Fine structure for 3D second order superintegrable systems: 3-parameter potentials.

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Abstract

A classical (or quantum) superintegrable system of second order is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent constants of the motion quadratic in the momenta, the maximum possible. For n=3 on conformally flat spaces with nondegenerate, i.e., 4-parameter potentials (the extreme case) we have worked out the structure and classified most of the possible spaces and potentials. Here we extend the analysis to a more degenerate class of functionally linearly independent superintegrable systems, the 3-parameter potential case. We show that for "true" 3-parameter potentials the algebra of constants of the motion no longer closes at order 6 but still all such systems are Stäckel transforms of systems on complex Euclidean space or the complex 3-sphere.

This is a significant step toward the complete structure analysis of all types of second order superintegrable systems.

1 Introduction

This paper is part of a series analyzing the structure of second order superintegrable systems [1, 2, 3, 4, 5] on conformally flat Riemannian manifolds. In 2D the structure is well understood and the possible systems have been classified, [6, 7, 8, 9]. The 3D case is more typical of the general nD problem and much remains to be done, although the basic outlines are now clear. To describe the situation we briefly review some important definitions. An n-dimensional complex Riemannian space is conformally flat if and only if it admits a set of local coordinates $x_1, \cdots, x_n$ such that the contravariant metric tensor takes the form $g^{ij} = \delta^{ij}/\lambda(\mathbf{x})$. A classical superintegrable system $\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x})$ on the phase space of this manifold is one that admits $2n-1$ functionally independent generalized symmetries (or constants of the motion) $\mathcal{S}_k, \quad k = 1, \cdots, 2n-1$ with $\mathcal{S}_1 = \mathcal{H}$ where the $\mathcal{S}_k$ are polynomials in the momenta p_j. That is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$ where

$$\{f, g\} = \sum_{j=1}^{n} (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$$

is the Poisson bracket for functions $f(\mathbf{x}, \mathbf{p}), g(\mathbf{x}, \mathbf{p})$ on phase space [10, 11, 12, 13, 14, 15, 16, 17]. It is easy to see that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such (in general nonpolynomial) symmetries always exist.

A system is second order superintegrable if the $2n-1$ functionally independent symmetries can be chosen to be quadratic in the momenta. Usually a superintegrable system is also required to be integrable, i.e., it is assumed that n of the constants of the motion are in involution, although we do not make that assumption here. Our ultimate goal is to develop tools that will enable us to study the structure of superintegrable systems of all orders and to develop a classification theory. We are starting with second order systems because it is the most tractable case. Thus each of the $2n-1$ symmetries takes the form $\mathcal{S} = \sum a^{ij}(\mathbf{x}) p_i p_j + W(\mathbf{x})$, quadratic in the momenta. There is an analogous definition for second-order quantum superintegrable systems with Schrödinger operator

$$H = \Delta + V(\mathbf{x}), \quad \Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j},$$

the Laplace-Beltrami operator plus a potential function, [18]. Here there are $2n-1$ second-order symmetry operators

$$S_k = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} a^{ij}_{(k)}) \partial_{x_j} + W^{(k)}(\mathbf{x}), \quad k = 1, \cdots, 2n-1$$

with $S_1 = H$ and $[H, S_k] \equiv HS_k - S_k H = 0$. A basic motivation for studying these systems is that they can be solved explicitly and in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.

For $n = 3$ the number of functionally independent second order symmetries is 5. The analysis of the corresponding superintegrable systems splits into two cases depending on whether these symmetries are functionally linearly independent (FLI) or not. A set of second order symmetries is functionally linearly independent (or continuously linearly independent) if the corresponding set of quadratic forms $a^{ij}(\mathbf{x})$ is linearly independent at each regular point $\mathbf{x}$, [1, 3, 5]. Otherwise the set is functionally linearly dependent (FLD). For nondegenerate, i.e., 4-parameter, potentials treated in [3] and for the 3-parameter potentials treated here we assume that the system admits 5 functionally linearly independent second order symmetries. Second order superintegrable systems for which this is not the case must necessarily involve potentials that satisfy a 1st order linear PDE [5]. An example of a system with 5 functionally linearly dependent second order symmetries is the Calogero potential [19, 20, 21, 5]. Another example is the second order superintegrable flat space system with 3-parameter potential $V(x, y, z) = \alpha/(x + iy)^2 + \beta/(x + iy) + \gamma/z^2$. Though this potential is 3-parameter we will not analyse it in this paper because the corresponding system is FLD.

In addition to admitting 5 functionally linearly independent second order symmetries, a 3-parameter potential must be expressible in the form

$$V(x, y, z) = \alpha V^{(1)}(x, y, z) + \beta V^{(2)}(x, y, z) + \gamma V^{(3)}(x, y, z) \tag{1}$$

where α, β, γ are parameters and the gradients of the $V^{(i)}$, $i = 1, 2, 3$, are assumed linearly independent for each (x, y, z) in some open set of C^3. Here we are ignoring a trivial additive parameter to the potential. Differentiating (1) with respect to x, y, z, respectively, we can solve the resulting 3 equations for α, β, γ as linear functions of V_1, V_2, V_3. (Here $V_1 = V_x$, $V_{12} = V_{xy}$, etc.) Now, differentiating (1) twice with respect to x and substituting the expressions for the parameters as functions of the first derivatives we obtain the parameter-free equation

$$V_{11} = \tilde{A}^{11}(x, y, z)V_1 + \tilde{B}^{11}(x, y, z)V_2 + \tilde{C}^{11}(x, y, z)V_3.$$

Similarly, we can derive the canonical system of 6 2nd order PDEs,

$$V_{ij} = \tilde{A}^{ij}V_1 + \tilde{B}^{ij}V_2 + \tilde{C}^{ij}V_3, \quad 1 \le i \le j \le 3, \tag{2}$$

satisfied by the potential. Note that the integrability conditions for this system will be satisfied identically. Further, the solution space of this system is exactly 4-dimensional and a solution is uniquely determined at a regular point (x_0, y_0, z_0) by specifying the values of V_1, V_2, V_3 and V at the point. (Here as usual we will ignore the additive constant and the freedom to specify V.)

Recall that the canonical system of 5 PDEs for a nondegenerate potential is

$$\begin{array}{rclcl} V_{22} & = & V_{11} & + & A^{22}V_1 + B^{22}V_2 + C^{22}V_3, \\ V_{33} & = & V_{11} & + & A^{33}V_1 + B^{33}V_2 + C^{33}V_3, \\ V_{ij} & = & & & A^{ij}V_1 + B^{ij}V_2 + C^{ij}V_3, \end{array} \tag{3}$$

where $1 \le i < j \le 3$. Clearly, a 3-parameter potential also satisfies (3) with the identifications

$$A^{jj} = \tilde{A}^{jj} - \tilde{A}^{11},\ B^{jj} = \tilde{B}^{jj} - \tilde{B}^{11},\ C^{jj} = \tilde{C}^{jj} - \tilde{C}^{11},\ j = 2, 3, \tag{4}$$

$$A^{ij} = \tilde{A}^{ij},\ B^{ij} = \tilde{B}^{ij},\ C^{ij} = \tilde{C}^{ij},\ 1 \le i < j \le 3.$$

If V is a nondegenerate, i.e., 4-parameter, potential then the integrability conditions for (3) are satisfied identically. If V is 3-parameter then the integrability conditions for (2) are satisfied identically, but the integrability conditions for (3) may not all be satisfied.

