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Some Applications of the Representation Theory of the Euclidean Group in Three-Space*

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Introduction

In recent years several authors have studied the theory of certain special functions through an analysis of the representation theory of the Lie groups and algebras with which these functions are associated. In particular, the rotation group and the Euclidean group in the plane have been so studied, [1], [3], [7]. Apparently, however, special functions associated with the representation theory of the Euclidean group in three-space have not been systematically investigated. This paper partially fills the gap.

We shall calculate the matrix elements of the unitary irreducible representations of the Euclidean group and determine complete sets of basis eigenfunctions for several realizations of these representations. By comparing the representation theory of the Euclidean group and that of its associated Lie algebra, we shall be able to derive recursion relations and addition theorems for these functions. In analogy with the situation for compact groups we shall also show that the matrix elements of the unitary irreducible representations obey a series of orthogonality and completeness relations. The special functions considered will be of two kinds: generalized spherical functions and the so-called spherical waves of definite energy and helicity, dealt with in scattering theory.

Most of the results concerning special functions presented in this paper are known in one form or another. However, we derive them here in an elegant and straight-forward manner and explicitly relate them to group theory. Also, the generalization of the addition theorem for spherical waves, equation (3.22), is new.

Since we are interested primarily in the group and algebraic problems, measure theoretical and convergence details will ordinarily be left to the reader. However, the needed rigor can easily be supplied.

Furthermore, we assume that the reader is familiar with the representation theory of the rotation group as given in [3] or [7].

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1. Resumé of the Representation Theory of $\mathscr{E}_3$

The Euclidean group in three-space may be defined as the set of pairs $\{\mathbf{a}, R\}$, where $\mathbf{a}$ is a real three-vector and R an element of the real proper orthogonal group in three dimensions, i.e., a real 3×3 matrix such that $RR^t = I$ and $\det R = 1$, [1], [2]. The multiplication law is

$$\{\mathbf{a}_1, R_1\}\{\mathbf{a}_2, R_2\} = \{\mathbf{a}_1 + R_1\mathbf{a}_2, R_1R_2\}. \tag{1.1}$$

In this paper we are primarily concerned with $\mathscr{E}_3$, the simply connected covering group of the Euclidean group. $\mathscr{E}_3$ may be defined as the set of pairs $\{\mathbf{a}, A\}$ where $\mathbf{a}$ is a real three-vector and A is an element of $SU(2)$, the group of 2×2 unitary matrices with determinant $+1$, [2]. The matrices $\pm A$ determine the same rotation $R(A)$ given by

$$A\mathbf{x} \cdot \boldsymbol{\sigma} A^* = (R(A)\mathbf{x}) \cdot \boldsymbol{\sigma}, \tag{1.2}$$

where $\boldsymbol{\sigma}$ stands for the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

The multiplication law for $\mathscr{E}_3$ is

$$\{\mathbf{a}_1, A_1\}\{\mathbf{a}_2, A_2\} = \{\mathbf{a}_1 + R(A_1)\mathbf{a}_2, A_1A_2\}. \tag{1.4}$$

In the following we shall usually write $A\mathbf{a}$ instead of $R(A)\mathbf{a}$.

This paper is concerned with calculations involving the unitary irreducible representations of $\mathscr{E}_3$ which act non-trivially on the translation subgroup of $\mathscr{E}_3$. We shall list these representations in a form suitable for computation.

For every vector

$$\mathbf{p} = (\rho \sin\theta \cos\varphi, \rho \sin\theta \sin\varphi, \rho\cos\theta), \quad 0 < \rho, 0 \leqq \theta \leqq \pi, 0 \leqq \varphi < 2\pi, \tag{1.5}$$

where ρ, θ, φ are spherical coordinates, we define the matrix $h(\mathbf{p})$ by

$$h(\mathbf{p}) = \begin{pmatrix} \cos\theta/2\, e^{i\varphi} & -i\sin\theta/2\, e^{i\varphi} \\ -i\sin\theta/2\, e^{-i\varphi} & \cos\theta/2\, e^{-i\varphi} \end{pmatrix}. \tag{1.6}$$

From (1.2) it follows that

$$R(h(\mathbf{p})) = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{1.7}$$

Setting $\hat{\mathbf{p}} = (0, 0, \rho)$ we have the relation $R(h(\mathbf{p}))\hat{\mathbf{p}} = \mathbf{p}$. For $A \in SU(2)$ the matrix

$$Q(\mathbf{p}, A) = h^{-1}(\mathbf{p})Ah(A^{-1}\mathbf{p}) \tag{1.8}$$

has the property $R(Q(\mathbf{p}, A)\hat{\mathbf{p}} = \hat{\mathbf{p}}$. Thus, $Q(\mathbf{p}, A)$ can be taken to be an element of $\mathscr{R}_2$, the two-sheeted covering group of the group of rotations about the z-axis. $\mathscr{R}_2$ is isomorphic to the multiplicative group of the complex numbers $e^{i\chi/2}$, $0 \leqq \chi \leqq 4\pi$. The unitary irreducible representations of $\mathscr{R}_2$ are one-dimensional and of the form

$$B_m : \chi \to e^{-im\chi}, \qquad 0 \leqq \chi < 4\pi, \tag{1.9}$$

where m can take the values $0, \pm\frac{1}{2}, \pm 1, \cdots$.

Recall that every unitary irreducible representation of $\mathscr{E}_3$ which acts trivially on the translation subgroup of $\mathscr{E}_3$, i.e., which maps this subgroup into the identity operator, is uniquely determined by an irreducible representation $D(l)$ of $SU(2)$. Here $2l = 0, 1, 2, \cdots$ and $\dim D(l) = 2l + 1$.

