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CONFORMAL KILLING TENSORS AND VARIABLE SEPARATION FOR HAMILTON–JACOBI EQUATIONS*

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Abstract. Every separable coordinate system for the Hamilton–Jacobi equation $g^{ij}W_iW_j=0$ corresponds to a family of $n-1$ conformal Killing tensors in involution, but the converse is false. For general n we find a practical characterization of those families of conformal Killing tensors that correspond to variable separation, orthogonal or not.

1. Introduction. This paper is devoted to the separation of variables problem for the Hamilton–Jacobi equation

$$g^{ij}\partial_{x^i}W\partial W_{x^j}=0, \qquad g^{ij}=g^{ji}, \qquad 1\le i,j\le n \tag{1.1}$$

and the explicit relation between variable separation and second order conformal Killing tensors on the (local) manifold V_n with metric tensor $\{g_{ij}\}$ analytic in the local coordinates $\{x^i\}$. (Here all coordinates and tensors are complex valued and we adopt the notation in Eisenhart's book [1].) Equation (1.1) is intimately related to the separation of variables problem for the Laplace or wave equation,

$$\frac{1}{\sqrt{g}}\partial_{x^i}\left(\sqrt{g}\,g^{ij}\partial_{x^j}\psi\right)=0, \qquad g=\det(g_{ij}). \tag{1.2}$$

It is straightforward to show that any coordinate system yielding (product) R-separation of (1.2) also yields (additive) separation of (1.1). (We have also shown for flat space and $n=3,4$ that the converse holds, i.e., the two equations separate in exactly the same coordinate systems, orthogonal or not [2], [3].)

In 1891 Stäckel [4] showed that (1.1) is additively separable in the orthogonal coordinate system $\{x^i\}$ if and only if there exists a nonzero function $Q(x^j)$ such that the metric $d\hat{s}^2$ where

$$ds^2=g_{ij}dx^i dx^j=H_j^2(dx^j)^2=Qh_j^2(dx^j)^2=Q\,d\hat{s}^2 \tag{1.3}$$

can be expressed in *Stäckel form*:

$$h_i^2=\frac{\Theta}{\Theta^{i1}}, \qquad 1=1,\cdots,n \tag{1.4}$$

where Θ is a Stäckel determinant, $\Theta=\det(\theta_{kl})$, $(\theta_{kl}(x^k))$ is a Stäckel matrix (row k depends only on the variable x^k), and Θ^{i1} is the $(i,1)$-cofactor of this matrix. Thus the condition for additive separation of (1.1) in coordinates $\{x^j\}$ is that ds^2 is conformal to a metric $d\hat{s}^2$ in Stäckel form. Separable solutions of (1.1) take the form $W=\sum_{i=1}^n B_i(x^i)$.

Moon and Spencer [5] show that (1.2) admits orthogonal R-separable solutions, i.e., solutions of the form $\psi=e^R\prod_{i=1}^n A_i(x^i)$ where R is a fixed function, if and only if (1) ds^2 is conformal (with factor Q^{-1}) to a Stäckel form metric $d\hat{s}^2$, (2) that

$$\frac{Q\mathcal{H}e^{2R}}{\Theta}=\prod_{i=1}^n f_i(x^i), \qquad \mathcal{H}=H_1H_2,\cdots,H_n, \tag{1.5}$$

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and (3) that e^R satisfy

$$\sum_{j=1}^{n} \frac{\Theta^{j1}}{\Theta f_j} \partial_{x^j}\left(f_j \partial_{x^j} e^{-R} \right) + \alpha e^{-R} = 0 \tag{1.6}$$

where α is a constant. (See, however, [6] for a discussion of condition (1.6).) In practice, to determine the orthogonal R-separable coordinate systems for the Laplace equation on the manifold V_n, one first finds all orthogonal separable systems for the Hamilton–Jacobi equation (1.1) and then determines for each system whether or not conditions (1.5) and (1.6) can be satisfied. For nonorthogonal coordinates the relationship is similar but more complicated, see [2], [3].

The relation between variable separation for (1.1) and conformal Killing tensors on V_n is most conveniently presented in terms of the symplectic structure on the cotangent bundle $\tilde{V}_n$ of this manifold. Corresponding to local coordinates $\{x^j\}$ on V_n we introduce coordinates $\{x^j, p_j\}$ on $\tilde{V}_n$. New coordinates $\{\hat{x}^k(x^l)\}$ on V_n correspond to coordinates $\{\hat{x}^k, \hat{p}_k\}$ on $\tilde{V}_n$ where $\hat{p}_k = p_l \partial x^l / \partial \hat{x}^k$. The *Poisson bracket* of two functions $F(x^j, p_j)$, $G(x^j, p_j)$ on $\tilde{V}_n$ is given by

$$[F, G] = \partial_{x^l} F \partial_{p_l} G - \partial_{p_l} F \partial_{x^l} G. \tag{1.7}$$

Let

$$H = g^{ij} p_i p_j. \tag{1.8}$$

A *first order symmetry* of (1.1) is a linear function L in the momenta p_j,

$$L = \xi^j(x) p_j \tag{1.9}$$

such that

$$[L, H] = \rho(x) H \tag{1.10}$$

for some analytic function ρ. Clearly, L is a symmetry if and only if $\{\xi^j\}$ is a conformal Killing vector for V_n [1]. Indeed it is straightforward to show that (1.10) is equivalent to

$$\xi_{i,j} + \xi_{j,i} = \rho g_{ij} \tag{1.11}$$

where $\xi_{i,j}$ is the jth covariant derivative of ξ_i. Similarly a *second order symmetry* of (1.1) is a quadratic function

$$A = a^{ij}(x) p_i p_j, \qquad a^{ij} = a^{ji}, \tag{1.12}$$

such that

$$[A, H] = \left(Q^l(x) p_l \right) H \tag{1.13}$$

where the $\{Q^l\}$ are analytic. Condition (1.13) is equivalent to

$$a_{ij,k} + a_{ki,j} + a_{jk,i} = \frac{1}{2}\left(Q_i g_{jk} + Q_k g_{ij} + Q_j g_{ki} \right), \tag{1.14}$$

i.e., $\{a^{ij}\}$ (or $\{a_{ij}\}$) is a *conformal Killing tensor of order* 2. It is obvious that $\rho(x)H$ is a (trivial) conformal Killing tensor for any analytic function ρ. Thus by addition of multiples ρH of H if necessary, one could assume that every nontrivial conformal Killing tensor is traceless, $a^l_l = 0$. (We shall ordinarily not make this assumption.) Note that then the Q_i can be expressed simply in terms of the components of a traceless A: $Q_i = (4/(n+2)) a^l_{1,l}$. For future use we also note that the condition for two quadratic functions A and $B = b^{ij} p_i p_j$ to be in involution, i.e., $[A, B] = 0$, is

$$a_{ij,l} b^l_k + a_{ki,l} b^l_j + a_{jk,l} b^l_i = b_{ij,l} a^l_k + b_{ki,l} a^l_j + b_{jk,l} a^l_i. \tag{1.15}$$

