + \frac{a}{uv}\partial_\varphi^2]\})\Psi = 0.$$

We look for separable solutions $\Psi = R^\ell U(u)V(v)e^{p\varphi}$ with $u = \mathrm{sn}^2(\mu,k)$ and take $a = \frac{1}{k^2}$, and with $U(u) = (\mathrm{sn}(\mu,k))^{-1/2}\hat{U}(\mu)$. The ODE satisfied by $\hat{U}$ is the Heun equation [8]

$$(\partial_\mu^2 + \frac{\frac{1}{4}+p^2}{\mathrm{sn}^2(\mu,k)} - k^2(\ell+\frac{1}{2})(\ell+\frac{3}{2})\mathrm{sn}^2(\mu,k) + \frac{1}{4}(1+k^2) + k^2\lambda)\hat{U}(\mu) = 0. \qquad (5)$$

An identical equation is satisfied by $\hat{V}(v)$ The operator characterising separation is [5]

$$\Lambda = (a+1)M_{XY}^2 + a(M_{XZ}^2 + M_{YZ}^2) + M_{XT}^2 + M_{YT}^2.$$

We can realise any homogeneous solution of degree ℓ as an integral transform from the model:

$$\Psi = \int\int (T + iX\sin\theta\cos\varphi' + iY\sin\theta\sin\varphi' + iZ\cos\theta)^\ell f(\theta,\varphi')d\theta d\varphi'.$$

We now seek eigenfunctions of Λ with eigenvalue λ in the model. If we look for solutions of the form $f(\theta,\varphi) = W(w)e^{p\varphi}$ where $\sin\theta = \mathrm{sn}(w,k)$ then with $W(w) = \mathrm{dn}(w,k)^{-\ell-3/2}\mathrm{cn}(w,k)^{1/2}\Omega(\omega)$ where $w = \omega + K$, we see that $\Omega(\omega)$ satisfies the (Heun) differential equation (5) with $\mu = \omega$, which is the exact same equation that arises from the separation of the Laplace equation. The product formula takes the form

$$(\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k))^{-1/2}M(\mu,k)M(\nu,k)e^{p\varphi} = c\int\int[-ik'\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{cn}(\omega',k)\cos(\varphi-\varphi')$$

$$-k\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{sn}(\omega',k) + \frac{i}{k'}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega',k)]^\ell \mathrm{sn}(\omega',k)^{1/2}M(\omega',k)e^{p\varphi'}d\omega' d\varphi',$$

Where the Heun function $M(z,k)$ is a solution of the ODE above. The φ' integration of this product formula could in principle be calculated. More results and detailed proofs will appear in forthcoming papers.

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