q-algebra and superalgebra tensor products, and special function identities

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Abstract

We work out the tensor products for distinct q-generalizations of euclidean, oscillator and $s\ell(2)$ type algebras and superalgebras, including cases where the method of highest weight vectors will not apply. In particular, we use the three-term recurrence relations for Askey-Wilson polynomials to decompose the tensor product of representations from the positive discrete series and representations from the negative discrete series. We show that various q-analogs of the exponential function can be used to mimic the exponential mapping from a Lie algebra to its Lie group and we compute the corresponding matrix elements of the "group operators" on these representation spaces. The most important q-series identities derived here are interpreted as the expansion of the matrix elements of a "group operator" (via the exponential mapping) in a tensor product basis in terms of the matrix elements in a reduced basis. They involve q-hypergeometric series with base q and $-q$, respectively, for the algebra and superalgebra cases, where $0 < q < 1$.

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1 Introduction

This is a brief survey of the work carried out by the authors and their collaborators, relating q-series identities and the representation theory of q-algebras and q-superalgebras. The motivation comes from the factorization method of Infeld-Hull-Schrödinger and its relation to the representation theory of the family of Lie algebras $\mathcal{G}(a,b)$, [1, 2, 3, 4]. Here, $\mathcal{G}(a,b)$ has basis $\{H, E_+, E_-, I\}$ and commutation relations

$$[H, E_\pm] = \pm E_\pm,\ [E_+, E_-] = aH + bI,\ [\mathcal{J}, I] = 0, \quad \forall \mathcal{J} \in \mathcal{G}(a,b). \tag{1}$$

For $a \neq 0$ this algebra is isomorphic to $s\ell(2) \oplus \{I\}$; for $a = 0, b \neq 0$ it is isomorphic to the oscillator algebra; and for $a = b = 0$ it is isomorphic to $m(2) \oplus \{I\}$, where $m(2)$ is the Euclidean algebra in the plane. See [4] for relations between matrix elements and tensor products of these Lie algebras, the factorization method, and special function identities.

Zhedanov and others have introduced a product of generalized $s\ell_q(2)$ algebras that 1) is a natural q-analog of (1), and 2) allows one to take tensor products of representations corresponding to two distinct algebras, [5, 6, 7]. Their generalization is an algebra (v,u) with generators H, E_+, E_- which obey the commutation relations

$$[H, E_+] = E_+, \quad [H, E_-] = -E_-, \quad [E_+, E_-] = -uq^{-H} - vq^H. \tag{2}$$

Here, u and v are real numbers and $0 < q < 1$. For $uv \neq 0$ this algebra is isomorphic to one of the true $s\ell_q(2)$ type algebras, for $uv = 0,\ u^2 + v^2 > 0$ it is isomorphic to a special realization of the q-oscillator algebra, and for $u = v = 0$ it is isomorphic to the Euclidean Lie algebra $m(2)$, [8]. This algebra has an invariant element

$$C = E_+E_- + \frac{vq^H - uq^{1-H}}{1-q}. \tag{3}$$

Furthermore, the family of algebras admits a multiplication $(v,u) \otimes (-u,t) \cong (v,t)$, defined by

$$F_\pm = \Delta(E_\pm) = E_\pm \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes E_\pm, \quad L = \Delta(H) = H \otimes I + I \otimes H. \tag{4}$$

The operators $F_\pm$, L satisfy the commutation relations (2). Using (4) we can easily define the tensor product $\rho \otimes \mu$ of a representation ρ of (v,u) and the representation μ of $(-u,t)$, thereby obtaining a representation of (v,t). This construction yields a convenient generalization of the tensor product computations in, for example, [9, 10, 11].

One can further extend this idea to q-superalgebras, such as generalizations of the $osp_q(1/2)$ algebra, [12, 13, 14]. Indeed, we define the q-superalgebra $[v,u]$ by the generators $H, V_\pm$ and relations

$$[H,V_\pm]=\pm\frac{1}{2}V_\pm,\quad \{V_+,V_-\}=-uq^{-2H}-vq^{2H},\tag{5}$$

where $[A,B]=AB-BA$ and $\{A,B\}=AB+BA$. Here, u,v are parameters. The "Casimir" operator C for $[v,u]$ is