We can now state the basic relation between separation of variables for (1.1) and conformal Killing tensors: To every orthogonal coordinate system $\{y^i\}$ which permits additive separation of variables in (1.1), there correspond $n-1$ second order conformal Killing tensors $A_1,\cdots,A_{n-1}$ which are in involution and such that $\{H,A_1,\cdots,A_{n-1}\}$ is linearly independent. The separable solutions $W=\Sigma_{k=1}^n W^{(k)}(y^k)$ are characterized by the relations

$$(1.16)\qquad H(y^j,p_j)=0,\qquad A_l(y^j,p_j)=\lambda_l,\qquad l=1,\cdots,n-1,\quad p_j=\partial_{y^j}W,$$

where $\lambda_1,\cdots,\lambda_{n-1}$ are the separation constants. The basis tensors A_l are of course not unique, but the space spanned by these tensors is uniquely determined. (A new proof of this correspondence is contained in Theorems 4 and 7 to follow. Expressions (1.16) are then obvious from Stäckel's construction.) For nonorthogonal separable coordinates, the same characterization is valid except that one or more of the A_l are conformal Killing vectors. For $n\le 4$ all possible separable systems and their corresponding conformal Killing tensors have been explicitly determined [2], [3].

A remaining problem with the theory is that there exist involutive families of $n-1$ conformal Killing tensors that are not related to any separable coordinate system. In this paper we give a complete solution to this problem. That is, we provide directly verifiable necessary and sufficient conditions for a family of conformal Killing tensors to determine a separable coordinate system for (1.1), and we show how to compute the separable coordinates from the given tensors. In §2 we study the case of orthogonal separable coordinates where the Killing tensor characterization is especially simple. Finally, in §3 we treat the general nonorthogonal case. The results of this paper are a nontrivial extension of the results in [7], [8] for the equation $g^{ij}\partial_i W\partial_j W=E$ with $E\neq 0$.

2. The orthogonal case. Let $\{x^j\}$ be a local orthogonal coordinate system on V_n and let $ds^2=g_{ij}dx^i dx^j=H_j^2(dx^j)^2$ be the metric for V_n as expressed in these coordinates. It follows from (1.3) and (1.4) that the Hamilton–Jacobi equation (1.1) is separable in the $\{x^i\}$ if and only if there exists an analytic function $Q(x)$ such that $H_j=Qh_j$ where the metric $d\hat{s}^2=h_j^2(dx^j)^2$ is in Stäckel form. We begin our study of such "conformally Stäckel" metrics by deriving a more convenient characterization for them.

It is well known that the metric $d\hat{s}^2$ is in Stäckel form with respect to the coordinates $\{x^j\}$ if and only if the conditions

$$(2.1)\qquad \frac{\partial^2\ln h_i^2}{\partial x^j\partial x^k}-\frac{\partial\ln h_i^2}{\partial x^j}\frac{\partial\ln h_i^2}{\partial x^k}+\frac{\partial\ln h_i^2}{\partial x^j}\frac{\partial\ln h_j^2}{\partial x^k}+\frac{\partial\ln h_i^2}{\partial x^k}\frac{\partial\ln h_k^2}{\partial x^j}=0,\qquad j\neq k,$$

are satisfied [1, App. 13]. Let $d\tilde{s}^2=K_j^2(dx^j)^2$ where $K_j^2=h_j^2/h_n^2$; in particular $K_n^2=1$. A straightforward computation using (2.1) yields

LEMMA 1. *If the metric* $d\hat{s}^2=h_j^2(dx^j)^2$ *is in Stäckel form then so is the metric* $d\tilde{s}^2=h_n^{-2}d\hat{s}^2$.

Now let $ds^2=H_j^2(dx^j)^2=Q\,d\hat{s}^2$. If $d\hat{s}^2$ is in Stäckel form, then by Lemma 1 the metric $H_n^{-2}ds^2=h_n^{-2}d\hat{s}^2$ is also in Stäckel form. Conversely, if $H_n^{-2}ds^2$ is in Stäckel form then $ds^2=H_n^2(H_n^{-2}ds^2)$ is conformal to a Stäckel form metric. This proves

LEMMA 2. $ds^2=H_j^2(dx^j)^2$ *is conformal to a Stäckel form metric if and only if the coefficients* H_j^2 *satisfy the conditions*

$$(2.2)\qquad \frac{\partial^2\ln K_i^2}{\partial x^j\partial x^k}-\frac{\partial\ln K_i^2}{\partial x^j}\frac{\partial\ln K_i^2}{\partial x^k}+\frac{\partial\ln K_i^2}{\partial x^j}\frac{\partial\ln K_j^2}{\partial x^k}+\frac{\partial\ln K_i^2}{\partial x^k}\frac{\partial\ln K_k^2}{\partial x^j}=0,\qquad j\neq k,$$

where $K_j^2=H_j^2/H_n^2$.

Note that for $i=n$, equations (2.2) are satisfied identically and for $k=n$ they read

$$\frac{\partial^2 \ln K_i^2}{\partial x^j \partial x^n}+\frac{\partial \ln K_i^2}{\partial x^j}\frac{\partial \ln\left(K_j^2/K_i^2\right)}{\partial x^n}=0, \qquad j\neq n. \tag{2.3}$$

THEOREM 1. *Let A be a second order conformal Killing tensor such that the n roots $\rho_1(x),\cdots,\rho_n(x)$ of the characteristic equation*

$$\det(a_{ij}-\rho g_{ij})=0 \tag{2.4}$$

are pairwise distinct. Furthermore, suppose the eigenvector fields corresponding to these n roots are normalizable, i.e., there exists a coordinate system $\{y^j\}$ on V_n such that

$$ds^2=g_{ij}dx^i dx^j=H_j^2\left(dy^j\right)^2, \qquad \psi=a_{ij}dx^i dx^j=\rho_j H_j^2\left(dy^j\right)^2. \tag{2.5}$$

Then the Hamilton–Jacobi equation (1.1) *is separable in the coordinates* $\{y^j\}$.