THEOREM 1. *Let ρ be a real positive number and M the manifold of all 3-vectors $\mathbf{p}$ such that $\mathbf{p} \cdot \mathbf{p} = \rho^2$. Denote by $H(\rho, m)$ the Hilbert space of Lebesgue square-integrable functions on M with inner product*

$$\langle f, g \rangle = \int_0^{2\pi} \int_0^{\pi} f(\mathbf{p}) g(\mathbf{p}) \sin\theta \, d\theta \, d\varphi, \qquad f, g \in H(\rho, m), \tag{1.10}$$

where $\mathbf{p}$ is given by (1.5) *and $2m$ is an integer. Then, the unitary representation U of $\mathscr{E}_3$ defined on $H(\rho, m)$ by*

$$[U(\mathbf{a}, A)f](\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{a}} B_m[h^{-1}(\mathbf{p}) A h(A^{-1}\mathbf{p})] f(A^{-1}\mathbf{p}) \tag{1.11}$$

is irreducible.

Every continuous unitary representation of $\mathscr{E}_3$ which acts non-trivially on the translation subgroup is unitary equivalent to a representation of the form (1.11) *for some choice of the constants ρ, m. Two such representations U_1 and U_2 are unitary equivalent if and only if $\rho_1 = \rho_2$ and $m_1 = m_2$. Thus, the representations are uniquely determined by the pair of numbers (ρ, m) where $\rho > 0$, $2m = 0, \pm 1, \pm 2, \cdots$.*

The proof of this theorem is well known, [2]. The matrices $h(\mathbf{p})$ were chosen in the form (1.6) to simplify the computations to follow.

Let U be an irreducible unitary representation of $\mathscr{E}_3$ on a Hilbert space H which is unitary equivalent to a representation (ρ, m) listed in Theorem 1. As is well known, under the restriction of U to the compact subgroup $SU(2)$ of $\mathscr{E}_3$, H breaks up into a direct sum of subspaces R_l, $2l = 0, 1, 2, \cdots$. $U/SU(2)$ leaves each R_l invariant and is unitary equivalent in $\mathscr{R}_l$ to a multiple of the irreducible representation $D(l)$ of $SU(2)$. Thus,

$$U/SU(2) \cong \sum_{2l=0}^{\infty} n_l D(l),$$

where the n_l are non-negative integers. In Section 4 we shall show that $n_l = 1$ for $l = |m|, |m| + 1, \cdots$, while $n_l = 0$ for all other values of l. Hence,

$$U/SU(2) \cong \sum_{l=|m|}^{\infty} D(l). \tag{1.12}$$

2. The Representation Theory of E_3

E_3, the Lie algebra of $\mathscr{E}_3$, is generated by the six elements $\mathscr{P}_i, \mathscr{J}_i, i = 1, 2, 3$, with commutation relations

$$(2.1)\qquad \begin{aligned}&[\mathscr{J}_i, \mathscr{J}_j] = \varepsilon_{ijk}\mathscr{J}_k, \quad [\mathscr{J}_i, \mathscr{P}_j] = \varepsilon_{ijk}\mathscr{P}_k,\\ &[\mathscr{P}_i, \mathscr{P}_j] = 0, \qquad i, j, k = 1, 2, 3,\end{aligned}$$

where ε_{ijk} is the completely anti-symmetric tensor such that $\varepsilon_{123} = +1$. The $\mathscr{J}_i$ generate the Lie algebra of $SU(2)$; the $\mathscr{P}_i$ generate the commutative Lie algebra of the translation subgroup of $\mathscr{E}_3$.

Consider a complex Hilbert space H and a representation ν of E_3 in terms of linear operators on H. Set

$$(2.2)\qquad \nu(\mathscr{P}) = P_k, \quad \nu(\mathscr{J}_k) = J_k, \qquad k = 1, 2, 3,$$

and define the operators $P^+, P^-, P^3, J^+, J^-, J^3$ on H by means of

$$(2.3)\qquad \begin{aligned}&J^{\pm} = \mp J_2 + iJ_1, \quad J^3 = iJ_3,\\ &P^{\pm} = \mp P_2 + iP_1, \quad P^3 = iP_3, \qquad i = \sqrt{-1}.\end{aligned}$$

Then we have

$$(2.4)\qquad \begin{aligned}&[J^3, J^{\pm}] = \pm J^{\pm}, \quad [J^3, P^{\pm}] = [P^3, J^{\pm}] = \pm P^{\pm},\\ &[J^+, P^+] = [J^-, P^-] = [J^3, P^3] = 0,\\ &[J^+, J^-] = 2J^3, \quad [J^+, P^-] = [P^+, J^-] = 2P^3,\end{aligned}$$

where $[A, B] = AB - BA$ for any two linear operators A and B on H. In general the operators (2.3) are unbounded and the commutation relations (2.4) are not everywhere defined on H.

As is well known, a continuous unitary representation U of $\mathscr{E}_3$ induces a representation ν of E_3. Further, U is irreducible if and only if ν is irreducible. Using these facts we can describe the irreducible representations of E_3 which are induced by the irreducible representations (ρ, m) of $\mathscr{E}_3$.

Theorem 2. *Let U be an irreducible unitary representation of $\mathscr{E}_3$ on a Hilbert space H, unitary equivalent to a representation (ρ, m) listed in Theorem 1. The irreducible representation ν of E_3 induced by U acts on H as follows: H has an ortho-normal basis consisting of the unit vectors $f_k^{(l)}$, $l = |m|, |m| + 1, \cdots, k = -l, -l + 1, \cdots, +l$. In terms of this basis we have the relations*

$$(2.5)\qquad \begin{aligned}&J^3 f_k^{(l)} = k f_k^{(l)}, \quad J^+ f_k^{(l)} = [(l + k + 1)(l - k)]^{1/2} f_{k+1}^{(l)},\\ &J^- f_k^{(l)} = [(l + k)(l - k + 1)]^{1/2} f_{k-1}^{(l)},\end{aligned}$$