$$C=V_+V_-+\frac{v}{1+q}q^{2H}+\frac{u}{1+q^{-1}}q^{-2H}\tag{6}$$

and satisfies the relations $[H,C]=0,\quad \{V_\pm,C\}=0$. Note that C^2 is an invariant operator, i.e., C^2 commutes with all elements of the $[v,u]$ algebra. We define the coproduct

$$L=\Delta(H)=H\otimes I+I\otimes H,\quad F_\pm=\Delta(V_\pm)=V_\pm\otimes q^H+q^{-H}\otimes V_\pm,\tag{7}$$

where

$$(A\otimes B)(C\otimes D)=(-1)^{p(B)p(C)}AC\otimes BD.\tag{8}$$

Here $p(A)$ is the *parity* of the operator A. In this case the bosonic variables H, q^H have parity 0 and the fermionic variables $V_\pm$ have parity 1, [15, 13]. It follows that this algebra admits a multiplication of the form

$$[v,u]\otimes[-u,t]=[v,t],\tag{9}$$

where the first factor corresponds to the algebra $[v,u]$ and the second factor to algebra $[-u,t]$. Indeed we have

$$[L,F_\pm]=\pm\frac{1}{2}F_\pm,\quad \{F_+,F_-\}=-tq^{-2L}-vq^{2L}.\tag{10}$$

In §2 we give some details on the representation theory of the q-algebra $(0,0)$ and its relation to the representation theory of the ordinary Lie algebra $m(2)$, and survey additional results for matrix elements and tensor products of q-algebras (v,u). In §3 we give a similar study for q-superalgebras $[v,u]$.. The notation used for q-series and q-integrals in this paper follows that of Gasper and Rahman [16].

2 Representations of the q-algebras (v, u)

The ordinary three dimensional Lie algebra $m(2)$ is determined by its generators H, E_+, E_- which obey the commutation relations

$$[H, E_+] = E_+, \quad [H, E_-] = -E_-, \quad [E_+, E_-] = 0.$$

We consider irreducible representations (ω) of $m(2)$, characterized by the positive number ω. The spectrum of H is the set $Z = \{m : m\ aninteger\}$ and the complex representation space has basis vectors f_m such that

$$E_\pm f_m = \omega f_{m\pm 1}, \ H f_m = m f_m, \ E_+ E_- f_m = \omega^2 f_m, \tag{11}$$

where $C \equiv E_+ E_-$ is an invariant operator. We can introduce an inner product such that $< f_n, f_{n'} >= \delta_{nn'}$, $n, n' \in Z$, and $H = H^*$, $E_+^* = E_-$.

Matrix elements $T_{m'm}$ of the complex motion group in the representation (ω) are typically defined by the expansions

$$\begin{aligned} e^{\beta E_+} e^{\alpha E_-} e^{\tau H} f_m &= \sum_{m'=-\infty}^{\infty} T_{m'm}(\alpha, \beta, \tau) f_{m'}, \\ T_{m'm}(\alpha, \beta, \tau) &= e^{[\tau m + i(\pi/2+\theta)(m-m')]} J_{m-m'}(\omega r), \end{aligned} \tag{12}$$

where $\alpha = ire^{i\theta}/2$, $\beta = ire^{-i\theta}/2$ and $J_n(r)$ is a Bessel function [4, 17]. The group multiplication property of the operators on the left-hand side of (12) leads to addition theorems for the matrix elements. For convenience we shall set $\tau = 0$.

We define the tensor product $(\omega_1) \otimes (\omega_2)$ of two irreducible representations (ω_1) and (ω_2) by the operators F_+, F_-, L:

$$F_\pm = E_\pm \otimes I + I \otimes E_\pm, \quad L = H \otimes I + I \otimes H.$$

Here, I is the identity operator on the corresponding representation space. The operators F_+, F_-, L again obey the $m(2)$ commutation relations and the representation is unitary. It is straightforward to decompose this representation into a direct integral of irreducible representations, [17]:

$$(\omega_1) \otimes (\omega_2) \cong \oplus \frac{1}{2\pi} \int_0^{2\pi} (\omega_\mu)\ d\mu, \tag{13}$$

where $\omega_\mu^2 = \omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos\mu$. From the proof of (13) we see that the matrix elements of the operator $e^{\beta F_+} e^{\alpha F_-}$ with respect to the tensor product basis $f_n^{\omega_1} \otimes f_{n'}^{\omega_2}$ and with respect to the reduced basis $f_m^{\omega_\mu}$ are related by the formula

$$J_{n'}(r')J_n(r'') = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n\phi''-n'\phi+n\phi)} J_{n+n'}(r)\, d\phi'' \tag{14}$$

where $re^{i\phi} = r' + r''e^{i\phi''}$.