Proof. Conditions (1.14) for A are equivalent to

$$\partial_{y^l}\ln\left(\frac{\rho_l-\rho_k}{H_k^2}\right)=0, \qquad l\neq k. \tag{2.6}$$

Setting $\mu_\alpha=\rho_\alpha-\rho_n$, $\alpha=1,\cdots,n-1$, we see that these equations can be written in the form

$$\begin{aligned}
&\text{a)} && \partial_{y^\alpha}\ln\left(\frac{\mu_\alpha}{H_n^2}\right)=0, \qquad \alpha=1,\cdots,n-1,\\
&\text{(2.7) b)} && \partial_{y^n}\ln\left(\frac{\mu_\alpha}{H_\alpha^2}\right)=0,\\
&\text{c)} && \partial_{y^\alpha}\ln\left(\frac{\mu_\alpha-\mu_\beta}{H_\beta^2}\right)=0, \qquad 1\leq\alpha,\beta\leq n-1, \quad \alpha\neq\beta,
\end{aligned}$$

or

$$\begin{aligned}
&\partial_\alpha\mu_\beta=(\mu_\beta-\mu_\alpha)\partial_\alpha\ln\left(H_\beta^2\right)+\mu_\alpha\partial_\alpha\ln H_n^2, \qquad \alpha\neq\beta,\\
&\partial_\alpha\mu_\alpha=\mu_\alpha\partial_\alpha\ln\left(H_n^2\right),\\
&\partial_n\mu_\alpha=\mu_\alpha\partial_n\ln\left(H_\alpha^2\right).
\end{aligned} \tag{2.8}$$

The integrability conditions $\partial_i\partial_j\mu_\alpha=\partial_j\partial_i\mu_\alpha$ for the system (2.8) can be written in the form

$$\begin{aligned}
&(\mu_\rho-\mu_\alpha)\left[\partial_{\alpha\beta}\ln\left(\frac{H_\beta^2}{H_n^2}\right)+\partial_\alpha\ln\left(\frac{H_\beta^2}{H_n^2}\right)\partial_\beta\ln\left(\frac{H_\alpha^2}{H_n^2}\right)\right]=0, \qquad \alpha\neq\beta,\\
&(\mu_\gamma-\mu_\alpha)\left[\partial_{\alpha\gamma}\ln\left(\frac{H_\beta^2}{H_n^2}\right)-\partial_\alpha\ln\left(\frac{H_\beta^2}{H_n^2}\right)\partial_\gamma\ln\left(\frac{H_\beta^2}{H_n^2}\right)\right.\\
&\qquad\left.+\partial_\alpha\ln\left(\frac{H_\beta^2}{H_n^2}\right)\partial_\gamma\ln\left(\frac{H_\alpha^2}{H_n^2}\right)+\partial_\alpha\ln\left(\frac{H_\gamma^2}{H_n^2}\right)\partial_\gamma\ln\left(\frac{H_\beta^2}{H_n^2}\right)\right]=0,\\
&\qquad\qquad \alpha,\beta,\gamma \text{ pairwise distinct},\\
&(\mu_\beta-\mu_\alpha)\left[\partial_{\alpha n}\ln\left(\frac{H_\beta^2}{H_n^2}\right)-\partial_\alpha\ln\left(\frac{H_\beta^2}{H_n^2}\right)\partial_n\left(\frac{H_\beta^2}{H_n^2}\right)\right.\\
&\qquad\left.+\partial_\alpha\ln\left(\frac{H_\beta^2}{H_n^2}\right)\partial_n\ln\left(\frac{H_\alpha^2}{H_n^2}\right)\right]=0, \qquad \alpha\neq\beta.
\end{aligned} \tag{2.9}$$

Since the ρ_i are pairwise distinct by assumption, we have $\mu_\beta - \mu_\alpha \neq 0$ for $\alpha \neq \beta$, so conditions (2.9) become

$$(2.10) \qquad \frac{\partial^2 \ln K_i^2}{\partial y^j \partial y^k} - \frac{\partial \ln K_i^2}{\partial y^j}\frac{\partial \ln K_i^2}{\partial y^k} + \frac{\partial \ln K_i^2}{\partial y^j}\frac{\partial \ln K_j^2}{\partial y^k} + \frac{\partial \ln K_i^2}{\partial y^k}\frac{\partial \ln K_k^2}{\partial y^j} = 0, \qquad j \neq k,$$

where $K_i^2 = H_i^2/H_n^2$, $i = 1, \cdots, n$. It follows from Lemma 2 that $H_j^2(dy^j)^2$ is conformal to a Stäckel metric, hence (1.1) separates in the coordinates $\{y^j\}$. Q.E.D.

Note that if (1.1) is separable in the coordinates $\{y^j\}$, then equations (2.10) hold and the integrability conditions for the system (2.8) are satisfied identically. Thus (2.8) admits a basis of $n-1$ vector solutions $\{\rho_j^{(\beta)}\}$, $\beta = 1, \cdots, n-1$. This proves

THEOREM 2. *Necessary and sufficient conditions that the metric* $ds^2 = g_{ij}dx^i dx^j = H_j^2(dy^j)^2$ *on* V_n *is conformal to a Stäckel form metric with respect to the coordinates* $\{y^j\}$ *are*:

1) *The space admits* $n-1$ *conformal Killing tensors* $a_{ij}^{(\beta)}$, $\beta = 1, \cdots, n-1$ *such that the n tensors* $\{g_{ij}, a_{ij}^{(\beta)}\}$ *form a linearly independent set at each point* x.

2) *The roots* $\rho^{(\beta)}$ *for each of the characteristic equations* $\det(a_{ij}^{(\beta)} - \rho^{(\beta)}g_{ij}) = 0$ *are simple.*

3)

$$(2.11) \qquad \left(a_{ij}^{(\beta)} - \rho_h^{(\beta)} g_{ij}\right)\lambda^i_{(h)} = 0, \qquad h = 1, \cdots, n, \quad \beta = 1, \cdots, n-1,$$

where $\rho_1^{(\beta)}, \cdots, \rho_n^{(\beta)}$ *are the roots of* $a_{ij}^{(\beta)}$ *and* $\lambda^i_{(h)} = \partial x^i / \partial y^h$.