$$(2.6)\qquad \begin{aligned}P^3 f_k^{(l)} = &-\rho\left[\frac{(l + m + 1)(l - m + 1)(l + k + 1)(l - k + 1)}{(l + 1)^2(2l + 3)(2l + 1)}\right]^{1/2} f_k^{(l+1)}\\ &-\frac{k\rho m}{l(l + 1)} f_k^{(l)} - \rho\left[\frac{(l + m)(l - m)(l + k)(l - k)}{l^2(2l + 1)(2l - 1)}\right]^{1/2} f_k^{(l-1)},\end{aligned}$$

$$P^{+}f_k^{(l)} = +\rho\left[\frac{(l+m+1)(l-m+1)(l+k+1)(l+k+2)}{(l+1)^2(2l+3)(2l+1)}\right]^{1/2} f_{k+1}^{(l+1)}$$

$$-[(l+k+1)(l-k)]^{1/2}\frac{m\rho}{l(l+1)} f_{k+1}^{(l)} \tag{2.7}$$

$$-\rho\left[\frac{(l+m)(l-m)(l-k)(l-k-1)}{l^2(2l+1)(2l-1)}\right]^{1/2} f_{k+1}^{(l-1)},$$

$$P^{-}f_k^{(l)} = -\rho\left[\frac{(l+m+1)(l-m+1)(l-k+2)(l-k+1)}{(l+1)^2(2l+3)(2l+1)}\right]^{1/2} f_{k-1}^{(l+1)}$$

$$-[(l+k)(l-k+1)]^{1/2}\frac{m\rho}{l(l+1)} f_{k-1}^{(l)} \tag{2.8}$$

$$+\rho\left[\frac{(l+m)(l-m)(l+k)(l+k-1)}{l^2(2l+1)(2l-1)}\right]^{1/2} f_{k-1}^{(l-1)},$$

$$P\cdot Pf_k^{(l)} = \sum_{i=1}^{3} P_iP_if_k^{(l)} = -\rho^2 f_k^{(l)}, \tag{2.9}$$

$$P\cdot Jf_k^{(l)} = \sum_{i=1}^{3} P_iJ_if_k^{(l)} = m\rho f_k^{(l)}. \tag{2.10}$$

We do not give a detailed proof of this theorem since very similar computations are carried out for the Lorentz group in [3] and [4]. The modification of these computations for the Euclidean group is simple. The basic idea of the proof comes from the remarks immediately following Theorem 1. From formula (1.12) and a knowledge of the representation theory of $SU(2)$, we can find an orthonormal basis $f_k^{(l)}$, $l = |m|, |m|+1, \cdots, -l \leqq k \leqq l$, for H such that equations (2.5) hold. Expressions (2.6) are derived by algebraic manipulations from the commutation relations (2.4).

3. Special Functions Associated with $\mathscr{E}_3$

We shall now use the results of Theorem 2 to compute the basis eigenvectors $f_k^{(l)}$ of the irreducible representations of $\mathscr{E}_3$ listed in Theorem 1. Given an irreducible unitary representation (ρ, m) in the form (1.11), we can easily determine the operators $J^+, J^-, J^3, P^+, P^-, P^3$ corresponding to the induced irreducible representation of E^3, [3], [5]. The results are

$$\begin{aligned} J^{\pm} &= e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + \frac{m}{\sin\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \\ J^3 &= -i\frac{\partial}{\partial\phi}, \\ P^{\pm} &= -\rho\sin\theta e^{\pm i\varphi}, \qquad P^3 = -\rho\cos\theta. \end{aligned} \tag{3.1}$$

The operators (3.1) satisfy the commutation relations (2.4). The basis eigenvectors $f_k^{(l)}$ will be functions of θ and φ. In fact, because of the relation $J^3 f_k^{(l)} = k f_k^{(l)}$ we can set

$$f_k^{(l)}(\mathbf{p}) = Q_{m,k}^l(\theta) e^{ik\varphi}, \qquad l = |m|, |m|+1, \cdots, \qquad k = -l, -l+1, \cdots, +l.$$

Using equations (2.5)–(2.8) and (3.1) we can determine the functions $Q_{m,k}^l(\theta)$. From (2.5) we see that $J^+ f_l^{(l)} = 0$. Hence, $Q_{m,l}^l(\theta)$ satisfies the equation

$$\left(\frac{d}{d\theta} + \frac{m}{\sin\theta} - k\cot\theta\right) Q_{m,l}^l(\theta) = 0. \tag{3.2}$$

The solution of (3.2) is

$$Q_{m,l}^l(\theta) = c_l \left[\sin\frac{\theta}{2}\right]^{l-m} \left[\cos\frac{\theta}{2}\right]^{l+m}, \tag{3.3}$$

where c_l is a constant. From equations (2.5) it follows that

$$Q_{m,l-h}^l(\theta) e^{i(l-h)\varphi} = \frac{1}{\prod_{s=l-h+1}^{l} [(l+s)(l-s+1)]^{1/2}} (J^-)^h [Q_{m,l}^l(\theta) e^{il\varphi}]. \tag{3.4}$$

Using (3.1) and induction on h we have

$$\begin{aligned} Q_{m,k}^l(\theta) &= 2^{-l} c_l \left[\frac{(l+k)!}{(2l)!\,(l-k)!}\right]^{1/2} (1-\cos\theta)^{-(k-m)/2} (1+\cos\theta)^{-(k+m)/2} \\ &\quad \times \frac{\partial^{l-k}}{\partial(\cos\theta)^{l-k}} [(1-\cos\theta)^{l-m} (1+\cos\theta)^{l+m}], \\ &\qquad l = |m|, |m|+1, \cdots, \qquad k = -l, -l+1, \cdots, +l. \end{aligned} \tag{3.5}$$

The constants c_l are determined subject to the requirements that

$$\int_0^{2\pi}\int_0^{\pi} |Q_{m,k}^l(\theta)|^2 \sin\theta \, d\theta \, d\varphi = 1, \qquad m, k = -l, -l+1, \cdots, +l, \tag{3.6}$$

and that relations (2.6)–(2.8) are satisfied.