In addition to the standard definition of the tensor product of two irreducible representations (ω_1) and (ω_2) of $m(2)$ there is a q-algebra definition given by the nontrivial Hopf algebra structure in which the coproduct is (4). With the q-analogs of the exponential function

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty},\ |x| < 1; \quad E_q(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}x^k}{(q;q)_k} = (-x;q)_\infty,$$

we define the following q-analogs of matrix elements of (ω), [18]:

$$e_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e,E)}(\alpha,\beta) f_{n'},\ |\omega\beta| < 1.$$

Setting $\alpha = ire^{i\psi}$, $\beta = ire^{-i\psi}$, we see that in terms of the new complex coordinates $[r, e^{i\psi}]$ we have

$$T_{n'n}^{(e,E)}[r, e^{i\psi}] = e^{i(\frac{\pi}{2}-\psi)(n'-n)} q^{(n'-n)/2} J_{n'-n}(r\omega q^{-\frac{1}{2}}; q),$$

$$J_\nu(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu {}_1\phi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix} ; q, qz^2 \right),$$

where $J_\nu(z;q)$ is the Hahn q-Bessel function. This and other q-analogs of $m(2)$ matrix elements obey group-like addition theorems [18].

The tensor product decomposition $(\omega_1) \otimes_q (\omega_2)$ is not unique, [18]. The simplest decompositions are

$$(\omega_1) \otimes_q (\omega_2) \equiv \sum_{p=-\infty}^{\infty} \oplus(\omega_1 q^{p/2}), \tag{15}$$

and another decompostion obtained by replacing ω_1 with ω_2 on the right-hand side of (15). A special case of the identity relating the matrix elements of the operator $e_q(\beta F_+)E_q(\alpha F_-)$ with respect to the tensor product basis and with respect to the reduced basis is

$$\begin{aligned}&(-q)^n J_{x-n}(Sq^{-y};q^2)J_n(RSq^n;q^2) = \\ &\sum_{k=-\infty}^{\infty} q^{2k} J_{k-n}(Rq^y;q^2)J_x(Sq^{k-y};q^2)J_k(Rq^{y+x};q^2),\end{aligned} \tag{16}$$

valid for $0 < Rq^{y+x+1} < 1$, $0 < Rq^{y+1} < 1$, $0 \le S$ and n, x, y integers. The case $S = q^z$ of this formula for z an integer is the addition theorem for Hahn q-Bessel functions derived by Koelink using the theory of quantum groups, [19]. (The computation of the matrix elements on the left hand side of this formula and in many formulas to follow is based heavily on the formal identities

$$E_q(X+Y) = E_q(Y)E_q(X), \quad e_q(X+Y) = e_q(X)e_q(Y), \tag{17}$$

valid for operators X, Y such that $YX = qXY$.)

The q-oscillator algebra is the associative algebra generated by the four elements H, E_+, E_-, $\mathcal{E}$ that obey the commutation relations

$$[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = -q^{-H}\mathcal{E}, \quad [\mathcal{E}, E_\pm] = [\mathcal{E}, H] = 0. \tag{18}$$

It admits a class of algebraically irreducible representations $\uparrow_{\ell,\lambda}$ where ℓ, λ are real numbers and $\ell > 0$, [**?**, 10]. These are defined on a vector space with basis $\{e_n : n = 0, 1, 2, \cdots\}$, such that

$$\begin{aligned} E_+ e_n &= \ell\sqrt{\frac{q^{-n-1}-1}{1-q}}e_{n+1}, \; E_- e_n = \ell\sqrt{\frac{q^{-n}-1}{1-q}}e_{n-1} \\ He_n &= (\lambda+n)e_n, \quad \mathcal{E}e_n = \ell^2 q^{\lambda-1}e_n. \end{aligned} \tag{19}$$