Note that condition 3) requires the vector fields $\lambda^i_{(1)}, \cdots, \lambda^i_{(n)}$ to be normal and to satisfy equations (2.11) for all β. Theorem 2 and its proof are patterned after the corresponding theorem due to Eisenhart which relates Killing tensors and (true) Stäckel forms [1], [9]. The theorem is not very useful in a practical sense because of the difficulty in deciding when the vector fields $\{\lambda^i_{(h)}\}$ defined by (2.11) are normalizable, i.e., when there exists an orthogonal coordinate system $\{y^i\}$ such that $\{\lambda^i_{(h)}\}$ is orthogonal to the coordinate surface $y^h = \text{const}$, for each $h = 1, \cdots, n$.

To solve this problem we recall some classical results in differential geometry that can be found in Eisenhart's book [1]. Given a family of orthogonal vector fields $\{\lambda^i_{(h)}(x), 1 \leq h \leq n\}$ we define their *coefficients of rotation* γ_{lhk} by

$$(2.12) \qquad \gamma_{lhk} = \lambda_{(l)i,j}\lambda^i_{(h)}\lambda^j_{(k)}, \qquad 1 < l, h, k \leq n;$$

see [1, p. 97]. A necessary and sufficient condition that there exist coordinates $\{y^h\}$ and nonzero invariant functions f_h such that $\lambda^i_{(h)} = (\partial x^i/\partial y^h) f_h$, $h = 1, \cdots, n$, is

$$(2.13) \qquad \gamma_{lhk} = 0, \qquad 1 \leq l, h, k \leq n, \quad h, k, l \text{ pairwise distinct.}$$

Let a_{ij} be a tensor field with n roots $\rho_1, \cdots, \rho_n$ (not necessarily distinct) and let $\{\lambda^i_{(h)}\}$ be a corresponding orthonormal set of eigenvectors:

$$(2.14) \qquad (a_{ij} - \rho_h g_{ij})\lambda^i_{(h)} = 0, \qquad h = 1, \cdots, n,$$

$$(2.15) \qquad \lambda^i_{(h)}\lambda_{(k)i} = \delta_{hk}, \qquad 1 \leq h, k \leq n.$$

It follows easily from (2.12), (2.14) and (2.15) that

$$(2.16) \qquad a_{ij,k}\lambda^i_{(h)}\lambda^j_{(l)}\lambda^k_{(m)} = (\rho_h - \rho_l)\gamma_{hlm}, \qquad h \neq l.$$

From (2.13) we find

THEOREM 3 (Eisenhart [1, p. 118]). *If* a_{ij} *has pairwise distinct roots* $\rho_1, \cdots, \rho_n$ *then the vector fields* $\{\lambda^i_{(h)}\}$ *are normalizable if and only if*

$$(2.17) \qquad a_{ij,k}\lambda^i_{(h)}\lambda^j_{(l)}\lambda^k_{(m)} = 0, \qquad i \leq h, l, m \leq n, \quad h, l, m \text{ distinct.}$$

This leads us to our fundamental result:

THEOREM 4. *Necessary and sufficient conditions that the orthogonal coordinate system* $\{y^j\}$ *be separable for the Hamilton–Jacobi equation* (1.1) *are the existence of* $n-1$ *quadratic functions* $A^{(\beta)}$, $\beta=1,\cdots,n-1$, (1.12), *such that*:

1) *The* $\{A^{(\beta)}\}$ *are second order symmetries of* (1.1), *i.e., the* $\{a_{ij}^{(\beta)}\}$ *are conformal Killing tensors.*

2) *The* $\{A^{(\beta)}\}$ *are in involution*: $[A^{(\alpha)},A^{(\beta)}]=0$, $1\leq\alpha,\beta\leq n-1$.

3) *The set* $\{H,A^{(1)},\cdots,A^{(n-1)}\}$ *is linearly independent* (*as n quadratic forms at each point* **x**).

4) *At least one of the quadratic forms, say* $A^{(1)}$, *has pairwise distinct roots.*

5) *In any local coordinate system* $\{x^j\}$ *the quadratic forms satisfy the algebraic commutation property*

$$a_{ij}^{(\alpha)}a_k^{(\beta)j}=a_{ij}^{(\beta)}a_k^{(\alpha)j}. \tag{2.18}$$

(*This property is independent of local coordinates.*)

Proof. We suppose that conditions 1)–5) are satisfied. Conditions 4) and 5) imply that the quadratic forms can be simultaneously diagonalized by a family of orthonormal vector fields. In local coordinates $\{x^j\}$ we have

$$\left(a_{ij}^{(\beta)}-\rho_h^{(\beta)}g_{ij}\right)\lambda^i_{(h)}=0, \qquad h=1,\cdots,n, \quad \beta=1,\cdots,n-1, \tag{2.19}$$

where $\rho_1^{(\beta)},\cdots,\rho_n^{(\beta)}$ are the roots of $a_{ij}^{(\beta)}$ and $\lambda^i_{(h)}\lambda_{(k)i}=\delta_{hk}$. Setting $\rho_h^{(n)}=1$, for $h=1,\cdots,n$ we can express condition 3) as

$$\det\left(\rho_m^{(l)}\right)\neq 0. \tag{2.20}$$

Furthermore, by (1.14), (1.15), (2.16), and (2.19), conditions 1) and 2) imply

$$\det\begin{pmatrix}\rho_l^{(\alpha)} & \rho_h^{(\alpha)} & \rho_m^{(\alpha)}\\ 1 & 1 & 1\\ \gamma_{mhl} & \gamma_{lmh} & \gamma_{hlm}\end{pmatrix}=0, \qquad 1\leq\alpha\leq n-1, \quad h,l,m \text{ distinct}, \tag{2.21}$$

and

$$\det\begin{pmatrix}\rho_l^{(\alpha)} & \rho_h^{(\alpha)} & \rho_m^{(\alpha)}\\ \rho_l^{(\beta)} & \rho_h^{(\beta)} & \rho_m^{(\beta)}\\ \gamma_{hlm}+\gamma_{lmh} & \gamma_{hlm}+\gamma_{mhl} & \gamma_{mhl}+\gamma_{lmh}\end{pmatrix}=0, \qquad 1\leq\alpha<\beta\leq n-1. \tag{2.22}$$

From (2.20) and (2.21) we have $\gamma_{mhl}=\gamma_{lmh}=\gamma_{hlm}$. Substituting this result into (2.22) and using (2.20) we find $\gamma_{mhl}=\gamma_{lmh}=\gamma_{hlm}=0$. Thus, by (2.13) the vector fields $\{\lambda^j_{(h)}\}$ are normalizable. It then follows from Theorem 2 that the $\{A^{(\beta)}\}$ determine an orthogonal separable coordinate system $\{y^j\}$.

