Four types of convergence

Math 5467

February 2, 2004

Let $f_1(t), f_2(t), \cdots$ be a sequence of real or complex valued functions defined on some interval J of the real line. We will distinguish four ways that we can say that this sequence converges to the function $f(t)$ defined on J:

$$\lim_{n\to\infty} f_n = f.$$

1. Pointwise convergence. We say that the $f_n(t)$ *converge pointwise* to $f(t)$ on J if for each $t \in J$ the sequence of numbers $\{f_n(t)\}$ converges to $f(t)$: $\lim_{n\to\infty} f_n(t) = f(t)$. To be precise, this means that for any $\epsilon > 0$ there is a $\delta(\epsilon, t) > 0$ such that

   $$|f_n(t) - f(t)| < \epsilon \qquad \text{whenever } n > \delta.$$

   In general, δ will depend on both ϵ and t.

2. Pointwise uniform convergence. We say that the $f_n(t)$ *converge pointwise uniformly* to $f(t)$ on J if for every $t \in J$ the sequence of numbers $\{f_n(t)\}$ converges to $f(t)$: $\lim_{n\to\infty} f_n(t) = f(t)$, in such a way that for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that

   $$|f_n(t) - f(t)| < \epsilon \qquad \text{whenever } n > \delta.$$

   Here the same δ works uniformly for all $t \in J$. Clearly, uniform convergence implies pointwise convergence, but the converse is false.

3. Convergence in the L^1 norm. Suppose the functions f_n, f belong to the normed space $L^1(J)$ of Lebesgue integrable functions. Recall that the norm of $g \in L^1(J)$ is given by the integral

   $$||g||_1 = \int_J |g(t)|\ dt.$$

We say that the f_n *converge in the L^1 norm* to f if for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that

$$||f_n - f||_1 = \int_J |f_n(t) - f(t)|\ dt < \epsilon \qquad \text{whenever } n > \delta.$$

4. Convergence in the L^2 norm. Suppose the functions f_n, f belong to the Hilbert space $L^2(J)$ of Lebesgue square integrable functions. Recall that the norm of $g \in L^2(J)$, $||g||_2 = \sqrt{(g, g)}$ is given by the integral

$$||g||_2^2 = \int_J |g(t)|^2\ dt.$$

We say that the f_n *converge in the L^2 norm* to f if for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that

$$||f_n - f||_2 = \sqrt{\int_J |f_n(t) - f(t)|^2\ dt} < \epsilon \qquad \text{whenever } n > \delta.$$

REMARKS Other than the fact that uniform convergence always implies pointwise convergence, these forms of convergence are somewhat independent of one another, as the homework exercises are designed to show. However, the following implications are easy to prove:

- If J is a bounded interval, then uniform convergence implies convergence in both the L^1 and L^2 norms.
- If J is a bounded interval, then L^2 convergence implies L^1 convergence. (Use the Schwarz inequality.)

The concept of uniform convergence is particularly useful when one wants to approximate functions $f(t)$ by continuous functions $f_n(t)$. Recall that a function $g(t)$ on J is *continuous at t_0* if $t_0 \in J$ and for any $\epsilon > 0$ there is a $\delta(\epsilon, t_0) > 0$ such that

$$|g(t) - g(t_0)| < \epsilon \qquad \text{whenever } |t - t_0| < \delta, \quad t \in J.$$

If g is continuous for all $t \in J$ we say simply that g is *continuous on J*.

Theorem 1 *Suppose the functions $f_1(t), f_2(t), \cdots$ are each continuous on the interval J and suppose $f_n(t) \to f(t)$ pointwise uniformly as $n \to \infty$. Then f is a continuous function on J.*

PROOF: Let $t_0 \in J$ Then for any $t \in J$ we can use the triangle inequality for the absolute value function to obtain

$$|f(t) - f(t_0)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(t_0)| + |f_n(t_0) - f(t_0)|.$$

Suppose we are given $\epsilon > 0$. Since the f_j converge uniformly, we can choose a number $\delta_1(\epsilon)$ such that $|f_n(s) - f(s)| < \frac{\epsilon}{3}$, for all $s \in J$, whenever $n > \delta_1$. Since the f_j are continuous, we can choose a number $\delta_2(t_0, n, \epsilon)$, for fixed n, such that $|f_n(t) - f_n(t_0)| < \frac{\epsilon}{3}$ whenever $|t - t_0| < \delta_2$. Thus if $n > \delta_1$ and $|t - t_0| < \delta_2$ we have

$$|f(t)-f(t_0)| \leq |f(t)-f_n(t)|+|f_n(t)-f_n(t_0)|+|f_n(t_0)-f(t_0)| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

It follows that f is continuous at any $t_0 \in J$. Q.E.D.

This result will prove very useful in our study of the approximation of functions.