A 3-parameter potential for a superintegrable system can arise in two distinct ways. First, the potential could be a linear restriction of a nondegenerate potential. For example, the potential

$$\tilde{V} = \alpha\left((x-iy)^3 + 6(x^2+y^2+z^2) + (x-iy) + (x-iy)^2 + 2(x+iy)\right) + \gamma(x-iy) + \delta z,$$

is the restriction $\beta = \alpha - \delta$ of the nondegenerate potential

$$V = \alpha\left((x-iy)^3 + 6(x^2+y^2+z^2)\right) + \beta\left((x-iy)^2 + 2(x+iy)\right)$$

$$+\gamma(x-iy) + \delta((x-iy)^2 + 2(x+iy) + z).$$

For the nondegenerate potential the canonical equations take the form (3) and it is not possible to solve for V_{11} in terms of V_1, V_2, V_3 whereas for the 3-parameter restriction we have a unique solution.

A second way the canonical equations (4) could arise is from the Bertrand-Darboux (B-D) equations for the 5 second order symmetries. As shown in [3], the rank of the second derivative terms in the 12 nontrivial B-D equations is 5, so the B-D equations must lead to the canonical conditions (3), plus possibly additional linear conditions involving only the first derivative terms V_j. However, for a 3-parameter potential no first order linear conditions can occur, since otherwise the potential would depend on strictly fewer than 3 parameters. Thus the equations must be (3) alone, but the integrability conditions are not identically satisfied. To obtain a 3-parameter potential the only possibility is that there is a single integrability condition and it is equivalent to $V_{11} = A^{11}V_1 + B^{11}V_2 + C^{11}V_3$. We call such superintegrable potentials **true** 3-parameter potentials.

The best known true 3-parameter system in Euclidean space is the extended Kepler-Coulomb potential:

$$V^{(1)} = \frac{\alpha}{\sqrt{x^2+y^2+z^2}} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} \tag{5}$$

The classical Hamilton-Jacobi equation, or the quantum Schrödinger equation, admits separable solutions in four coordinate systems: spherical, spheroconical, prolate spheroidal and parabolic coordinates. The bound states are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another. However, the space of second order symmetries is only 5-dimensional (as compared to 6 for nondegenerate potentials) and, although there are useful identities among the generators and commutators that enable one to derive spectral properties algebraically, there is no finite quadratic algebra structure. Two other examples that can be made real in real Euclidean space are given by Evans:

$$V^{(2)} = \frac{\alpha x}{y^2\sqrt{x^2+y^2}} + \frac{\beta}{y^2} + \gamma z, \tag{6}$$

$$V^{(3)} = \frac{\alpha x}{y^2\sqrt{x^2+y^2}} + \frac{\beta}{y^2} + \frac{\gamma}{z^2}. \tag{7}$$

A complex Euclidean example that is real in Minkowski space is

$$V^{(4)} = \frac{\alpha}{(x+iy)^2} + \beta(x^2+y^2+z^2) + \gamma z. \tag{8}$$

A complex sphere example, real on the hyperboloid $u_1^2 - u_2^2 + u_3^2 - u_4^2 = 1$, is

$$V^{(5)} = \frac{a}{(s_1+is_2)\sqrt{s_1^2+s_2^2}} + \frac{b}{(s_1+is_2)^2} + \frac{c}{(s_3-is_4)^2} \tag{9}$$

where $s_1^2 + s_2^2 + s_3^3 + s_4^2 = 1$. For all these cases the space of second order symmetries is exactly 5-dimensional and the quadratic algebra does not close, although it has a rich structure.

In this paper we shall show clearly how to distinguish between restrictions of systems with nondegenerate potentials and systems with true 3-parameter potentials just from the structure of their second order symmetry algebras. Further we will uncover the structure of the higher order symmetries for true 3-parameter potentials and prove multiseparability and lack of closure of the algebra of symmetries. We treat only the classical case here. Proofs of the analogous results for quantum systems are straight forward, just as in [5].

2 Conformally flat spaces in three dimensions

We adopt the notation for a classical superintegrable system with 3-parameter potential on a conformally flat space, as given in the introduction. Thus the system will admit 5 functionally linearly independent second order symmetries (or constants of the motion)

$$\mathcal{S} = \sum_{k,j=1}^{3} a^{kj}(x,y,z)p_k p_j + W(x,y,z) \equiv \mathcal{L} + W, \quad a^{jk} = a^{kj}$$

. The symmetry condition $\{\mathcal{H}, \mathcal{S}\} = 0$ where $\mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/\lambda(x,y,z) + V(x,y,z)$ leads to the second order Killing tensor equations

$$\begin{array}{lcl} a_i^{ii} & = & -G_1 a^{1i} - G_2 a^{2i} - G_3 a^{3i} \\ 2a_i^{ij} + a_j^{ii} & = & -G_1 a^{1j} - G_2 a^{2j} - G_3 a^{3j}, \quad i \neq j \\ a_k^{ij} + a_j^{ki} + a_i^{jk} & = & 0, \quad i,j,k \text{ distinct} \end{array} \tag{10}$$

and

$$W_k = \lambda \sum_{s=1}^{3} a^{sk} V_s, \quad k = 1,2,3. \tag{11}$$

(Here a subscript j denotes differentiation with respect to x_j and $\lambda = \exp G$.) The requirement that $\partial_{x_\ell} W_j = \partial_{x_j} W_\ell,\ \ell \neq j$ leads from (11) to the second order B-D partial differential equations for the potential.

$$\sum_{s=1}^{3} \left[V_{sj}\lambda a^{s\ell} - V_{s\ell}\lambda a^{sj} + V_s \left((\lambda a^{s\ell})_j - (\lambda a^{sj})_\ell \right) \right] = 0. \tag{12}$$

We assume five functionally linearly independent constants of the motion (including the Hamiltonian itself). Thus the Hamilton-Jacobi equation admits four additional constants of the motion:

$$\mathcal{S}_h = \sum_{j,k=1}^{3} a_{(h)}^{jk} p_k p_j + W_{(h)} = \mathcal{L}_h + W_{(h)}, \qquad h = 1, \cdots, 4.$$

Assuming also that V is 3-parameter, we substitute the requirement for a 3-parameter potential (2) into the B-D equations (12) and obtain three equations for the derivatives a_i^{jk}, the first of which is

$$(a_3^{11} - a_1^{31})V_1 + (a_3^{12} - a_1^{32})V_2 + (a_3^{13} - a_1^{33})V_3 \tag{13}$$
$$+a^{12}(A^{23}V_1 + B^{23}V_2 + C^{23}V_3) - (a^{33} - a^{11})(A^{13}V_1 + B^{13}V_2 + C^{13}V_3)$$
$$-a^{23}(A^{12}V_1 + B^{12}V_2 + C^{12}V_3) + a^{13}(A^{33}V_1 + B^{33}V_2 + C^{33}V_3)$$
$$= (-G_3a^{11} + G_1a^{13})V_1 + (-G_3a^{12} + G_1a^{23})V_2 + (-G_3a^{13} + G_1a^{33})V_3,$$

and the other two are obtained in a similar fashion.