Conversely, given an orthogonal separable coordinate system $\{y^j\}$ for (1.1), we see from the definition of separability, (e.g., (3.5)), that $H=fH'$ for some function f where H' is in Stäckel form with respect to these coordinates. It follows from [7, Thm. 6], that there exist Killing tensors (with respect to H') $A_1,\cdots,A_{n-1}$ that satisfy properties 2)–5). It is obvious that the A_j are conformal Killing tensors for H. Q.E.D.

3. The general case. We now examine the separation for variables problem for (1.1) for the more general case in which the separable coordinates may be nonorthogonal. Our definition of variable separation is identical with that presented in [2], [3] and is based on a division of the separable coordinates into three classes: *ignorable*, *essential of type* 1 and *essential of type* 2. Let $\{x^j\}$ be a coordinate system on V_n with contravariant metric tensor (g^{ij}) and such that the first n_1 coordinates x^a are essential of type 1, the next n_2 coordinates x^r are essential of type 2, and the last n_3 coordinates x^α are ignorable, $n=n_1+n_2+n_3$. (In the following, indices a,b,c range from 1 to n, indices r,s,t range from n_1+1 to n_1+n_2, indices α,β,γ range from n_1+n_2+1 to n, and indices i,j,k range from 1 to n.) This means that in terms of the coordinates $\{x^j\}$ the metric satisfies $g^{ik}=Q\hat{g}^{ik}$ where $\partial_\alpha \hat{g}^{ik}=0$, $\alpha=n_1+n_2+1,\cdots,n$, and that the separation equations take the form

$$W_a^2+\sum_{\alpha,\beta} A_a^{\alpha,\beta}(x^a)W_\alpha W_\beta=\Phi_a(x^a,\boldsymbol{\lambda}), \tag{3.1}$$

$$2\sum_{\alpha} B_r^\alpha(x^r)W_r W_\alpha+\sum_{\alpha,\beta} C_r^{\alpha,\beta}(x^r)W_\alpha W_\beta=\Phi_r(x^r,\boldsymbol{\lambda}), \tag{3.2}$$

$$W_\alpha=\lambda_\alpha. \tag{3.3}$$

Here $A_a^{\alpha,\beta}(=A_a^{\beta,\alpha})$, $C_r^{\alpha,\beta}(=C_r^{\beta,\alpha})$ and Φ_i are defined and analytic in a neighborhood $N\subset C^{n_1+n_2}$ of some given point $(x_0^1,\cdots,x_0^{n_1+n_2})$. Furthermore,

$$\Phi_i(x^i,\boldsymbol{\lambda})=\sum_{j=2}^{n_1+n_2}\lambda_j\theta_{ij}(x^i), \qquad i=1,\cdots,n_1+n_2, \tag{3.4}$$

where the complex parameters $\lambda_1,\cdots,\lambda_n$ are arbitrary and the vectors $\partial_{\lambda_j}\tilde{\Phi}$, $j=2,\cdots,n_1+n_2$ are linearly independent for $\mathbf{x}\in N$.

We say that the coordinates $\{x^j\}$ are *separable* for the H–J equation

$$\sum g^{ij}\partial_i W\partial_j W=0 \tag{3.5}$$

if there exist analytic functions A,B,C,Φ above and functions $U_a(x^i)$, $V_r(x^i)$, analytic in N, such that (3.5) can be written in the form

$$\sum_a U_a\Phi_a+\sum_r V_r\Phi_r=0 \tag{3.6}$$

(identically in the parameters $\lambda_2,\cdots,\lambda_{n_1+n_2}$), where $W=\sum_{j=1}^n W^{(j)}(x^j)$, $W_i=\partial_i W=\partial_i W^{(i)}$.

The functions U_a, V_r are uniquely determined by (3.6) up to an arbitrary multiplicative factor $Q(\mathbf{x})$. To analyse the structure of these solutions it is convenient to introduce an $(n_1+n_2)\times(n_1+n_2)$ Stäckel matrix $(\theta_{ij}(x^i))$, $i,j=1,\cdots,n_1+n_2$ whose first column (not unique) is subject only to the condition $\Theta=\det(\theta_{ij})\neq 0$ and whose remaining columns are determined by (3.4). Then

$$U_a=\frac{Q\Theta^{a1}}{\Theta}, \qquad V_r=\frac{Q\Theta^{r1}}{\Theta} \tag{3.7}$$

where Θ^{lm} is the (lm)-cofactor of the matrix (θ_{ij}). The nonzero components of the contravariant metric tensor are thus

$$\begin{aligned}
&g^{ab}=\left(\frac{Q\Theta^{a1}}{\Theta}\right)\delta^{ab}, \qquad g^{r\alpha}=g^{\alpha r}=\left(\frac{Q\Theta^{r1}}{\Theta}\right)B_r^\alpha(x^r),\\
&\frac{1}{2}g^{\alpha\beta}=Q\left(\sum_a A_a^{\alpha,\beta}(x^a)\frac{\Theta^{a1}}{\Theta}+\sum_r C_r^{\alpha,\beta}(x^r)\frac{\Theta^{r1}}{\Theta}\right), \qquad \alpha\neq\beta,\\
&g^{\alpha\alpha}=Q\left(\sum_a A_a^{\alpha,\alpha}(x^a)\frac{\Theta^{a1}}{\Theta}+\sum_r C_r^{\alpha,\alpha}(x^r)\frac{\Theta^{r1}}{\Theta}\right).
\end{aligned} \tag{3.8}$$