Setting $k = l$ in (3.6) we have

$$8\pi |c_l|^2 \int_0^{\pi/2} \left[\sin\frac{\theta}{2}\right]^{2l-2m+1} \left[\cos\frac{\theta}{2}\right]^{2l+2m+1} d\left(\frac{\theta}{2}\right) = 1. \tag{3.7}$$

We can evaluate this integral to obtain

$$|c_l| = \left[\frac{(2l+1)!}{4\pi(l-m)!\,(l+m)!}\right]^{1/2}. \tag{3.8}$$

Equation (2.7) yields the relation

$$\sin\theta \, Q_{m,l}^l(\theta) = -\left[\frac{(l+m+1)(l-m+1)(2)}{(2l+3)(l+1)}\right]^{1/2} Q_{m,l+1}^{l+1} \tag{3.9}$$

Using (3.3) and (3.9) we can calculate the ratio between c_l and c_{l+1}. The result is that we can set

$$c_l = (-1)^l\left[\frac{(2l+1)!}{4\pi(l-m)!\,(l+m)!}\right]^{1/2}.$$

Thus

$$\text{(3.10)}\quad \begin{aligned} Q^l_{m,k} &= \frac{(-1)^l}{2^{l+1}}\left[\frac{(l+k)!\,(2l+1)}{\pi(l-m)!\,(l+m)!\,(l-k)!}\right]^{1/2}(1-\cos\theta)^{(m-k)/2} \\ &\quad\times (1+\cos\theta)^{-(m+k)/2}\frac{d^{l-k}}{d(\cos\theta)^{l-k}}\left[(1-\cos\theta)^{l-m}(1+\cos\theta)^{l+m}\right]. \end{aligned}$$

Note that $Q^l_{0,k}(\theta)e^{-ik\varphi} = Y_{l,k}(\theta)$, where the $Y_{l,k}(\theta)$ are spherical harmonics, [3].

The functions $Q^l_{m,k}(\theta)$ are closely related to the generalized spherical functions, [3]. It is shown in [3] that the matrix elements of the irreducible unitary representation $D(l)$ of $SU(2)$, parametrized by the Euler angles φ_1, θ, φ_2, are given by

$$\text{(3.11)}\quad \begin{aligned} T^l_{m,k}(\varphi_1,\theta,\varphi_2) &= e^{-im\varphi_1}P^l_{m,k}(\theta)e^{-ik\varphi_2}, \\ &\qquad m, k = -l, -l+1, \cdots, l, \\ 0 \leqq \varphi_1 < 4\pi,\ &0 \leqq \theta < \pi,\ 0 \leqq \varphi_2 \leqq 2\pi. \end{aligned}$$

From the fact that these T matrices are unitary and form an irreducible representation of $SU(2)$, one can derive the following familiar relations:

$$\text{(3.12)}\quad P^l_{m,k}(\theta) = P^l_{k,m}(\theta) = P^l_{-m,-k}(\theta),$$

$$\text{(3.13)}\quad \begin{aligned} P^{j_1}_{m_1,k_1}(\theta)P^{j_2}_{m_2,k_2}(\theta) &= \sum_j (j_1j_2m_1m_2\,|\,j, m_1+m_2)(j_1j_2k_1k_2\,|\,j, k_1+k_2) \\ &\qquad \times P^j_{m_1+m_2,k_1+k_2}(\theta), \end{aligned}$$

where the quantities $(j_1j_2m_1m_2\,|\,j, m_1+m_2)$ are Clebsch-Gordon coefficients, [1]. The P and Q functions are related by the simple formula

$$\text{(3.14)}\quad P^l_{m,k}(\theta) = e^{i\pi(k+m)/2}\sqrt{\frac{4\pi}{2l+1}}\,Q^l_{m,k}(\theta).$$

Returning to the problem of calculating the basis eigenvector of the representation (ρ, m) given in Theorem 1, we have

$$\text{(3.15)}\quad f^{(l)}_k(\mathbf{p}) = Q^l_{m,k}(\theta)e^{ik\varphi}, \quad \begin{aligned} l &= |m|, |m|+1, \cdots, \\ k &= -l, -l+1, \cdots, +l, \end{aligned}$$

where $Q^l_{m,k}(\theta)$ is given by (3.10). By substituting (3.15) and (3.1) into equations (2.5)–(2.8) the reader can derive a series of recursion relations satisfied by the functions $Q^l_{m,k}(\theta)$. We omit this simple exercise.

By definition of the eigenvector $f^{(l)}_k$, the relation

$$\text{(3.16)}\quad [U(\mathbf{0}, A)f^{(l)}_k](\mathbf{p}) = \sum_{k'=-l}^{l} T^l_{k,k'}(A)f^{(l)}_k(\mathbf{p})$$

holds, where the T matrices are given by (3.11) when $SU(2)$ is parametrized by the Eulerian angles. Thus, the matrix elements of $U(\mathbf{0}, A)$ are

$$\langle f_{k'}^{(l')}, U(\mathbf{0}, A) f_k^{(l)} \rangle = T_{k,k'}^l(A)\, \delta_{ll'}\,, \tag{3.17}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $H(\rho, m)$.

From (1.11) it follows that the matrix elements of $U(\mathbf{r}, I)$ are given by

$$\begin{aligned} \langle f_{k'}^{(l)}, U(\mathbf{r}, I) f_k^l \rangle &= [l', k' \,|\rho, m|\, l, k](\mathbf{r}) \\ &= \int_0^{2\pi}\int_0^{\pi} Q_{m,k'}^{l'}(\theta) e^{ik'\varphi} e^{i\mathbf{p}\cdot\mathbf{r}} Q_{m,k}^l(\theta) e^{ik\varphi} \sin\theta \, d\theta\, d\varphi\,. \end{aligned} \tag{3.18}$$