The integrability conditions satisfied by the 3-parameter potential equations can be expressed as follows. We introduce the vector $\mathbf{v} = (V_1, V_2, V_3)^{\mathrm{T}}$, and the matrices $\tilde{\mathbf{A}}^{(j)}$, $j = 1, 2, 3$, where

$$\tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} \tilde{A}^{11}, & \tilde{B}^{11}, & \tilde{C}^{11}, \\ \tilde{A}^{12}, & \tilde{B}^{12}, & \tilde{C}^{12}, \\ \tilde{A}^{13}, & \tilde{B}^{13}, & \tilde{C}^{13} \end{pmatrix}, \quad \tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} \tilde{A}^{12}, & \tilde{B}^{12}, & \tilde{C}^{12}, \\ \tilde{A}^{22}, & \tilde{B}^{22}, & \tilde{C}^{22}, \\ \tilde{A}^{23}, & \tilde{B}^{23}, & \tilde{C}^{23} \end{pmatrix},$$

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} \tilde{A}^{13}, & \tilde{B}^{13}, & \tilde{C}^{13}, \\ \tilde{A}^{23}, & \tilde{B}^{23}, & \tilde{C}^{23}, \\ \tilde{A}^{33}, & \tilde{B}^{33}, & \tilde{C}^{33} \end{pmatrix}.$$

Then

$$\partial_{x_j}\mathbf{v} = \tilde{\mathbf{A}}^{(j)}\mathbf{v} \qquad j = 1, 2, 3. \tag{14}$$

The integrability conditions for this system are

$$\tilde{\mathbf{A}}_i^{(j)} - \tilde{\mathbf{A}}_j^{(i)} = \tilde{\mathbf{A}}^{(i)}\tilde{\mathbf{A}}^{(j)} - \tilde{\mathbf{A}}^{(j)}\tilde{\mathbf{A}}^{(i)} \equiv [\tilde{\mathbf{A}}^{(i)}, \tilde{\mathbf{A}}^{(j)}]. \tag{15}$$

The integrability conditions (15) are analytic expressions in x_1, x_2, x_3 and must hold identically. For convenience in the computation to follow we define

$$\mathcal{U}^1 = \tilde{\mathbf{A}}_2^{(3)} - \tilde{\mathbf{A}}_3^{(2)} = \tilde{\mathbf{A}}^{(i)}\tilde{\mathbf{A}}^{(j)} - [\tilde{\mathbf{A}}^{(2)}, \tilde{\mathbf{A}}^{(3)}] \tag{16}$$

plus cyclic permutations, so that the conditions are $\mathcal{U}^1 = \mathcal{U}^2 = \mathcal{U}^3 = 0$

There are $6 \times 3 = 18$ first partial derivatives a_i^{jk} of a quadratic symmetry. There are 10 Killing tensor conditions (10) for these derivatives and $3 \times 3 = 9$ conditions from the 3 B-D equations (13) and its companions. (Since each derivative V_j can be prescribed arbitrarily at a point, each B-D equation yields 3 conditions.) Thus we can use 18 conditions to solve for all of the a_i^{jk} of and obtain

$$\begin{aligned}
a_1^{11} &= -G_1a^{11} - G_2a^{12} - G_3a^{13} \qquad (17)\\
a_2^{22} &= -G_1a^{12} - G_2a^{22} - G_3a^{23},\\
a_3^{33} &= -G_1a^{13} - G_2a^{23} - G_3a^{33},\\
3a_1^{12} &= a^{12}A^{22} - (a^{22} - a^{11})A^{12} - a^{23}A^{13} + a^{13}A^{23}\\
&+ G_2a^{11} - 2G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a_2^{11} &= -2a^{12}A^{22} + 2(a^{22} - a^{11})A^{12} + 2a^{23}A^{13} - 2a^{13}A^{23}\\
&- 2G_2a^{11} + G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a_3^{13} &= -a^{12}C^{23} + (a^{33} - a^{11})C^{13} + a^{23}C^{12} - a^{13}C^{33}\\
&- G_1a^{11} - G_2a^{12} - 2G_3a^{13} + G_1a^{33},\\
3a_1^{33} &= 2a^{12}C^{23} - 2(a^{33} - a^{11})C^{13} - 2a^{23}C^{12} + 2a^{13}C^{33}\\
&- G_1a^{11} - G_2a^{12} + G_3a^{13} - 2G_1a^{33},\\
3a_2^{23} &= a^{23}(B^{33} - B^{22}) - (a^{33} - a^{22})B^{23} - a^{13}B^{12} + a^{12}B^{13}\\
&- G_1a^{13} - 2G_2a^{23} - G_3a^{33} + G_3a^{22},\\
3a_3^{22} &= -2a^{23}(B^{33} - B^{22}) + 2(a^{33} - a^{22})B^{23} + 2a^{13}B^{12} - 2a^{12}B^{13}\\
&- G_1a^{13} + G_2a^{23} - G_3a^{33} - 2G_3a^{22},\\
3a_1^{13} &= -a^{23}A^{12} + (a^{11} - a^{33})A^{13} + a^{13}A^{33} + a^{12}A^{23}\\
&- 2G_1a^{13} - G_2a^{23} - G_3a^{33} + G_3a^{11},\\
3a_3^{11} &= 2a^{23}A^{12} + 2(a^{33} - a^{11})A^{13} - 2a^{13}A^{33} - 2a^{12}A^{23}\\
&+ G_1a^{13} - G_2a^{23} - G_3a^{33} - 2G_3a^{11},\\
3a_2^{33} &= -2a^{13}C^{12} + 2(a^{22} - a^{33})C^{23} + 2a^{12}C^{13} - 2a^{23}(C^{22} - C^{33})\\
&- G_1a^{12} - G_2a^{22} + G_3a^{23} - 2G_2a^{33},\\
3a_3^{23} &= a^{13}C^{12} - (a^{22} - a^{33})C^{23} - a^{12}C^{13} - a^{23}(C^{33} - C^{22})\\
&- G_1a^{12} - G_2a^{22} - 2G_3a^{23} + G_2a^{33},\\
3a_2^{12} &= -a^{13}B^{23} + (a^{22} - a^{11})B^{12} - a^{12}B^{22} + a^{23}B^{13}\\
&- G_1a^{11} - 2G_2a^{12} - G_3a^{13} + G_1a^{22},\\
3a_1^{22} &= 2a^{13}B^{23} - 2(a^{22} - a^{11})B^{12} + 2a^{12}B^{22} - 2a^{23}B^{13}\\
&- G_1a^{11} + G_2a^{12} - G_3a^{13} - 2G_1a^{22},\\
3a_1^{23} &= a^{12}(B^{23} + C^{22}) + a^{11}(B^{13} + C^{12}) - a^{22}C^{12} - a^{33}B^{13}\\
&+ a^{13}(B^{33} + C^{23}) - a^{23}(C^{13} + B^{12})\\
&- 2G_1a^{23} + G_2a^{13} + G_3a^{12}.\\
3a_3^{12} &= a^{12}(-2B^{23} + C^{22}) + a^{11}(C^{12} - 2B^{13}) - a^{22}C^{12} + 2a^{33}B^{13}\\
&+ a^{13}(-2B^{33} + C^{23}) + a^{23}(-C^{13} + 2B^{12})
\end{aligned}$$

$$
\begin{aligned}
&- \quad 2G_3a^{12} + G_2a^{13} + G_1a^{23}. \\
3a_2^{13} \quad &= \quad a^{12}(B^{23} - 2C^{22}) + a^{11}(B^{13} - 2C^{12}) + 2a^{22}C^{12} - a^{33}B^{13} \\
&+ \quad a^{13}(B^{33} - 2C^{23}) + a^{23}(2C^{13} - B^{12}) \\
&- \quad 2G_2a^{13} + G_1a^{23} + G_3a^{12}.
\end{aligned}
$$