Furthermore,

$$\sum_{l=1}^{n_1+n_2} \frac{\Theta^{lm}}{\Theta}\Phi_l = \begin{cases} 0 & \text{if } m=1, \\ \lambda_m & \text{otherwise,} \end{cases} \tag{3.9}$$

so,

$$\begin{aligned} &H(\mathbf{x},\mathbf{p}) \equiv g^{ij}p_i p_j = 0, \\ &A_m(\mathbf{x},\mathbf{p}) \equiv a^{ij}_{(m)} p_i p_j = \lambda_m, \qquad m = 2,\cdots,n_1+n_2, \\ &L_\alpha(\mathbf{x},\mathbf{p}) \equiv p_\alpha = \lambda_\alpha, \qquad p_i = \partial_{x^i} W \end{aligned} \tag{3.10}$$

where the nonzero terms of the symmetric quadratic form $(a^{ij}_{(m)})$ are

$$\begin{aligned} &a^{ab}_{(m)} = \left(\frac{\Theta^{am}}{\Theta}\right)\delta^{ab},\ a^{r\alpha}_{(m)} = \left(\frac{\Theta^{rm}}{\Theta}\right)B^\alpha_r, \\ &\frac{1}{2}a^{\alpha\beta}_{(m)} = \sum_c A^{\alpha,\beta}_c \frac{\Theta^{cm}}{\Theta} + \sum_r C^{\alpha,\beta}_r \frac{\Theta^{rm}}{\Theta}, \qquad \alpha \neq \beta, \\ &a^{\alpha\alpha}_{(m)} = \sum_c A^{\alpha,\alpha}_c \frac{\Theta^{cm}}{\Theta} + \sum_r C^{\alpha,\alpha}_r \frac{\Theta^{rm}}{\Theta}. \end{aligned} \tag{3.11}$$

It follows immediately from [8, Thm. 2] that

$$\begin{aligned} &\text{(a)} \qquad A_m, L_\alpha \ \text{ are conformal Killing tensors,} \\ &\text{(b)} \qquad [A_m A_l] = 0, \quad [A_m, L_\alpha] = 0, \quad [L_\alpha, L_\beta] = 0. \end{aligned} \tag{3.12}$$

Note that while relations (3.6) determine the coordinates and the metric in an essentially unique manner, there is some freedom of choice for the conformal Killing tensors A_m, due to the nonuniqueness of the first column in the Stäckel matrix. (This freedom is due to the fact that we may replace A_m by $A_m + f(\mathbf{x})H$ without altering relations (3.10).)

We shall now analyse the structure of these separation equations and their relationship to the commutation properties (3.12). First we derive practical, necessary and sufficient conditions to determine if a given coordinate system $\{x^j\}$ yields separation for the Hamilton–Jacobi equation (1.1). Let g^{ij} be the components of the contravariant metric tensor in these coordinates. It is convenient to reorder the coordinates in a standard form. Let n_3 be the number of ignorable variables x^α. Of the remaining $n-n_3$ variables, suppose n_2 variables x^r have the property $g^{rr}=0$ and the remaining n_1 variables x^a satisfy $g^{aa}\neq 0$. We relable the variables so that $1 \leq a \leq n_1$, $n_1+1 \leq n \leq n_1 + n_2$, and $n_1+n_2+1 \leq \alpha \leq n$.

THEOREM 5. *Suppose* (g^{ij}) *is in standard form with respect to the variables* $\{x^i\}$. *The Hamilton–Jacobi equation* (1.1) *is separable for this system if and only if*:

1) *The contravariant metric assumes the form*

$$(g^{ij}) = \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \left(\begin{array}{ccc} \delta^{ab}H_a^{-2} & 0 & 0 \\ 0 & 0 & H_r^{-2}B_r^\alpha \\ 0 & H_r^{-2}B_r^\alpha & g^{\alpha\beta} \end{array}\right) \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \end{array}$$

where $B^\alpha_r = B^\alpha_r(x^r)$.

2) *The metric* $d\tilde{s}^2=\sum_{a=1}^{n_1}H_a^2(dx^a)^2+\sum_{r=n_1+1}^{n_1+n_2}H_r^2(dx^r)^2$ *is conformal to a Stäckel form metric, i.e., relations* (2.2) *hold for* $K_j^2=H_j^2/H_{n_1+n_2}^2$.

3) *For each* $g^{\alpha\beta}(\mathbf{x})$, $g^{\alpha\beta}H_{n_1+n_2}^2$ *is a Stäckel multiplier for the metric* $d\tilde{s}^2/H_{n_1+n_2}^2$, *i.e.*,

$$\partial_{ij}g^{\alpha\beta}+\partial_i g^{\alpha\beta}\partial_j\ln H_i^2+\partial_j g^{\alpha\beta}\partial_i\ln H_j^2+g^{\alpha\beta}\left(\partial_{ij}\ln H_j^2+\partial_i\ln H_j^2\,\partial_j\ln H_i^2\right)=0.$$

Proof. This result follows directly from [8, Thm. 1] and Lemma 2.

THEOREM 6. *Let* (g^{ij}) *be the metric tensor on* V_n *in the coordinates* $\{x^i\}$. *If the Hamilton–Jacobi equation* (1.1) *is separable in these coordinates then there exist a function* $Q(\mathbf{x})$ *and a* κ*-dimensional vector space* $\mathcal{A}$ *of second order conformal Killing tensors on* V_n *such that*:

1) *Each* L_α *and* $A\in\mathcal{A}$ *is a* (*true*) *Killing tensor for the Hamiltonian* $\hat{H}$, *where* $H=Q(\mathbf{x})\hat{H}$, *and* $\hat{H}\in\mathcal{A}$.

2) $[A,B]=0$, $[L_\alpha,L_\beta]=0$, $[L_\alpha,A]=0$ *for all* $A,B\in\mathcal{A}$.

3) *For each of the* n_1 *essential coordinates of type* 1, x^a, *the form* dx^a *is a simultaneous eigenform for each* $A\in\mathcal{A}$, *with simple root* ρ_a^A.

4) *For each of the* n_2 *essential coordinates of type* 2, x^r, *the form* dx^r *is a simultaneous eigenform for every* $A\in\mathcal{A}$, *with root* ρ_r^A *of multiplicity* 2. *The root* ρ^A *corresponds to only one eigenform.*

5) $\partial_i(a^{\alpha\beta}-\rho_i^A g^{\alpha\beta})=0$, $i=1,\cdots,n_1+n_2$, $A\in\mathcal{A}$.