Since $\mathcal{E}$ is a constant for the representations $\uparrow_{\ell,\lambda}$, they can be considered as representations of the algebras $(0, \ell q^{\lambda-1})$. Similarly, the q-oscillator algebra admits a class of algebraically irreducible representations $\downarrow_{\ell,\lambda}$. These are defined on a vector space with basis $\{h_m : -m = 0, 1, 2, \cdots\}$, such that

$$\begin{aligned} E_+ h_m &= \ell\sqrt{\frac{q^{m}-1}{1-q}}h_{m+1}, \; E_- h_m = \ell\sqrt{\frac{q^{m-1}-1}{1-q}}h_{m-1} \\ Hh_m &= (-\lambda+m)h_m, \quad \mathcal{E}h_m = -\ell^2 q^{\lambda-1}h_m. \end{aligned} \tag{20}$$

Note that the $\downarrow_{\ell,\lambda}$ can be considered as representations of $(-\ell q^{\lambda-1}, 0)$.

Now we consider the tensor product representation

$$\uparrow_{\ell,\lambda}(0, \ell q^{\lambda-1}) \otimes \downarrow_{\ell,\lambda}(-\ell q^{\lambda-1}, 0) \tag{21}$$

of the Euclidean Lie algebra $(0,0)$. The invariant operator is $C = F_+F_-$.

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $s \geq 0$ are just those linear combinations of the basis vectors $H_n^s = e_n \otimes h_m$ where $n + m = s,\ -m = 0, 1, \cdots$. For $s < 0$, they are linear combinations of the basis vectors $H_n^s = e_n \otimes h_m$ where now $n + m = s,\ n = 1, 2, \cdots$. Applying the symmetric operator C to the ON set $\{H_n^s\}$ we find

$$\begin{aligned}(1-q)CH_n^s &= q^{-\lambda-n+(s-1)/2}\ell^2\left[(1-q^{-n-1})(1-q^{s-n-1})\right]^{\frac{1}{2}} H_{n+1}^s \\ &+ q^{-\lambda-n+(s+1)/2}\ell^2\left[(1-q^{s-n})(1-q^{-n})\right]^{\frac{1}{2}} H_{n-1}^s \\ &- [q^{-\lambda+s-n}\ell^2(1-q^{-n-1}) + q^{-\lambda-n}\ell^2(1-q^{s-n})]H_n^s. \end{aligned} \tag{22}$$

If we introduce a spectral transform for a self-adjoint extension of this operator so that C corresponds to multiplication by the transform variable x, then (22) takes the form of a three term recurrence relation for orthogonal polynomials $H_n^s(x)$ of order n in x. Indeed, comparing (22) with the three term recurrence relation for the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array}; q, -(1-q)xq^{n+\alpha+1}\right), \tag{23}$$

[16], page 194, we get a match with $C = \ell^2 q^{-\lambda}x$ and the identification

$$H_n^s(x,t) = (-1)^n x^{-s/2} t^s \left[\frac{(q;q)_n q^n (1-q)^{1-s}}{(q^{1-s};q)_n (q;q)_{-s} q^{(s-s^2)/2}\log q^{-1}}\right]^{\frac{1}{2}} L_n^{(-s)}(x;q), \tag{24}$$

where $t = e^{i\phi}$ and $x = \cos\theta$, we can verify that (22) holds, as well as the orthogonality relations

$$\begin{aligned}< H_n^s, H_{n'}^{s'} > &= \delta_{n,n'}\delta_{s,s'}, \quad < f, g > = \frac{1}{2\pi}\int_0^\infty \rho(x)\, dx \int_0^{2\pi} d\phi\, f(x,t)\overline{g(x,t)} \\ \rho(x) &= \frac{1}{(-(1-q)x;q)_\infty}. \end{aligned} \tag{25}$$

For $s \geq 0$,

$$H_n^s(x,t) = (-1)^M x^{s/2} t^s \left[\frac{(q;q)_M q^M (1-q)^{1+s}}{(q^{1+s};q)_M (q;q)_s q^{-(s+s^2)/2} \log q^{-1}} \right]^{\frac{1}{2}} L_M^{(s)}(x;q), \quad (26)$$

where $M = -m = 0, 1, \cdots$, and $n = M + s$. Furthermore, in terms of the new variables x, t the action of the operators $F_\pm, L$ is

$$F_+ = q^{-\lambda/2} \ell t x^{1/2}, \quad F_- = q^{-\lambda/2} \ell t^{-1} x^{1/2}, \quad L = t\frac{\partial}{\partial t}. \quad (27)$$