The remaining condition is

$$
\begin{aligned}
a^{11}(\tilde{C}^{12} - \tilde{B}^{13}) + a^{22}(\tilde{A}^{23} - \tilde{C}^{12}) + a^{33}(\tilde{B}^{13} - \tilde{A}^{23}) + a^{12}(\tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11} - \tilde{B}^{23}) \\
+a^{13}(\tilde{C}^{23} + \tilde{B}^{11} - \tilde{B}^{33} - \tilde{A}^{12}) + a^{23}(\tilde{B}^{12} + \tilde{A}^{33} - \tilde{A}^{22} - \tilde{C}^{13}) = 0,
\end{aligned} \tag{18}
$$

which we can regard as an obstruction to extending the assumed 5-dimensional space of second order symmetries to the full 6-dimensional space of quadratic forms. (Note that the analogous obstruction equation appears for the nondegenerate potential case in [3], but there the linear integrability conditions for the nondegenerate potential cause the obstruction to vanish identically.)

Since the system (17) is in involution, a second order symmetry is determined up to a trivial added constant by the value of the quadratic form $a^{ij}(\mathbf{x}_0)$ at a single regular point. Thus the dimension of the space of second order symmetries (≥ 5 by assumption) cannot exceed 6. We will show that the dimension is 6 if and only if the 3-parameter potential is a restriction of a nondegenerate potential. Further, the obstruction (18) vanishes identically if and only if the dimension is 6. This is a satisfying solution of our first structure problem. Unfortunately, the proof is lengthy.

3 No obstruction implies nondegenerate potential

Suppose the obstruction (18) vanishes identically. Then the conditions

$$
\begin{aligned}
\tilde{C}^{12} = \tilde{B}^{13} = \tilde{A}^{23},\ \tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11} - \tilde{B}^{23} = 0, \\
\tilde{C}^{23} + \tilde{B}^{11} - \tilde{B}^{33} - \tilde{A}^{12} = 0,\ \tilde{B}^{12} + \tilde{A}^{33} - \tilde{A}^{22} - \tilde{C}^{13} = 0
\end{aligned} \tag{19}
$$

must hold identically.

To determine the dimension of the solution space of second order symmetries we need to study the integrability conditions for equations (17). For the integrability conditions we define the vector-valued function

$$
\mathbf{h}(x, y, z) = \left(a^{11}, a^{12}, a^{13}, a^{22}, a^{23}, a^{33}\right)^{\mathrm{T}}
$$

and directly compute the 6×6 matrix functions $\mathcal{A}^{(j)}$ to get the first-order system

$$\partial_{x_j}\mathbf{h} = \mathcal{A}^{(j)}\mathbf{h} \qquad j = 1, 2, 3, \tag{20}$$

under the assumptions (19). The integrability conditions for this system are

$$\mathcal{A}^{(j)}_i\mathbf{h} - \mathcal{A}^{(i)}_j\mathbf{h} = \mathcal{A}^{(i)}\mathcal{A}^{(j)}\mathbf{h} - \mathcal{A}^{(j)}\mathcal{A}^{(i)}\mathbf{h} \equiv [\mathcal{A}^{(i)}, \mathcal{A}^{(j)}]\mathbf{h}. \tag{21}$$

In terms of the 6×6 matrices

$$\mathcal{S}^{(1)} = \mathcal{A}^{(3)}_2 - \mathcal{A}^{(2)}_3 - [\mathcal{A}^{(2)}, \mathcal{A}^{(3)}], \qquad \mathcal{S}^{(2)} = \mathcal{A}^{(1)}_3 - \mathcal{A}^{(3)}_1 - [\mathcal{A}^{(3)}, \mathcal{A}^{(1)}],$$

$$\mathcal{S}^{(3)} = \mathcal{A}^{(2)}_1 - \mathcal{A}^{(1)}_2 - [\mathcal{A}^{(1)}, \mathcal{A}^{(2)}],$$

the integrability conditions are

$$\mathcal{S}^{(1)}\mathbf{h} = \mathcal{S}^{(2)}\mathbf{h} = \mathcal{S}^{(3)}\mathbf{h} = 0 \tag{22}$$

We can proceed in analogy with the proof in [3] of the $5 \Longrightarrow 6$ Theorem for nondegenerate potentials.

Assume first that the system of equations (17) admits a 6-parameter family of solutions a^{jk}. Thus at any regular point we can prescribe the values of the a^{jk} arbitrarily. This means that (21) or (22) holds identically in $\mathbf{h}$. Thus $\mathcal{S}^{(1)} = \mathcal{S}^{(2)} = \mathcal{S}^{(3)} = 0$. Using these expressions, we can perform a tedious but straightforward Maple-assisted computation that yields

1. An expression for each of the first partial derivatives $\partial_\ell \tilde{A}^{ij}$, $\partial_\ell \tilde{B}^{ij}$, $\partial_\ell \tilde{C}^{ij}$, for the 18 functions as homogeneous polynomials of order at most two in the $\tilde{A}^{i'j'}$, $\tilde{B}^{i'j'}$, $\tilde{C}^{i'j'}$. There are $54 = 3 \times 18$ such expressions in all. An example is

$$\tilde{B}^{12}_2 = \frac{2}{3}\tilde{A}^{12}\tilde{B}^{12} - \frac{1}{6}\tilde{B}^{12}G_2 - \frac{5}{6}G_1\tilde{A}^{12} - \frac{1}{6}G_1G_2$$

$$+\frac{1}{3}(\tilde{B}^{22} - \tilde{B}^{11})\tilde{B}^{12} + \frac{1}{3}(\tilde{B}^{22} - \tilde{B}^{11})G_1 + \frac{1}{3}\tilde{A}^{23}\tilde{B}^{23} - \frac{7}{6}G_3\tilde{A}^{23} + \frac{1}{2}G_{12}.$$

