Math 2243. Lecture 020 An example

Problem 1 *let* $V = P_3$ *be the vector space of polynomials in* t *of degree* ≤ 3 *and let* $T : V \to V$ *be the linear operator* $Tp(t) = p(-t)$ *for any polynomial* $p(t) = at^3 + bt^2 + ct + d \in P_3$*. The set*

$$\{\mathbf{e}_1 = 1, \quad \mathbf{e}_2 = t, \quad \mathbf{e}_3 = t^2, \quad \mathbf{e}_4 = t^3\}$$

is the standard basis for V*.*

a. *What is the matrix of* T *with respect to this basis?*

b. *Compute the kernel of* T *and its dimension.*

c. *Compute the eigenvalues and eigenspaces of* T*.*

d. *Compute the range (i.e. the image) of* T *and its dimension.*

Solution:

a.

$$T(\mathbf{e}_j) = T(t^{j-1}) = (-t)^{j-1} = (-1)^{j-1}t^{j-1}.$$

Thus

$$T(\mathbf{e}_1) = \mathbf{e}_1, \quad T(\mathbf{e}_2) = -\mathbf{e}_2, \quad T(\mathbf{e}_3) = \mathbf{e}_3, \quad T(\mathbf{e}_4) = -\mathbf{e_4},$$

and the matrix of T in this basis is

$$A = (T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3), T(\mathbf{e}_4)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

b. Clearly $Tp(t) = p(-t) = 0$ for all t if and only if $p(t) \equiv 0$, so the kernel of T consists of only the zero function, $\ker(T) = \{0\}$. Another way to see this is to note that A is invertible. The dimension of the kernel (the nullity) is zero.

c. From part a., we see that the eigenvalues of T (or A) are $\lambda = \pm 1$ with eigenspaces

$$E_1 = \text{ span}\{\mathbf{e}_1, \mathbf{e}_3\}, \qquad E_{-1} = \text{ span}\{\mathbf{e}_2, \mathbf{e}_4\},$$

d. If $p(t) = at^3 + bt^2 + ct + d \in P_3$ then $p(t) = Tq(t)$ where $q(t) = -at^3 + bt^2 - ct + d \in P_3$. Thus Range$(T) = P_3$. Another way to see this is to use the fact that Nullity(T) + Rank$(T) = 4$, the dimension of P_3. Since the nullity is 0, the rank is 4, so the range has the full dimension of P_3.