Operators (27) acting on the space of square integrable functions of the periodic variable ϕ define the unitary irreducible representation $(\sqrt{q^{-\lambda} x}\ell)$ of the Euclidean Lie algebra $(0,0)$. Thus,

$$\uparrow_{\ell,\lambda} (0, \ell q^{\lambda-1}) \otimes \downarrow_{\ell,\lambda} (-\ell q^{\lambda-1}, 0) \cong \int_0^\infty \oplus (\sqrt{q^{-\lambda} x}\ell)\ \rho(x)\ dx. \quad (28)$$

The functions $H_n^s(x,t)$ are the Clebsch-Gordan coefficients for this decomposition. However, as is well known [16, 20], the measure for which the q-Laguerre polynomials are orthogonal is not unique. Indeed C has no unique self-adjoint extension (the deficiency indices are (1,1)). Thus, there is a multiplicity of possible self-adjoint extensions for C and each such extension defines a different tensor product. For each of these cases the $H_n^s(x,t)$ are the Clebsch-Gordan coefficients, with the proviso that the coefficients satisfy orthogonality but not completeness relations. (See [8] for another example.) These decompositions can be used just as before to obtain identities relating the matrix elements of the operator $E_q(\beta F_+) e_q(\alpha F_-)$ with respect to the tensor product basis and the reduced basis. The identities express products of q-Laguerre polynomials as integrals or sums over Hahn q-Bessel functions, with expansion coefficients that are themselves products of q-Laguerre polynomials.

Related results have been derived in the following papers. In [7] the Clebsch-Gordan coefficients for the tensor product of representations in the positive discrete series is calculated for the case $(v,u) \otimes (-u,w)$. The method of highest weights is used and the tensor product expansion is unique. The results are expressed in terms of q-Hahn polynomials. Here the eigenspace of L with eigenvalue α is finite dimensional, so one is considering finite discrete Askey-Wilson polynomials. In [8] results similar to the above have been worked out for the tensor product of a representation from the positive discrete series and one from the negative discrete series calculated for the case $(v,u)\otimes(-u,w)$. Results from the theory of Askey-Wilson polynomials are employed, and the expansion is unique.

In [10] we considered the tensor product representation

$$(\omega)(0,0)\otimes\uparrow_{\ell,\lambda}(0,\ell q^{\lambda-1})$$

of the q-oscillator algebra $(0,\ell q^{\lambda-1})$ and worked out the related matrix element identity. The direct sum decomposition corresponded to the orthogonality relations for the polynomials

$$H_n(x) = {}_2\phi_1\left(\begin{matrix} q^{-n}, & \frac{\ell^2}{x} \\ 0 & \end{matrix}; q, \frac{-xq}{\omega^2(1-q)}\right) = q^n {}_2\phi_1\left(\begin{matrix} q^{-n}, & \frac{-\omega^2(1-q)}{x} \\ 0 & \end{matrix}; q, \frac{xq}{\ell^2}\right).$$

These polynomials are orthogonal with respect to a measure with support at the points

$$x = q^m\ell^2,\ -(1-q)q^m\omega^2, \quad m=0,1,2,\cdots.$$

In [21] we considered tensor products of various discrete analogs of the Euclidean and oscillator algebras. For general tensor products of (v,u) algebras where the representation for each factor is bounded neither above or below, the situation is more complicated. The general spectral theory for the second order difference operator C has to be employed, such as sketched in [22], and in a certain sense the tensor product is no longer clearly defined [23]. Nevertheless, these tensor products can be used to derive useful special function identities, as we shall show in forthcoming work.

3 Representations of the q-supersymmetric algebras $[v,u]$

Next we consider a degenerate case of the tensor product of two q-superalgebras, a representation of the q-supereuclidean algebra $[0,0]$ of the form $\downarrow'_\xi [0,u]\otimes\uparrow'_\lambda [-u,0]$. Here the family of algebraically irreducible representations (bounded below) $\uparrow'_\lambda$, of the q-superalgebra $[-u,0]$, where $u>0$, is defined as follows, [24]. An orthogonal basis for the representation space is $\{e_n : n=0,1,\cdots\}$

where $(p=-q)$

$$
\begin{aligned}
V_{-}e_n &= \left[\frac{(1-p^n)uq^{-2\lambda}}{1+q}\right]^{\frac{1}{2}} e_{n-1}, \quad He_n = (-\lambda + \frac{n}{2})e_n \\
V_{+}e_n &= \left[\frac{(1-p^{n+1})uq^{-2\lambda}}{1+q}\right]^{\frac{1}{2}} (-1)^n e_{n+1}.
\end{aligned} \tag{29}
$$