2. Exactly 5 quadratic identities for the independent functions (in addition

to the linear identities (19)):

$$
\begin{array}{lrl}
(a) & -A^{23}B^{33}-A^{12}A^{23}+A^{13}B^{12}+B^{22}A^{23}+B^{23}A^{33} & \\
 & +\frac{1}{2}A^{22}G_3-\frac{1}{2}A^{33}G_3-\frac{1}{2}B^{12}G_3-\frac{1}{2}G_1G_3 & \\
 & -\frac{1}{2}A^{13}G_1+\frac{3}{2}G_{13}-\frac{1}{2}A^{23}G_2-A^{22}B^{23} & =0,\\
(b) & \left(A^{33}\right)^2+B^{12}A^{33}-A^{33}A^{22}-B^{33}A^{12}-C^{33}A^{13}+B^{22}A^{12} & \\
 & -B^{12}A^{22}+A^{13}B^{23}-\left(A^{12}\right)^2+ & \\
 & \frac{3}{2}G_{22}-\frac{1}{2}G_y^2-\frac{3}{2}G_{33}+\frac{1}{2}A^{13}G_3+\frac{1}{2}B^{33}G_2+ & \\
 & -\frac{1}{2}A^{22}G_1+\frac{1}{2}A^{33}G_1-\frac{1}{2}B^{23}G_3-\frac{1}{2}B^{22}G_2+\frac{1}{2}C^{33}G_3+\frac{1}{2}(G_3)^2 & =0,\\
(c) & -\left(B^{33}\right)^2-B^{33}A^{12}+B^{33}B^{22}+B^{12}A^{33}+B^{23}C^{33}-\left(B^{23}\right)^2 & \\
 & +\left(B^{12}\right)^2+\frac{1}{2}(G_1)^2-\frac{3}{2}G_{11}+\frac{3}{2}G_{33} & \\
 & -\frac{1}{2}B^{33}G_2-\frac{1}{2}A^{33}G_1-\frac{1}{2}(G_3)^2-\frac{1}{2}C^{33}G_3 & =0,\\
(d) & -B^{12}A^{23}-A^{33}A^{23}+A^{13}B^{33}+A^{12}B^{23} & \\
 & +\frac{3}{2}G_{23}-\frac{1}{2}A^{23}G_1-\frac{1}{2}A^{12}G_3 & \\
 & -\frac{1}{2}B^{23}G_2-\frac{1}{2}G_2G_3-\frac{1}{2}B^{33}G_3 & =0,\\
(e) & A^{12}B^{12}+C^{33}A^{23}-A^{23}B^{23}+B^{33}A^{22}-B^{33}A^{33} & \\
 & +\frac{3}{2}G_{12}-\frac{1}{2}G_1G_2-\frac{1}{2}A^{12}G_1 & \\
 & -\frac{1}{2}B^{12}G_2-\frac{1}{2}A^{23}G_3 & =0
\end{array}
\tag{23}
$$

There are *no nontrivial conditions* in which some derivative of G is involved as a factor in each term and no conditions other than those described here..

Theorem 1 ($5 \Longrightarrow 6$) *Let V be a 3-parameter potential corresponding to a conformally flat space in 3 dimensions that is superintegrable, i.e., suppose V satisfies the equations (2), where conditions (19,15,4) hold, and there are 5 functionally linearly independent constants of the motion. Then the space of second order symmetries for the Hamiltonian $\mathcal{H}=(p_x^2+p_y^2+p_z^2)/\lambda(x,y,z)+V(x,y,z)$ (excluding multiplication by a constant) is of dimension $D=6$. At any regular point (x_0,y_0,z_0) and given constants $\alpha^{kj}=\alpha^{jk}$ there is exactly one symmetry $\mathcal{S}$ (up to an additive constant) such that $a^{kj}(x_0,y_0,z_0)=\alpha^{kj}$.*

PROOF OF THEOREM: The proof takes many steps, most of which have to be carried out with computer algebra software. We give the logic behind the proof and describe the steps in order.

If there is only a 5-parameter family of solutions then (22) holds only for the $\mathbf{h}$ that lie in a 5-dimensional space. By appropriate Euclidean transformation of coordinates, if necessary, we can use Gauss-Jordan elimination and show that there is a basis for the space of the form $\tilde{\mathbf{h}}^j$, $j=1,\cdots 5$ where

$$\left(\tilde{\mathbf{h}}^1,\tilde{\mathbf{h}}^2,\tilde{\mathbf{h}}^3,\tilde{\mathbf{h}}^4,\tilde{\mathbf{h}}^5\right)=$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \alpha_1(x,y,z) & \alpha_2(x,y,z) & \alpha_3(x,y,z) & \alpha_4(x,y,z) & \alpha_5(x,y,z) \end{pmatrix}.$$

Here we mean that if $\mathbf{h}$ belongs to the solution space then there are unique differentiable functions $g_j(x,y,z)$ such that $\mathbf{h} = \sum_{j=1}^{5} g_j \tilde{\mathbf{h}}^j$. It follows that the integrability conditions become

$$\mathcal{S}^{(\ell)}_{kj} + \alpha_j \mathcal{S}^{(\ell)}_{k6} = 0, \quad \ell = 1, \cdots, 3, \quad k = 1, \cdots, 6, \quad j = 1, \cdots, 5. \tag{24}$$

Further, the conditions (22) must hold. The question that we need to decide is whether the conditions (22) and (24) imply

$$\mathcal{S}^{(1)} = \mathcal{S}^{(2)} = \mathcal{S}^{(3)} = 0.$$

Some of the elements of the matrices $\mathcal{S}^{(i)}$ vanish identically. Indeed

$$\mathcal{S}^{(1)}_{16} = \mathcal{S}^{(1)}_{26} = \mathcal{S}^{(1)}_{46} = \mathcal{S}^{(2)}_{14} = \mathcal{S}^{(2)}_{34} = \mathcal{S}^{(2)}_{64} = \mathcal{S}^{(3)}_{41} = \mathcal{S}^{(3)}_{51} = \mathcal{S}^{(3)}_{61} \equiv 0$$

Also

$$\mathcal{S}^{(2)}_{16} \equiv \mathcal{S}^{(1)}_{15}, \ \mathcal{S}^{(2)}_{66} \equiv -\mathcal{S}^{(2)}_{61}, \mathcal{S}^{(3)}_{46} \equiv -\mathcal{S}^{(1)}_{43},$$

This implies that the following conditions must hold no matter what are the values of the α_j:

$$\mathcal{S}^{(1)}_{ij} = 0, \quad i = 1,2,4 \quad 1 \le j \le 6, \qquad \mathcal{S}^{(2)}_{1j} = \mathcal{S}^{(3)}_{4j} = 0, \quad 1 \le j \le 6. \tag{25}$$

Our strategy is to use these identities and the potential integrability conditions (15) step by step to solve for as many of the 54 independent partial derivatives

$$\partial_k D^{ij}, \ 1 \le i \le j \le 3, \quad D = A, B, C,$$

as we can. In each case we will obtain an expression for the derivative as a polynomial in the 18 variables D^{ij} with coefficients in the linear and zero order terms that involve derivatives of G. Then to substitute these results back into the potential equations to obtain new results.