Writing $\mathbf{r}$ in spherical coordinates,

$$\mathbf{r} = (r \sin\theta_r \cos\varphi_r,\, r\sin\theta_r \sin\varphi_r,\, r\cos\theta_r)\,,$$

we make use of the well-known formula

$$e^{i\mathbf{p}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{\nu=-l}^{l} j_l(\rho r)\, \overline{Y}_{l,\nu}(\theta_r, \varphi_r) Y_{l,\nu}(\theta, \varphi)\,, \tag{3.19}$$

where the j_l are spherical Bessel functions. Substituting (3.19) into the last equation in (3.18), applying (3.13) and simplifying, we obtain

$$\begin{aligned} [l', k' \,|\rho, m|\, l, k](\mathbf{r}) &= \sqrt{4\pi} \sum_{h=0}^{\infty} \left[\frac{(2h+1)(2l'+1)}{2l+1}\right]^{1/2} \\ &\times e^{i\pi h/2} j_h(\rho r) Y_{h,k-k'}(\theta_r, \varphi_r)(h, l', 0, m \mid l, m)(h, l', k-k', k' \mid l, k)\,. \end{aligned} \tag{3.20}$$

The fact that the U operators form a unitary representation of $\mathscr{E}_3$ allows us to derive a number of relations satisfied by the functions (3.20). Thus, $U^*(\mathbf{r}, I) = U(-\mathbf{r}, I)$ implies

$$[l', k' \,|\rho, m|\, l, k](-\mathbf{r}) = \overline{[l, k \,|\rho, m|\, l', k'](\mathbf{r})}\,. \tag{3.21}$$

Further, the group property

$$U(\mathbf{r}_1, I) U(\mathbf{r}_2, I) = U(\mathbf{r}_1 + \mathbf{r}_2, I)$$

leads to

$$\begin{aligned} &[l', k' \,|\rho, m|\, l, k](\mathbf{r}_1 + \mathbf{r}_2) \\ &\quad = \sum_{n=|m|}^{\infty} \sum_{\nu=-n}^{n} [l', k' \,|\rho, m|\, n, \nu](\mathbf{r}_1)[n, \nu \,|\rho, m|\, l, k](\mathbf{r}_2)\,. \end{aligned} \tag{3.22}$$

When $m = l' = k' = 0$, (3.22) reduces to the addition theorem for spherical waves

$$\begin{aligned} j_l(\rho r) Y_{l,k}(\theta_r, \varphi_r) &= \sum_{n=0}^{\infty} \sum_{\nu=-n}^{n} \sum_{h=0}^{\infty} \sqrt{4\pi}\, (i)^{n+h-l} \left|\frac{(2h+1)(2n+1)}{2l+1}\right|^{1/2} \\ &\times j_n(\rho r_1) j_h(\rho r_2) Y_{n,\nu}(\theta_{r_1}, \varphi_{r_1}) Y_{h,k-\nu}(\theta_{r_2}, \varphi_{r_2}) \\ &\times (h, n, 0, 0 \mid l, 0)(h, n, k-\nu, \nu \mid l, k) \end{aligned} \tag{3.23}$$

which was first derived by Friedman and Russek, [6]. Here, $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$.

The group property

$$U(\mathbf{r}, A) = U(\mathbf{0}, A)U(A^{-1}\mathbf{r}, I) = U(\mathbf{r}, I)U(\mathbf{0}, A)$$

leads to the relations

$$\sum_{\nu'=-l}^{l} T^l_{k,\nu'}(A)[l, \nu' \,|\rho, m|\, j, \nu](A^{-1}\mathbf{r}) = \sum_{\mu'=-j}^{j} [l, k \,|\rho, m|\, j, \mu'](\mathbf{r})\, T^j_{\mu'\nu}(A) \,. \tag{3.24}$$

Fix l and consider the $2l + 1$ component quantity

$$\chi^{(\rho,m)}_{l;j\nu}(\mathbf{r}) = ([l, s \,|\rho, m|\, j, \nu](\mathbf{r})), \qquad s = -l, -l+1, \cdots, +l, \tag{3.25}$$

for some j, ν. Define the action W of $\mathscr{E}_3$ on $\chi(\mathbf{r})$ by (in matrix notation)

$$[W(\mathbf{a}, A)\chi^{(\rho,m)}_{l;j\nu}](\mathbf{r}) = T^l(A)\chi^{(\rho,m)}_{l;j,\nu}(A^{-1}(\mathbf{r} + \mathbf{a})) \,. \tag{3.26}$$

Then, (3.24) shows that $\chi(\mathbf{r})$ is a spinor field of weight l and that in fact under the action of $SU(2)$ it transforms like the eigenvector $f^{(j)}_\nu$ of the irreducible representation $D(j)$. Further, (3.22) shows that

$$[W(\mathbf{a}, I)\chi^{(\rho,m)}_{l;j\nu}](\mathbf{r}) = \sum_{n=|m|}^{\infty} \sum_{s=-n}^{\infty} [n, s \,|\rho, m|\, j, \nu](\mathbf{a})\,\chi^{(\rho,m)}_{l;n,s}(\mathbf{r}) \,. \tag{3.27}$$

Let $M(\rho, m, l)$ be the complex linear manifold generated by the l-spinor functions $\chi^{(\rho,m)}_{l;j,\nu}$, $j = |m|, |m| + 1, \cdots, \nu = -j, -j + 1, \cdots, +j$. We can uniquely define an inner product $\langle .\,, .\rangle$ on $M(\rho, m, l)$, linear in the second argument, conjugate linear in the first, by requiring that

$$\langle \chi^{(\rho,m)}_{l;j_1,\nu_1}, \chi^{(\rho,m)}_{l;j_2,\nu_2} \rangle = \delta_{j_1 j_2}\delta_{\nu_1\nu_2} \tag{3.28}$$

for all admissible j_1, j_2, ν_1, ν_2. Completing $M(\rho, m, l)$ with respect to $\langle .\,, .\rangle$ we obtain the Hilbert space $H(\rho, m, l)$. Denote again by W the action of $\mathscr{E}_3$ on $H(\rho, m, l)$ induced by (3.26). Then, it follows easily from (3.21)–(3.24) that W is a unitary irreducible representation of $\mathscr{E}_3$ on $H(\rho, m, l)$, unitary equivalent to the representation (ρ, m). In fact, the unitary equivalence maps the basis vector $f^{(j)}_\nu$ of (ρ, m) into the l-spinor $\chi^{(\rho,m)}_{l;j,\nu}(\mathbf{r})$.