6) $g^{ab}=0$ *if* $a\neq b$; $g^{ar}=g^{a\alpha}=g^{rs}=0$.

7) $\kappa=n+n_3(n_3-1)/2$.

These results are readily obtained from the following theorem. Let $\{x^i\}$ be a local coordinate system for V_n with coordinates divided into three classes containing n_1, n_2 and n_3 variables, respectively. (We call these variables essential of types 1 and 2 or ignorable, respectively, even though they may have nothing to do with variable separation.) Let $H=g^{ij}p_ip_j$.

THEOREM 7. *Suppose there exists a* κ*-dimensional space* $\mathcal{A}$ *of second order conformal Killing tensors and an* n_3*-dimensional space of Killing vectors with basis* $L_\alpha=p_\alpha$, $\alpha=n_1+n_2+1,\cdots,n$. *Furthermore, suppose conditions* 2)–7) *of Theorem* 6 *are satisfied. Then the Hamilton–Jacobi equation* (1.1) *is separable in the coordinates* $\{x^i\}$. *There exists a Stäckel matrix* $(\theta_{ij}(x^i))$ *such that the Killing tensors* A_1, A_m, $m=2,\cdots,n_1+n_2$, (3.10) *and* $L_\alpha L_\beta=p_\alpha p_\beta$, $n_1+n_2+1\le\alpha\le\beta\le n$, *form a basis for* $\mathcal{A}$.

Proof. Most of the proof follows closely that of [8, Thm. 3], with the added complication that the elements of $\mathcal{A}$ are conformal, rather than true, Killing tensors. Conditions 3), 4) and 6) imply that for any $A\in\mathcal{A}$ we have

$$(a^{ij})=\begin{array}{cc} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} & \\ \begin{pmatrix} \delta^{ab}\rho_a H_a^{-2} & 0 & 0 \\ 0 & 0 & \rho_r g^{r\alpha} \\ 0 & \rho_r g^{\alpha r} & a^{\alpha\beta} \end{pmatrix} & \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \end{array}. \tag{3.13}$$

If $(\rho^A)=(\rho^B)$ for $A,B\in\mathcal{A}$ it follows from (3.13) and condition 5) that $A-B$ is a linear combination of the $n_3(n_3+1)/2$ conformal Killing tensors $L_\alpha L_\beta=p_\alpha p_\beta$, $\alpha\le\beta$.

The condition (1.13) can be written as

$$\begin{aligned} a^{ij}\partial_j g^{kl}+a^{lj}\partial_j g^{ik}+a^{kj}\partial_j g^{li}-g^{ij}\partial_j a^{kl}-g^{lj}\partial_j a^{ik}-g^{kj}\partial_j a^{li}& \\ =Q^i g^{kl}+Q^l g^{ik}+Q^k g^{li}&. \end{aligned} \tag{3.14}$$

Setting $(i,k,l)=(a,b,b)$ in (3.14) we obtain

$$\partial_a(\rho_b-\rho_a)=(\rho_b-\rho_a)\partial_a \ln H_b^{-2}, \qquad \partial_a\rho_a=-Q^aH_a^2. \tag{3.15}$$

Setting $(i,k,l)=(a,r,\alpha)$ we find

$$\partial_a(\rho_r-\rho_a)=(\rho_r-\rho_a)\partial_a \ln g^{r\alpha} \quad \text{if } g^{r\alpha}\not\equiv 0 \tag{3.16}$$

and for $(i,k,l)=(a,a,\alpha)$,

$$\partial_r\rho_a+(\rho_a-\rho_r)\partial_r \ln H_a^{-2}+g_{r\beta}Q^\beta, \qquad g^{\alpha s}g_{s\beta}Q^\beta=Q^\alpha. \tag{3.17}$$

The case $(i,k,l)=(a,a,r)$ leads to $Q^r=0$ and $(i,k,l)=(r,\alpha,\beta)$ leads to

$$\begin{aligned}(g^{\beta s}g^{\alpha r}+g^{\alpha s}g^{\beta r})\partial_s\rho_r+(\rho_r-\rho_s)(g^{\beta s}\partial_s g^{\alpha r}+g^{\alpha s}\partial_s g^{\beta r})&\\ +Q^\alpha g^{r\beta}+Q^\beta g^{r\alpha}=0 &\qquad (\text{sum on } s).\end{aligned} \tag{3.18}$$

Multiplying both sides of (3.18) by $g_{R\alpha}g_{S\beta}$ and summing on α and β we find

$$\begin{aligned}\delta^r_R\partial_S\rho_R+\delta^r_S\partial_R\rho_S+(\rho_r-\rho_S)g_{R\alpha}\partial_S g^{\alpha r}+(\rho_r-\rho_R)g_{S\beta}\partial_R g^{\beta r}&\\ +g_{R\alpha}Q^\alpha\delta^r_S+g_{S\beta}Q^\beta\delta^r_R=0.&\end{aligned} \tag{3.19}$$

For $R=S=r$ in this expression we find

$$\partial_r\rho_r+g_{r\beta}Q^\beta=0. \tag{3.20}$$

Furthermore, for $r=S$, $r\neq R$ in (3.19) we obtain

$$\partial_R(\rho_S-\rho_R)=(\rho_R-\rho_S)g_{S\beta}\partial_R g^{\beta S}. \tag{3.21}$$

Substitution of (3.20) and (3.21) into (3.18), elimination of all derivative terms $\partial_i\rho_j$ and computation of the coefficient of ρ_s in the resulting equation lead to

$$g_{r\gamma}\partial_s g^{\gamma r}=\partial_s(\ln g^{\alpha r}) \quad \text{if } r\neq s \text{ and } g^{\alpha r}\not\equiv 0. \tag{3.22}$$

Since this expression is independent of α, we can set

$$g^{\alpha r}=B^\alpha_r(x^r)H_r^{-2}. \tag{3.23}$$

Expressions (3.15)–(3.17) and (3.20)–(3.23) lead to

$$\partial_i(\rho_j-\rho_i)=(\rho_j-\rho_i)\partial_i \ln H_j^2, \qquad i,j=1,\cdots,n_1+n_2. \tag{3.24}$$

Comparing this equation with (2.6) we see that the metric $d\hat{s}^2=\Sigma_{i=1}^{n_1+n_2}H_i^2(dx^i)^2$ is conformal to a Stäckel form metric.