For the first step, our initial results imply that the quadratic identity (d) holds, further by substitution the identity $\mathcal{U}^3_{12} = 0$ implies that the quadratic identity (a) holds. However, (a) is equivalent to $\mathcal{S}^{(3)}_{2,6} = 0$ which in turn implies $\mathcal{S}^{(3)}_{2j} = 0, \ 1 \le j \le 6$. The fact that $\mathcal{S}^{(3)}_{46} = 0$ implies by substitution that $\mathcal{S}^{(3)}_{16} = 0$, so $\mathcal{S}^{(3)}_{1j} = 0, \ 1 \le j \le 6$. Then quadratic identity (e) is implied by $\mathcal{S}^{(3)}_{13} = 0$. The potential identities $\mathcal{U}^1_{33} = 0$, $\mathcal{U}^2_{23} = 0$ and substitution imply $\mathcal{S}^{(3)}_{36} = \mathcal{S}^{(2)}_{56} = 0$, so $\mathcal{S}^{(3)}_{3j} = \mathcal{S}^{(2)}_{5j} = 0,\ 1 \le j \le 6$. Similarly, the result $\mathcal{S}^{(3)}_{46} = 0$, and substitution implies $\mathcal{S}^{(2)}_{66} = 0$, so $\mathcal{S}^{(2)}_{6j} = 0,\ 1 \le j \le 6$. The result $\mathcal{S}^{(1)}_{43} = 0$, and substitution implies $\mathcal{S}^{(1)}_{36} = 0$, so $\mathcal{S}^{(1)}_{3j} = 0,\ 1 \le j \le 6$, and and substitution implies $\mathcal{S}^{(2)}_{26} = 0$, so $\mathcal{S}^{(2)}_{2j} = 0,\ 1 \le j \le 6$. From this we can obtain the remaining quadratic identities (b) and (c). The process finally ends with the result

$$\mathcal{S}^{(1)} = \mathcal{S}^{(2)} = \mathcal{S}^{(3)} = 0,$$

hence that the integrability conditions are satisfied identically and there is a 6-parameter family of symmetries. Further, each of the 54 independent partial derivatives

$$\partial_k D^{ij},\ 1 \le i \le j \le 3, \quad D = A, B, C,$$

is expressed explicitly as a polynomial in the 18 variables. Q.E.D.

Suppose we have a superintegrable 3-parameter potential with no obstruction. Is it a restriction of a nondegenerate (4-parameter) potential or is it truly 3-parameter? To help answer this question we call call on some of the results of papers [3] and [4]. The results from [3] on the structure of the third order constants of the motion for nondegenerate superintegrable systems needed only the assumption of a 3-parameter potential with no obstructions for their proof.

Theorem 2 *Let $\mathcal{K}$ be a third order constant of the motion for a conformally flat superintegrable system with 3-parameter potential V and no obstructions:*

$$\mathcal{K} = \sum_{k,j,i=1}^{3} a^{kji}(x,y,z)p_k p_j p_i + \sum_{\ell=1}^{3} b^{\ell}(x,y,z)p_\ell.$$

Then

$$b^{\ell}(x,y,z) = \sum_{j=1}^{3} f^{\ell,j}(x,y,z)V_j(x,y,z)$$

with $f^{\ell,j} + f^{j,\ell} = 0, \quad 1 \le \ell, j \le 3$. The a^{ijk}, b^ℓ are uniquely determined by the four numbers

$$f^{1,2}(x_0,y_0,z_0),\ f^{1,3}(x_0,y_0,z_0),\ f^{2,3}(x_0,y_0,z_0),\ f^{1,2}_3(x_0,y_0,z_0)$$

at any regular point (x_0,y_0,z_0) of V.

Corollary 1 *For a system satisfying the assumptions of the theorem, the space of truly third order constants of the motion is 4-dimensional and is spanned by Poisson brackets of the second order constants of the motion.*

Correspondingly, the proofs of the following results from [4] need only the assumptions of 3-parameter superintegrability with no obstructions:

Theorem 3 *Let V be a superintegrable 3-parameter potential with no obstruction in a 3D conformally flat space. Then V defines a multiseparable system.*

Corollary 2 *For a system satisfying the assumptions of the theorem, there is a continuous 1-parameter (or multi-parameter) family of separable systems for V, spanning at least a 5-dimensional subspace of symmetries.*

4 The 3D Stäckel transform

The Stäckel transform [22] or coupling constant metamorphosis [23] plays a fundamental role in relating superintegrable systems on different manifolds. Suppose we have a superintegrable system

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{\lambda(x,y,z)} + V(x,y,z) \tag{26}$$

in local orthogonal coordinates, with 3-parameter potential $V(x,y,z)$:

$$V_{ij} = A^{ij}V_1 + B^{ij}V_2 + C^{ij}V_3,\ V_{22} = V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3, \tag{27}$$

$1 \le i \le j \le 3$, and suppose $U(x,y,z)$ is a particular solution of equations (27), nonzero in an open set. Then the transformed system $\tilde{H} = (p_1^2 + p_2^2 + p_3^2)/\tilde{\lambda} + \tilde{V}$ with nondegenerate potential $\tilde{V}(x,y,z)$:

$$\tilde{V}_{ij} = \tilde{A}^{ij}\tilde{V}_1 + \tilde{B}^{ij}\tilde{V}_2 + \tilde{C}^{ij}\tilde{V}_3 \tag{28}$$

is also superintegrable, where

$$\tilde{\lambda} = \lambda U,\ \tilde{V} = \frac{V}{U},\ \tilde{A}^{11} = A^{11} - 2\frac{U_1}{U}, \tilde{B}^{22} = A^{11} - 2\frac{U_2}{U},\ \tilde{C}^{33} = C^{33} - 2\frac{U_3}{U},$$

$$\tilde{B}^{11} = B^{11},\ \tilde{C}^{11} = C^{11},\ \tilde{A}^{22} = A^{22},\quad \tilde{C}^{22} = C^{22},$$

$$\tilde{A}^{33} = A^{33},\ \tilde{B}^{33} = B^{33},\ \tilde{A}^{13} = A^{13} - \frac{U_3}{U},\ \tilde{C}^{13} = C^{13} - \frac{U_1}{U},$$

$$\tilde{A}^{12} = A^{12} - \frac{U_2}{U},\ \tilde{B}^{12} = B^{12} - \frac{U_1}{U},$$

$$\tilde{B}^{23} = B^{23} - \frac{U_3}{U},\ \tilde{C}^{23} = C^{23} - \frac{U_2}{U},$$

and $\tilde{A}^{23} = A^{23}$, $\tilde{B}^{13} = B^{13}$, $\tilde{C}^{12} = C^{12}$. Let $S = \sum a^{ij}p_ip_j + W = S_0 + W$ be a second order symmetry of H and $S_U = \sum a^{ij}p_ip_j + W_U = S_0 + W_U$ be the special case that is in involution with $(p_1^2 + p_2^2 + p_3^2)/\lambda + U$. Then $\tilde{S} = S_0 - \frac{W_U}{U}H + \frac{1}{U}H$ is the corresponding symmetry of $\tilde{H}$. Since one can always add a constant to a nondegenerate potential, it follows that $1/U$ defines an inverse Stäckel transform of $\tilde{H}$ to H. See [22] for many examples of this transform.

The next result follows immediately from the proof of the corresponding theorem in [4].