Note that we can construct a representation of $\mathscr{E}_3$ unitary equivalent to (ρ, m) for each value of $l = |m|, |m| + 1, \cdots$.

Fixing l again, we observe that the action of $\mathscr{E}_3$ on $H(\rho, m, l)$ induces an irreducible representation of E_3. The infinitesimal operators corresponding to this representation are

$$\begin{aligned}
J^\pm &= e^{\pm i\varphi}\left[\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi}\right] + S^\pm \,, \\
J^3 &= -i\frac{\partial}{\partial\varphi} + S^3 \,, \\
P^\pm &= e^{\pm i\varphi}\left[i \sin\theta \frac{\partial}{\partial r} + i\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \mp \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right] , \\
P^3 &= i\cos\theta\frac{\partial}{\partial r} - i\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \,,
\end{aligned} \tag{3.29}$$

where $\mathbf{r} = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta)$ and

$$S^{\pm}[l, k \,|\rho, m|\, j, \nu] = [(l \pm k)(l \mp k + 1)]^{1/2}[l, k \mp 1 \,|\rho, m|\, j, \nu]\,, \tag{3 30}$$
$$S^{3}[l, k \,|\rho, m|\, j, \nu] = k[l, k \,|\rho, m|\, j, \nu]\,.$$

By replacing the eigenvectors $f_\nu^{(j)}$ with the l-spinors $\chi_{l;j,\nu}^{(\rho,m)}$ and substituting the expressions (3.29) into equations (2.5)–(2.8) the reader can easily obtain a series of recursion relations for the matrix elements $[l, k \,|\rho, m|\, j, \nu](\mathbf{r})$. Note also that (2.9) yields the equation

$$(\nabla^2 + \rho^2)[l, k \,|\rho, m|\, j, \nu](\mathbf{r}) = 0\,, \tag{3.31}$$

where ∇^2 is the Laplacian.

4. Completeness and Orthogonality Relations

Because of the group relation $U(\mathbf{r}, A) = U(\mathbf{r}, I)U(\mathbf{0}, A)$ it follows easily that the general matrix element $\langle f_s^{(h)}, U(\mathbf{r}, A) f_\nu^{(j)}\rangle$ corresponding to the representation (ρ, m), is given by

$$\begin{aligned}\langle f_s^{(h)}, U(\mathbf{r}, A) f_k^{(j)}\rangle &= \{h, s \,|\rho, m|\, j, k\}(\mathbf{r}, A)\\ &= \sum_{\nu=-j}^{j} [h, s \,|\rho, m|\, j, \nu](\mathbf{r})\, T^{j}_{\nu,k}(A)\,.\end{aligned} \tag{4.1}$$

It is well known that the matrix elements $T^{j}_{\nu,k}(A)$ defined for $j = \frac{1}{2}, 1, \frac{3}{2}, \cdots, \nu, k = -j, -j+1, \cdots, +j$ satisfy the orthogonality relations

$$\int_{SU(2)} \overline{T^{j_1}_{\nu_1,k_1}(A)}\, T^{j_2}_{\nu_2,k_2}(A)\, dA = \frac{\delta_{j_1 j_2}\delta_{\nu_1\nu_2}\delta_{k_1k_2}}{2j_1 + 1}\,, \tag{4.2}$$

where dA is the Haar measure on $SU(2)$, normalized so that the volume of $SU(2)$ is 1, [7]. We shall give a similar set of orthogonality relations for the matrix elements (4.1).

The Haar measure on $\mathscr{E}_3$ is given by $dA\, d^3\mathbf{r}$, where $d^3\mathbf{r}$ is the Haar measure on the translation subgroup of $\mathscr{E}^3$. The following orthogonality relations hold:

$$\begin{aligned}&\iiint_{-\infty}^{\infty} d^3\mathbf{r} \int_{SU(2)} dA \{h_1, s_1 \,|\rho_1, m_1|\, j_1 k_1\}(\mathbf{r}, A)\{h_2, s_2 \,|\rho_2, m_2|\, j_2, k_2\}(\mathbf{r}, A)\\ &\qquad = 4\pi^2 \delta_{h_1h_2}\delta_{s_1s_2}\delta_{j_1j_2}\delta_{k_1k_2}\delta_{m_1m_2} \frac{\delta(\rho_1 - \rho_2)}{\rho_1^2}\,,\end{aligned} \tag{4.3}$$

where $h_i, j_i = |m_i|, |m_i| + 1, \cdots, \quad s_i = -h_i, -h_i + 1, \cdots, +h_i, \quad k_i = -j_i, -j_i + 1, \cdots, +j_i, \rho_i > 0, i = 1, 2$. One obtains this result by substituting the expressions (4.1) and (3.18) for the matrix elements in (4.3) and by making use of (4.2) and the Fourier integral theorem.

Another well known property of the matrix elements $T^j_{\nu,k}(A)$ is that they form a complete orthogonal basis for the Hilbert space of functions defined on $SU(2)$ and square integrable with respect to the Haar measure dA, [7]. The completeness relations satisfied by the functions $\{h, s\,|\rho, m|\,j, k\}(\mathbf{r}, A)$ are somewhat more complicated.