The representation space is a Hilbert space with indefinite inner product

$$
< e_n, e_{n'} >= \delta_{nn'}(-1)^{n(n-1)/2}, \qquad n, n' = 0, 1, \cdots, \tag{30}
$$

and $V_{+} = (V_{-})^{*}$, $H^{*} = H$. We need the following matrix elements of $\uparrow'_{\lambda}$:

$$
e_{-q}(\beta V_{-})E_{-q}(\alpha V_{+})e_n = \sum_{n'=0}^{\infty} T'^{(e-,E+)}_{n'n}(\alpha,\beta)e_{n'}, \tag{31}
$$

$$
\begin{aligned}
T'^{(e-,E+)}_{n'n}(\alpha,\beta) &= q^{\frac{1}{2}(n'-n)(n'+n-1)}\alpha^{n'-n}\left(\frac{uq^{-2\lambda}}{1+q}\right)^{\frac{1}{2}(n'-n)}\frac{\sqrt{(p^{n+1};p)_{n'-n}}}{(p;p)_{n'-n}} \\
&\times \Big[\frac{1-i}{2}{}_1\phi_1\left(\begin{array}{c} p^{n'+1} \\ p^{n'-n+1} \end{array} ;p; \frac{-iq^{n'}\alpha\beta u}{p^n q^{2\lambda}(1+q)}\right) \\
&+ \frac{1+i}{2}{}_1\phi_1\left(\begin{array}{c} p^{n'+1} \\ p^{n'-n+1} \end{array} ;p; \frac{iq^{n'}\alpha\beta u}{p^n q^{2\lambda}(1+q)}\right)\Big].
\end{aligned} \tag{32}
$$

Here,

$$
< T'e_n, e_{n'} >= T'_{n'n}(-1)^{n(n-1)/2}. \tag{33}
$$

The motivation for the choice of base $p=-q$ comes from the relations

$$
(V_{\pm}\otimes q^H)(q^{-H}\otimes V_{\pm}) = -q^{\pm 1}(q^{-H}\otimes V_{\pm})(V_{\pm}\otimes q^H).
$$

We also consider a second family of algebraically irreducible representations (bounded above) $\downarrow'_{\xi}$, of the q-superalgebra $[0,u]$, where $u>0$, defined as follows. A convenient orthogonal basis for the representation space with indefinite inner product is $\{j_m : m = 0, 1, \cdots\}$ where

$$
< j_m, j_{m'} >= \delta_{mm'}(-1)^{m(m+1)/2}, \qquad m, m' = 0, 1, \cdots, \tag{34}
$$

$$
\begin{aligned}
V_{-}j_m &= \left[\frac{(1-p^{m+1})uq^{2\xi}}{1+q}\right]^{\frac{1}{2}} j_{m+1}, \quad Hj_m = -(\xi+\frac{m}{2})j_m \\
V_{+}j_m &= \left[\frac{(1-p^{m})uq^{2\xi}}{1+q}\right]^{\frac{1}{2}} j_{m-1}
\end{aligned} \tag{35}
$$

We have $V_+ = (V_-)^*$ and $H^* = H$.

For this representation we consider the matrix elements

$$
(e-,E+): \quad e_p(\beta V_-)E_p(\alpha V_+)j_m = \sum_{m'} S'^{(\xi)}_{m'm}(\alpha,\beta)j_{m'},
$$

$$
\begin{aligned}
S'^{(\xi)}_{m'm}(\alpha,\beta) &= (\beta)^{m'-m}(\frac{uq^{2\xi}}{1+q})^{\frac{1}{2}(m'-m)}\frac{\sqrt{(p^{m+1};p)_{m'-m}}}{(p;p)_{m'-m}} \\
&\times \; {}_2\phi_1\left(\begin{matrix} p^{-m}, & 0 \\ p^{m'-m+1} & \end{matrix}; p; \frac{-\alpha\beta p^m u}{q^{-2\xi}(1+q)}\right).
\end{aligned} \tag{36}
$$