Theorem 4 *Every superintegrable system with 3-parameter potential and no obstruction on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.*

Using Theorem 4 we can show that a 3-parameter superintegrable potential with no obstruction is a restriction of a nondegenerate potential. In [3] we derived the integrability conditions for the nondegenerate potential equations (3). We recall the form of these equations (the details are in [3]). We introduce the vector $\mathbf{w} = (V_1, V_2, V_3, V_{11})^{\mathrm{T}}$, and the matrices $\mathbf{A}^{(j)}$, $j = 1, 2, 3$, such that

$$\partial_{x_j}\mathbf{w} = \mathbf{A}^{(j)}\mathbf{w} \qquad j = 1, 2, 3. \tag{29}$$

The integrability conditions for this system are

$$\mathbf{A}_i^{(j)} - \mathbf{A}_j^{(i)} = [\mathbf{A}^{(i)}, \mathbf{A}^{(j)}]. \tag{30}$$

What we must show is that the 5 no obstruction equations (19), the 108 integrability conditions

$$\mathcal{S}^{(1)} = \mathcal{S}^{(2)} = \mathcal{S}^{(3)} = 0$$

for the 6-dimensional family of symmetries and the 27 integrability conditions (15) for the 3-parameter potential imply the integrability conditions (30) for a nondegenerate potential.

First we assume that we are in complex Euclidean space, so that $G(x, y, z) \equiv 0$. Then, from the proof of Theorem 1 we can express each of the 54 independent partial derivatives

$$\partial_k\tilde{D}^{ij},\ 1 \le i \le j \le 3, \quad D = \tilde{A}, \tilde{B}, \tilde{C},$$

explicitly as a quadratic polynomial in the 18 variables $\tilde{D}^{\ell m}$, and we can obtain the 5 fundamental quadratic identities (23). Finally, from the integrability conditions $\partial_\ell\partial_k\tilde{D}^{ij} = \partial_k\partial_\ell\tilde{D}^{ij}$ we obtain 54 cubic polynomial identities. These second and third order identities generate a polynomial ideal Σ

in the 18 variables $\tilde{D}^{\ell m}$. Substituting the quadratic expressions for the $\partial_k \tilde{D}^{ij}$ into the integrability conditions (30) we can show that these conditions are equivalent to 7 quadratic and cubic identities $Ij = 0,\ j = 1, \cdots, 7$ in the variables $\tilde{D}^{\ell m}$. Making use of the Gröbner basis package in the symbol manipulation program MAPLE, we have verified that $Ij \in \Sigma$ for $j = 1, \cdots, 7$. Thus conditions (30) must hold for Euclidean space potentials.

We cannot use the same reasoning for metrics with general G because we do not know the explicit form of G. However, from Theorem 4 the only remaining case we need study is the 3-sphere. As was argued in Section 4.5 of [4], at any regular point on the sphere we can always choose Cartesian-like local coordinates such that

$$G(x, y, z) = -2\ln(1 + \frac{x^2 + y^2 + z^2}{4})$$

and the regular point is $(0, 0, 0)$. Thus at this point

$$G = G_j = 0,\ G_{ij} = -\delta_{ij}, \quad 1 \le i, j \le 3.$$

With this choice we can carry out the exact analog of the Euclidean space demonstration and use the Gröbner basis package in MAPLE to verify that $Ij \in \Sigma$ for $j = 1, \cdots, 7$, so that conditions (30) hold for 3-sphere potentials.

Thus, by exploitation of the integrability conditions for the potential and for equations (17) we obtain the following results:

Theorem 5 *A 3-parameter potential for a 3D second order superintegrable FLI system is a restriction of a nondegenerate potential if and only if the obstruction (18) vanishes identically. If the obstruction doesn't vanish then the space of second order symmetries is 5-dimensional and the system is uniquely determined by the values of* $\tilde{D}^{ij}, i \le j,\ D = A, B, C$ *at a single regular point.*

5 Structure theory for true 3-parameter potentials

For a superintegrable FLI system with true 3-parameter potential there is a nontrivial obstruction (18) which we can express in the form

$$\text{trace } (\mathcal{A}\mathcal{J}) = 0 \tag{31}$$

where $\mathcal{A} = (a^{ij})$ and

$$\mathcal{J} = \begin{pmatrix} -\tilde{B}^{13} + \tilde{C}^{12} & \frac{1}{2}(\tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11} - \tilde{B}^{23}) & \frac{1}{2}(\tilde{C}^{23} - \tilde{B}^{33} + \tilde{B}^{11} - \tilde{A}^{12}) \\ \frac{1}{2}(\tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11} - \tilde{B}^{23}) & \tilde{A}^{23} - \tilde{C}^{12} & \frac{1}{2}(\tilde{B}^{12} + \tilde{A}^{33} - \tilde{A}^{22} - \tilde{C}^{13}) \\ \frac{1}{2}(\tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11} - \tilde{B}^{23}) & \frac{1}{2}(\tilde{B}^{12} + \tilde{A}^{33} - \tilde{A}^{22} - \tilde{C}^{13}) & \tilde{B}^{13} - \tilde{A}^{23} \end{pmatrix} \tag{32}$$

$$= \begin{pmatrix} X_1 & Y_1 & Y_2 \\ Y_1 & X_2 & Y_3 \\ Y_2 & Y_3 & -X_1 - X_2 \end{pmatrix}$$

where the X_j, Y_k have obvious definitions. Under a complex rotation of coordinates with 3×3 orthogonal matrix O the quadratic form of the symmetry transforms as $O \to O\mathcal{A}O^{\rm tr}$, so $\mathcal{J} \to O\mathcal{J}O^{\rm tr}$. Thus the traceless symmetric obstruction $\mathcal{J}$ transforms as the 5-dimensional irreducible representation of the complex rotation group. It follows that about any regular point $\mathbf{x}_0$ we can always find a rotation that fixes this point and such that *all* the matrix elements of $\mathcal{J}$ are nonzero. In this section we can always assume that we have determined a regular point and a coordinate system centered at this point for which all the elements of $\mathcal{J}$ are nonvanishing.

At a regular point $\mathbf{x}_0$ we can use our knowledge of the obstruction to determine an explicit basis for the second order symmetries. The five basis elements take the form $\mathcal{S}^{(\ell,m)} = \sum_{ij} \mathcal{A}^{(\ell,m)}_{ij} p_i p_j + W^{(\ell,m)}$ where $W^{(\ell,m)}(\mathbf{x}_0) \equiv 0$ and

$$\mathcal{A}^{(1,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{C^{12}-B^{13}}{A^{23}-B^{13}} \end{pmatrix}, \quad \mathcal{A}^{(2,2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{A^{23}-C^{12}}{A^{23}-B^{13}} \end{pmatrix}, \tag{33}$$

$$\mathcal{A}^{(1,2)} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{A^{13}+C^{22}-C^{11}-B^{23}}{A^{23}-B^{13}} \end{pmatrix}, \ \mathcal{A}^{(1,3)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & \frac{C^{23}+B^{11}-B^{33}-A^{12}}{A^{23}-B^{13}} \end{pmatrix},$$

$$\mathcal{A}^{(2,3)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{B^{12}+A^{33}-A^{22}-C^{13}}{A^{23}-B^{13}} \end{pmatrix}.$$

Next we investigate the space of third order constants of the motion, assuming a true 3-parameter potential V. We have

$$\mathcal{K} = \sum_{k,j,i=1}^{3} a^{kji}(x,y,z) p_k p_j p_i + b^{\ell}(x,y,z) p_\ell, \tag{34}$$

which must satisfy $\{\mathcal{H}, \mathcal{K}\} = 0$. Here a^{kji} is symmetric in the indices k, j, i.