Let H be the Hilbert space of complex functions $f(\mathbf{r}, A)$ defined on $\mathscr{E}_3$ and square integrable with respect to the Haar measure $dA\, d^3\mathbf{r}$. The inner product is given by

$$\langle f, g\rangle = \int_{\mathscr{E}_3} \overline{f(\mathbf{r}, A)}\, g(\mathbf{r}, A)\, dA\, d^3\mathbf{r}, \qquad f, g \in H. \tag{4.4}$$

THEOREM 3. *Let $f \in H$. Then*

$$\begin{aligned} f(\mathbf{r}, A) = \sum_{2n=-\infty}^{\infty} \sum_{j=|n|}^{\infty} \sum_{h=|n|}^{\infty} \sum_{\nu=-l}^{l} \sum_{m=-j}^{j} \int_0^\infty \rho^2\, d\rho\, b^n_{j,m;l,\nu}(\rho) \\ \times \{j, m\,|\rho, n|\, l, \nu\}(\mathbf{r}, A)\,, \end{aligned} \tag{4.5}$$

where the coefficients $b(\rho)$ are given by

$$b^n_{j,m;l,\nu}(\rho) = \frac{1}{4\pi^2} \iint_{\mathscr{E}_3} \overline{\{j, m\,|\rho, n|\, l, \nu\}(\mathbf{r}, A)} f(\mathbf{r}, A) dA\, d^3\mathbf{r}\,. \tag{4.6}$$

Furthermore,

$$\begin{aligned} &\frac{1}{4\pi^2} \iint |f(\mathbf{r}, A)|^2\, dA\, d^3\mathbf{r} \\ &\quad = \sum_{2n=-\infty}^{\infty} \sum_{j=|n|}^{\infty} \sum_{h=|n|}^{\infty} \sum_{\nu=-l}^{l} \sum_{m=-j}^{j} \int_0^\infty \rho^2\, d\rho\, |b^n_{j,m;l,\nu}(\rho)|^2 < \infty\,. \end{aligned} \tag{4.7}$$

The convergence in expressions (4.5) *and* (4.6) *is in the mean.*

Proof: We give an outline of the proof, leaving the measure theoretic and convergence details to the reader. We define a representation U of $\mathscr{E}_3$ on H, the left-regular representation, by means of the formula

$$\begin{aligned} [U(\mathbf{a}, A)f](\mathbf{r}, B) = f(A^{-1}(\mathbf{r} + \mathbf{a}), A^{-1}B)\,, \\ f \in H,\ A, B \in SU(2)\,. \end{aligned} \tag{4.8}$$

From the definition of the inner product (4.4) it is easily checked that U is unitary.

Now consider the Hilbert space H', an exact copy of H. On H' we define the unitary representation U' of $\mathscr{E}_3$:

$$\begin{aligned} [U'(\mathbf{a}, A)g](\mathbf{p}, B) = e^{i\mathbf{p}\cdot A^{-1}\mathbf{a}} g(A^{-1}\mathbf{p}, A^{-1}B)\,, \\ g \in H',\ A, B \in SU(2)\,, \end{aligned} \tag{4.9}$$

The representations U and U' are unitary equivalent. In fact, the mapping

$$V : H \to H'$$

defined by

$$(VF)(\mathbf{p}, B) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} e^{-i\mathbf{p}\cdot\mathbf{r}} f(\mathbf{r}, B) d^3\mathbf{r}\,, \qquad f \in H\,, \tag{4.10}$$

is a unitary transformation of H onto H' such that $U(\mathbf{a}, A) = V^{-1}U'(\mathbf{a}, A)V$ for all $(\mathbf{a}, A) \in \mathscr{E}_3$. Recall that V^{-1} is given by

$$[V^{-1}g](\mathbf{r}, B) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} e^{+i\mathbf{r}\cdot\mathbf{p}} g(\mathbf{p}, B) d^3\mathbf{p}\,, \qquad g \in H'. \tag{4.11}$$

Under the restriction of U' to the compact subgroup $SU(2)$, H' splits into a direct sum of invariant subspaces, each subspace transforming according to an irreducible representation $D(j)$ of $SU(2)$. We shall exhibit this decomposition explicitly.

Let R be a subspace of H' transforming according to the representation $D(j)$. Then, there is a $(2j+1)$-dimensional basis $X_m^{(j)}(\mathbf{p}, B)$, $m = -j, -j+1, \cdots, j$, for R such that the action of $SU(2)$ on this basis is given by

$$\begin{aligned} [U(\mathbf{0}, A)X_m^{(j)}](\mathbf{p}, B) &= X_m^{(j)}(A^{-1}\mathbf{p}, A^{-1}B) \\ &= \sum_{n=-j}^{j} T^j_{m,n}(A^{-1}) \cdot X_n^{(j)}(\mathbf{p}, B)\,. \end{aligned} \tag{4.12}$$

As in Section 1, given $\mathbf{p}$ we define the positive number ρ by $\mathbf{p}\cdot\mathbf{p} = \rho^2$ and set $\hat{\mathbf{p}} = (0, 0, \rho)$. (From now on we assume $\mathbf{p} \neq 0$ since the set of points for which $\mathbf{p} = 0$ is of measure zero.) We recall that the matrix $h(\mathbf{p}) \in SU(2)$ defined by (1.6) has the property that $h(\mathbf{p})\hat{\mathbf{p}} = \mathbf{p}$. Using these facts, we see that (4.12) can be written in the form

$$X_m^{(j)}(\mathbf{p}, B) = \sum_{n=-j}^{j} T^j_{m,n}(h(\mathbf{p}))X_n^{(j)}(\hat{\mathbf{p}}, h^{-1}(\mathbf{p})B)\,, \qquad B \in SU(2)\,. \tag{4.13}$$

Thus, the functions $X_m^{(j)}$ are completely determined by their values on the manifold consisting of the points $(\hat{\mathbf{p}}, B)$, $B \in SU(2)$. Since the functions $T^j_{n,m}$ form a complete orthogonal basis for functions on $SU(2)$ square integrable with respect to Haar measure, we have the expansion

$$X_m^{(j)}(\hat{\mathbf{p}}, B) = \sum_{2l=0}^{\infty} \sum_{k=-l}^{l} \sum_{\nu=-l}^{l} \alpha^l_{k,\nu}(\rho) T^l_{k,\nu}(B), \tag{4.14}$$

where the $\alpha^l_{k,\nu}$ are functions of ρ. Choose $C \in SU(2)$ such that $C\hat{\mathbf{p}} = \hat{\mathbf{p}}$. It is easily shown that $T^l_{n,m}(C) = \delta_{nm}e^{-im\varphi}$ for some φ such that $0 \leqq \varphi < 4\pi$, [3]. From (4.12) we obtain the relation

$$X_m^{(j)}(\hat{\mathbf{p}}, C^{-1}B) = e^{+im\varphi}X_m^{(j)}(\hat{\mathbf{p}}, B)\,. \tag{4.15}$$

Comparing (4.15) with (4.14) we conclude that $\alpha^l_{k,\nu}(\rho) = 0$ unless $k = m$. Furthermore, $l + j$ must be an integer.