Now we form the tensor product representation

$$
\downarrow'_\xi [0,u] \otimes \uparrow'_\lambda [-u,0] \cong [0,0] \tag{37}
$$

of $[0,0]$ using the definition (7). Here $C = F_+F_-$. Note that $F_+^* = F_-$, $L^* = L$ and that a basis for the indefinite inner product space is $\{j_m \otimes e_n, \quad m,n = 0,1,\cdots\}$ where

$$
< j_m \otimes e_n, j_{m'} \otimes e_{n'} >=< j_m, j_{m'} >< e_n, e_{n'} >= \delta_{mm'}\delta_{nn'}(-1)^{\frac{1}{2}(m+n)(m-n+1)}.
$$

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $-\lambda-\xi+\alpha$, $\alpha \geq 0$ are just those linear combinations of the basis vectors $J^\alpha_m = j_m \otimes e_n$ where $n = 2\alpha+m$, $m = 0,1,\cdots$. For $\alpha < 0$, they are linear combinations of the basis vectors $J^\alpha_m = j_m \otimes e_n$ where $n = 2\alpha+m$, $n = 0,1,\cdots$. Taking the case $\alpha \geq 0$ and applying C to the orthogonal set $\{J^\alpha_m\}$ we find

$$
\begin{aligned}
\frac{(1+q)q^{2(\lambda-\xi)}}{u}CJ^\alpha_m &= (-1)^\alpha p^m q^{\frac{\alpha}{2}+\frac{1}{2}}\Big[(1-p^{m+1})(1-p^{\alpha+m+1})\Big]^{\frac{1}{2}}J^\alpha_{m+1} \\
&+ \; p^m q^{\frac{\alpha}{2}-\frac{1}{2}}\Big[(1-p^m)(1-p^{\alpha+m})\Big]^{\frac{1}{2}}J^\alpha_{m-1} \\
&- \; p^m\Big[q^\alpha(i-p^{m+1})+(-1)^\alpha(1-p^{\alpha+m})\Big]J^\alpha_m.
\end{aligned} \tag{38}
$$

If we introduce the spectral transform of the self adjoint operator so that C corresponds to multiplication by the transform variable x, then (38) takes the form of a three term recurrence relation for orthogonal polynomials $J^{\alpha}_{m}(x)$ of order m in x. Indeed, comparing (38) with the three term recurrence relation for the continuous Askey-Wilson polynomials [16], we see that we get a match for a special case of the little q-Jacobi polynomials

$$p_n(x;a,b;Q) = {}_2\phi_1 \left(\begin{matrix} Q^{-n}, & abQ^{n+1} \\ aQ & \end{matrix} ; Q, xQ \right) \tag{39}$$

where $Q = p$, $a = p^{\alpha}$ and $C \sim xuq^{2(\xi-\lambda)}(-1)^{\alpha+1}/(1+q)$. Making the identification

$$J^{\alpha}_{m}(x,s) = \left[\frac{q^{-m(\alpha+1)}(p^{\alpha+1};p)_{\infty}(p^{\alpha+1};p)_m}{(p;p)_m} \right]^{\frac{1}{2}} p_m(x;p^{\alpha},0;p)s^{\alpha} \tag{40}$$

we have

$$< J^{\alpha}_{m}, J^{\alpha'}_{m} >= \delta_{\alpha\alpha'} \frac{(-1)^{[\frac{\alpha}{2}]}}{2} \left[1 - (-1)^{\alpha} + (-1)^{m} + (-1)^{\alpha+m} \right] \tag{41}$$

where

$$< f(x)s^{\alpha}, g(x)s^{\alpha'} > \equiv \delta_{\alpha\alpha'} (-1)^{\alpha(\alpha-1)/2} \sum_{y=0}^{\infty} \frac{p^{y(\alpha+1)}}{(p;p)_y} f(p^y)g(p^y). \tag{42}$$