The integrability conditions $\partial_i\partial_j a^{\alpha\beta}=\partial_j\partial_i a^{\alpha\beta}$ for condition 5) are simply that $g^{\alpha\beta}H^2_{n_1+n_2}$ is a Stäckel multiplier for the metric $d\hat{s}^2/H^2_{n_1+n_2}$. Thus, the Hamilton–Jacobi equation separates in the coordinates **x**. Q.E.D.

Remark 1. It is sufficient to require that condition 5) of Theorem 6 be valid for $i=n_1+1,\cdots,n_1+n_2$ since the requirement that the elements of $\mathcal{A}$ be conformal Killing tensors with $(i,j,k)=(a,\alpha,\beta)$ in (3.14) yields this condition for $i=1,\cdots,n_1$.

Remark 2. Most of the conditions $[A,B]=0$, $A,B\in\mathcal{A}$ (this is just (3.14) with g^{ij} replaced by b^{ij} and $Q^i=0$) are satisfied as a consequence of (3.24) and condition 5). However, the cases $(i,k,l)=(a,a,a)$ and $(i,k,l)=(r,\alpha,\beta)$ lead to the additional requirements

$$\mu_i\partial_i\rho_i=\rho_i\partial_i\mu_i, \qquad i=1,\cdots,m,\ (m+n_1+n_2) \tag{3.25}$$

where A has roots ρ_i and B has roots μ_i.

It is now easy to formulate and prove our main result, the characterization of those involutive families of conformal Killing tensors that correspond to variable separation for the Hamilton–Jacobi equation.

Let $\{x^j\}$ be a local coordinate system on the Riemannian manifold V_n and let $\theta_{(j)}=\lambda_{i(j)}dx^i$, $1\leq j\leq n$, be a local basis of one-forms on V_n. The dual basis of vector fields is $X^{(h)}=\Lambda^{i(h)}\partial_{x^i}$, $1\leq h\leq n$, where $\Lambda^{i(h)}\lambda_{i(j)}=\delta_{(j)}^{(h)}$. We say that the forms $\{\theta_{(j)}\}$ are *normalizable* if there exist local analytic functions $g_{(j)}$, y^j such that $\theta_{(j)}=g_{(j)}dy^j$, (no sum).

THEOREM 8. *Suppose there exists a κ-dimensional vector space $\mathcal{Q}$ of second order conformal Killing tensors on V_n such that*:

1) $[A,B]=0$ *for each* $A,B\in\mathcal{Q}$.

2) *There is a basis of one-forms* $\theta_{(h)}=\lambda_{i(h)}dx^i$, $1\leq h\leq n$, *such that*:

   a) *The* n_1 *forms* $\theta_{(a)}$, $1\leq a\leq n_1$, *are simultaneous eigenforms for every* $A\in\mathcal{Q}$ *with root* ρ_a^A:

$$\left(a^{ij}-\rho_a^A g^{ij}\right)\lambda_{j(a)}=0.$$

   b) *The* n_2 *forms* $\theta_{(r)}$, $n_1+1\leq r\leq n_1+n_2$, *are simultaneous eigenforms for every* $A\in\mathcal{Q}$ *with root* ρ_r^A:

$$\left(a^{ij}-\rho_r^A g^{ij}\right)\lambda_{j(r)}=0.$$

The root ρ_r^A *has multiplicity* 2 *but corresponds to only one eigenform.*

3) $X^{(h)}(\lambda_{i(\alpha)}a^{ij}\lambda_{j(\beta)}-\rho_h^A\lambda_{i(\alpha)}g^{ij}\lambda_{j(\beta)})=0$, $h=n_1+1,\cdots,n_1+n_2$, *for all* $A\in\mathcal{Q}$ *and all* $\alpha,\beta=n_1+n_2+1,\cdots,n$.

4) $[L_\alpha,L_\beta]=0$ *where* $L_\alpha=\Lambda^{i(\alpha)}p_i$ *and each* L_α *is a conformal Killing vector.*

5) $[A,L_\alpha]=0$ *for each* $A\in\mathcal{Q}$.

6) $\kappa=n+n_3(n_3-1)/2$ *where* $n_3=n-n_1-n_2$.

7) $G_{(ab)}\equiv\lambda_{i(a)}g^{ij}\lambda_{j(b)}=0$ *if* $1\leq a<b\leq n$, *and* $G_{(ar)}=G_{(a\alpha)}=G_{(rs)}=0$ *for* $1\leq a\leq n_1$, $n_1+1\leq r,s\leq n_1+n_2$, $n_1+n_2+1\leq\alpha\leq n$.

Then there exist local coordinates $\{y^j\}$ *for* V_n *such that* $\theta_{(j)}=f^{(j)}(\mathbf{y})\,dy^j$ *for suitably chosen functions* $f^{(j)}$, *and the Hamilton–Jacobi equation* (1.1) *is separable in these coordinates. Conversely, to every separable coordinate system* $\{y^j\}$ *for the Hamilton–Jacobi equation there corresponds a family* $\mathcal{Q}$ *of conformal Killing tensors on* V_n *with properties* 1)–7).

Proof. This result follows immediately from Theorem 7, once we show that the $\theta_{(h)}$ are normalizable.

The rest of the proof coincides almost word for word with the proof of [8, Thm. 4]. To see this, we remark that the proof of [8, Thm. 4] exploits the relations $[A,B]=0$ for $A,B\in\mathcal{Q}$, identical to those in the present case, and the relations $[A,H]=0$. In the present case, A is only a conformal Killing tensor so $[A,H]=0$ is replaced by (3.14). Multiplying (3.14) by $\lambda_{(m_1)i}\lambda_{(m_2)k}\lambda_{(m_3)l}$ and summing on i,k,l we obtain an identity $E^{A,H}_{m_1,m_2,m_3}$, the right-hand side of which is $\lambda_{(m_1)i}Q^iG_{(m_2m_3)}+\lambda_{(m_2)k}Q^kG_{(m_1m_2)}+\lambda_{(m_3)l}Q^lG_{(m_2m_2)}$. Examining each step in the proof of [8, Thm. 4], we see that the analogy of this identity is needed only in those instances where m_1, m_2, m_3 are such that the right-hand side of $E^{A,H}_{m_1,m_2,m_3}$ vanishes. Q.E.D.

Examples illustrating the practical application of Theorems 4 and 8 can easily be obtained from the corresponding examples in [7] and [8].

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