Substituting (4.14) into (4.13) and using the fact that the T matrices form a representation of $SU(2)$, we see that the functions $X^{(j)}_m$ have the form

$$X^{(j)}_m(\mathbf{p}, B) = \sum_l \sum_{n=-j}^{j} \sum_{\nu=-l}^{l} \sum_{s=-l}^{l} \alpha^n_{j,m;l,\nu}(\rho)\, T^j_{m,n}(h(\mathbf{p})) \times T^l_{n,s}(h^{-1}(\mathbf{p}))\, T^l_{s,\nu}(B) \,. \tag{4.16}$$

Conversely, it is easy to show that the quantity on the right side of (4.16) transforms properly under $SU(2)$. From the parametrization of $h(\mathbf{p})$ by Euler angles and equations (3.12), (3.15), the T matrices can be expressed in terms of the Q functions, with the result

$$T^j_{m,n}(h(\mathbf{p}))\, T^l_{n,s}(h^{-1}(\mathbf{p})) = e^{i\varphi(s-m)}(-1)^{2j} \frac{4\pi}{\sqrt{(2j+1)(2l+1)}}\, Q^j_{m,n}(\theta) Q^l_{n,s}(\theta) \,, \quad 0 \leqq \theta \leqq \pi,\ 0 \leqq \varphi < 2\pi \,. \tag{4.17}$$

If $f \in H$, the preceding considerations show that $Vf \in H'$ can be expanded in the form

$$(Vf)(\mathbf{p}, B) = \sum_{j,m,n,l,\nu,s} \cdots \sum \alpha^n_{j,m;l,\nu}(\rho)\, T^j_{m,n}(h(\mathbf{p}))\ T^l_{n,s}(h^{-1}(\mathbf{p}))\, T^l_{s,\nu}(B) \,. \tag{4.18}$$

Application of the transformation V^{-1} to Vf and comparison with equations (3.18) and (4.17) yield the expansion (4.5), where

$$b^n_{j,m;l,\nu}(\rho) = \left[\frac{2}{\pi(2j+1)(2l+1)}\right]^{1/2} (-1)^{2j}\, \alpha^n_{j,m;l,\nu}(\rho) \,.$$

Equation (4.6) follows from the orthogonality relations (4.3).

It is now convenient to derive equation (1.12). We follow the notation of Section 1. Suppose the functions $f^{(j)}_\nu \in H(\rho, m)$, $\nu = -j, -j+1, \cdots, +j$, form a basis for the representation $D(j)$ of $SU(2)$. Thus,

$$B_m[h^{-1}(\mathbf{r}) A h(A^{-1}\mathbf{r})] f^{(j)}_\nu(A^{-1}\mathbf{r}) = \sum_{\mu=-j}^{j} T^{(j)}_{\mu,\nu}(A) f^{(j)}_\mu(\mathbf{r}) \quad \text{for all } \mathbf{r} \in M,\ A \in SU(2) \,. \tag{4.19}$$

Set $A^{-1} = h(\mathbf{p})$, $r = \hat{\mathbf{p}}$ to obtain

$$f^{(j)}_\nu(\mathbf{p}) = \sum_{\mu=-j}^{j} T^j_{\mu,\nu}(h^{-1}(\mathbf{p})) f^{(j)}_\mu(\hat{\mathbf{p}}) \,, \qquad \mathbf{p} \in M \,, \tag{4.20}$$

where the quantities $f^{(j)}_\mu(\hat{\mathbf{p}})$ are constants. Choose $C \in SU(2)$ such that $C\hat{\mathbf{p}} = \hat{\mathbf{p}}$. Then, $T^j_{\mu,\nu}(C) = e^{-i\nu\varphi}\delta_{\mu\nu}$ for some φ, $0 \leqq \varphi < 4\pi$. In (4.19) we set $A = C$, $\mathbf{r} = \hat{\mathbf{p}}$ to derive

$$e^{-im\varphi} f^{(j)}_\nu(\hat{\mathbf{p}}) = e^{-i\nu\varphi} f^{(j)}_\nu(\hat{\mathbf{p}}) \,. \tag{4.21}$$

We conclude that $f_\nu^{(j)}(\hat{\mathbf{p}}) = 0$ for $\nu \neq m$ and that

$$f_\nu^{(j)}(\mathbf{p}) = cT^j_{m,\nu}(h^{-1}(\mathbf{p})) \,, \tag{4.22}$$

where c is a constant. Conversely, the quantity on the right side of (4.22) transforms correctly under $SU(2)$. This proves (1.12).

5. Concluding Remarks

The l-spinor functions $\chi^{(\rho,m)}_{l;j,\nu}(\mathbf{r})$ defined by (3.25) are of special importance in mathematical physics. It is known that an l-spinor $\theta(\mathbf{r})$ satisfying the wave equation $(\nabla^2 + \rho^2)\theta(\mathbf{r}) = 0$ can be expanded as a countable linear combination of the spinors $\chi^{(\rho,m)}_{l;j,\nu}(\mathbf{r})$ where $\nu = -j, -j+1, \cdots, +j$, $j = |m|, |m|+1, \cdots$, $m = -l, -l+1, \cdots, +l$, [8]. Such an expansion is useful because of the simple transformation properties of the χ's under the action of $\mathscr{E}_3$. Thus, the results obtained in Section 3 yield recursion relations and addition theorems for spinor-valued solutions of the wave equation.

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