For fixed integer α odd, this defines a true inner product for little q-Jacobi polynomials; for α even it is just a bilinear product since the discrete measure can be negative. It is not dificult to verify that on this basis the superalgebra acts as follows:

$$F_{+} = q^{\xi-\lambda}\sqrt{\frac{u}{q+1}}sR_s, \quad F_{-} = q^{\xi-\lambda}\sqrt{\frac{u}{q+1}}\frac{x}{s}, \quad L = -(\xi+\lambda) + \frac{1}{2}s\partial_s \tag{43}$$

where $R_sG(s) = G(-s)$. For fixed x this representation is equivalent to the irreducible representation $(\omega) = (q^{\xi-\lambda}\sqrt{\frac{xu}{q+1}})$ of $[0,0]$, as studied in [24]. Thus we have derived the decomposition

$$\downarrow'_{\xi} [0,u] \otimes \uparrow'_{\lambda} [-u,0] \cong \oplus \sum_{y=0}^{\infty} (q^{\xi-\lambda}\sqrt{\frac{p^y u}{q+1}})[0,0] \tag{44}$$

Again, the functions $J^h_m(x,s)$ are essentially the Clebsch-Gordan coefficients for this decomposition; the orthogonality and completeness relations for the corresponding little p-Jacobi polynomials are the unitarity conditions for the C-G coefficients. Also, the decomposition (44) can be used to obtain an identity relating the matrix elements (32), (36) and the adjoint of (43). We can compute the matrix element

$$T'_{m'n',mn}(\alpha,\beta) =< e_p(\beta F_-)E_p(\alpha F_+)j_m\otimes e_n, j_{m'}\otimes e_{n'} >$$

in two diferent ways. On one hand we have the representation

$$\begin{aligned} T'_{m'n',mn}(\alpha,\beta) &= < e_p(\beta F_-)E_p(\alpha F_+)J^h_m, J^{h'}_{m'} > \\ &= < U'^{(\cdot)}_{h'h}(\alpha,\beta)K^h_m(\cdot), K^{h'}_{m'}(\cdot) >, \end{aligned} \tag{45}$$

where $J^h_m(x,s) = K^h_m(x)s^h$, $U'^{(\omega)}_{h'h}(\alpha,\beta) = x^{(h-h')/2}L'_{h'h}(\alpha,\beta)$, $n = h+m$, $n' = h'+m'$ and the L' are essentially the adoint matrix elements to (43), [24]:

$$\begin{aligned} L'^{(x)}_{h'h}(\alpha,\beta) &= \frac{(\beta q^{\xi-\lambda}\sqrt{\frac{ux}{1+q}})^{h-h'}}{(p;p)_{h-h'}} \\ &\times \Big[{}_1\phi_1\left(\begin{matrix}0\\ p^{h-h'+1}\end{matrix};p,\frac{i\alpha\beta ux(-1)^{h+1}}{q^{2\lambda-2\xi}(1+q)}\right)\frac{1-i}{2} \\ &+ {}_1\phi_1\left(\begin{matrix}0\\ p^{h-h'+1}\end{matrix};p,(-1)^h\frac{i\alpha\beta q^{2\xi-2\lambda}ux}{1+q}\right)\frac{1+i}{2}\Big]. \end{aligned} \tag{46}$$

On the other hand,

$$\begin{aligned} &2T'_{m'n',mn}(\alpha,\beta) = \\ =\ & S'^{(\xi)}_{m'm}(\alpha q^{-\lambda+\frac{n'}{2}},\beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}},\beta(-1)^m q^{\xi+\frac{m}{2}}) \\ +\ & S'^{(\xi)}_{m'm}(\alpha q^{-\lambda+\frac{n'}{2}},\beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}},-\beta(-1)^m q^{\xi+\frac{m}{2}}) \\ +\ & S'^{(\xi)}_{m'm}(-\alpha q^{-\lambda+\frac{n'}{2}},\beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}},\beta(-1)^m q^{\xi+\frac{m}{2}}) \\ -\ & S'^{(\xi)}_{m'm}(-\alpha q^{-\lambda+\frac{n'}{2}},\beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}},-\beta(-1)^m q^{\xi+\frac{m}{2}}). \end{aligned} \tag{47}$$

In [25] we carried out the analogous constructions for tensor products of positive discrete series representations of the general algebra $[v,u]$ for

$v^2 + u^2 > 0$. Here the method of highest weights can be employed. In [24] irreducible representations of the "Euclidean" superalgebra $[0,0]$ were studied and the (nonunique) tensor product decomposition for

$$[0,0] \otimes [0,0] \cong [0,0]$$

was worked out. Also, the product $[v,0] \otimes [0,0] \cong [v,0]$ was studied.

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