The required conditions are

$$
\begin{aligned}
a_i^{iii} &= -\frac{3}{2}\sum_s a^{sii}(\ln\lambda)_s, \qquad (35)\\
3a_i^{jii} + a_j^{iii} &= -3\sum_s a^{sij}(\ln\lambda)_s, \quad i\neq j\\
a_i^{ijj} + a_j^{iij} &= -\frac{1}{2}\sum_s a^{sjj}(\ln\lambda)_s - \frac{1}{2}\sum_s a^{sii}(\ln\lambda)_s, \quad i\neq j\\
2a_i^{ijk} + a_j^{kii} + a_k^{jii} &= -\sum_s a^{sjk}(\ln\lambda)_s, \quad i,j,k \text{ distinct}
\end{aligned}
$$

$$
b_k^j + b_j^k = 3\lambda\sum_s a^{skj}V_s, \quad j\neq k, \quad j,k = 1,2,3, \qquad (36)
$$

$$
b_j^j = \frac{3}{2}\lambda\sum_s a^{sjj}V_s - \frac{1}{2}\sum_s b^s(\ln\lambda)_s,, \quad j = 1,2,3,
$$

and

$$
\sum_s b^s V_s = 0. \qquad (37)
$$

The a^{kji} is just a third order Killing tensor. As usual, we require that the highest order terms, the a^{kji} in the constant of the motion, be independent of the three independent parameters in V. However, the b^ℓ must depend on these parameters. We set

$$
b^\ell(x,y,z) = \sum_{j=1}^{3} f^{\ell,j}(x,y,z)V_j(x,y,z).
$$

Substituting this expression into (37) we see that

$$
f^{\ell,j} + f^{j,\ell} = 0, \quad 1\leq \ell, j\leq 3.
$$

Further

$$
b_j^i = \sum_{\ell\neq i}(f_j^{i,\ell}V_\ell + f^{i,\ell}V_{j\ell}),
$$

where the subscript j denotes the partial derivative with respect to x_j. Substituting these results and expressions (3) into the defining equations (36) and equating coefficients of V_1, V_2, V_3, respectively, we obtain the independent conditions ($G_s \equiv (\ln\lambda)_s$):

$$
\begin{aligned}
\lambda a^{111} &= \frac{2}{3}(f^{1,2}\tilde{A}^{12} + f^{1,3}\tilde{A}^{13}) + \frac{1}{3}\sum_{s=1}^{3} f^{s,1}G_s, \qquad (38)\\
\lambda a^{222} &= \frac{2}{3}(-f^{1,2}\tilde{B}^{12} + f^{2,3}\tilde{B}^{23}) + \frac{1}{3}\sum_{s=1}^{3} f^{s,2}G_s,
\end{aligned}
$$

$$
\begin{aligned}
\lambda a^{333} &= \frac{2}{3}(-f^{1,3}\tilde{C}^{13} - f^{2,3}\tilde{C}^{23}) + \frac{1}{3}\sum_{s=1}^{3} f^{s,3}G_s, \\
\lambda a^{112} &= \frac{2}{9}\left(f^{1,2}(\tilde{A}^{22} - \tilde{A}^{11} + \tilde{B}^{12}) + f^{1,3}(\tilde{A}^{23} + \tilde{B}^{13}) + f^{2,3}\tilde{A}^{13}\right) + \frac{1}{9}\sum_{s=1}^{3} f^{s,2}G_s, \\
\lambda a^{113} &= \frac{2}{9}\left(f^{1,2}(\tilde{A}^{23} + \tilde{C}^{12}) + f^{1,3}(\tilde{A}^{33} - \tilde{A}^{11} + \tilde{C}^{13}) - f^{2,3}\tilde{A}^{12}\right) + \frac{1}{9}\sum_{s=1}^{3} f^{s,3}G_s, \\
\lambda a^{122} &= \frac{2}{9}\left(f^{1,2}(-\tilde{A}^{12} + \tilde{B}^{22} - \tilde{B}^{11}) + f^{1,3}\tilde{B}^{23} + f^{2,3}(\tilde{A}^{23} + \tilde{B}^{13})\right) + \frac{1}{9}\sum_{s=1}^{3} f^{s,1}G_s, \\
\lambda a^{223} &= \frac{2}{9}\left(-f^{1,2}(\tilde{B}^{13} + \tilde{C}^{12}) - f^{1,3}\tilde{B}^{12} + f^{2,3}(-\tilde{B}^{22} + \tilde{B}^{33} + \tilde{C}^{23})\right) + \frac{1}{9}\sum_{s=1}^{3} f^{s,3}G_s, \\
\lambda a^{133} &= \frac{2}{9}\left(f^{1,2}\tilde{C}^{23} + f^{1,3}(-\tilde{A}^{13} + \tilde{C}^{33} - \tilde{C}^{11}) - f^{2,3}(\tilde{A}^{23} + \tilde{C}^{12})\right) + \frac{1}{9}\sum_{s=1}^{3} f^{s,1}G_s, \\
\lambda a^{233} &= \frac{2}{9}\left(-f^{1,2}\tilde{C}^{13} - f^{1,3}(\tilde{B}^{13} + \tilde{C}^{12}) + f^{2,3}(-\tilde{B}^{23} - \tilde{C}^{22} + \tilde{C}^{33})\right) + \frac{1}{9}\sum_{s=1}^{3} f^{s,2}G_s, \\
\lambda a^{123} &= \frac{2}{9}\left(f^{1,2}(\tilde{C}^{22} - \tilde{C}^{11}) + f^{1,3}(\tilde{B}^{33} - \tilde{B}^{11}) + f^{2,3}(-\tilde{B}^{12} + \tilde{C}^{13})\right),
\end{aligned}
$$

$$
\begin{aligned}
f_1^{1,2} &= \frac{1}{3}\left(f^{1,2}(\tilde{A}^{22} - \tilde{A}^{11} - 2\tilde{B}^{12}) + f^{1,3}(\tilde{A}^{23} - 2\tilde{B}^{13}) + f^{2,3}\tilde{A}^{13}\right) - \frac{1}{3}\sum_{s=1}^{3} f^{s,2}G_s, \qquad (39) \\
f_2^{1,2} &= \frac{1}{3}\left(f^{1,2}(-2\tilde{A}^{12} - \tilde{B}^{22} + \tilde{B}^{11}) - f^{1,3}\tilde{B}^{23} + f^{2,3}(2\tilde{A}^{23} - \tilde{B}^{13})\right) + \frac{1}{3}\sum_{s=1}^{3} f^{s,1}G_s, \\
f_1^{1,3} &= \frac{1}{3}\left(f^{1,2}(\tilde{A}^{23} - 2\tilde{C}^{12}) + f^{1,3}(\tilde{A}^{33} - \tilde{A}^{11} - 2\tilde{C}^{13}) - f^{2,3}\tilde{A}^{12}\right) - \frac{1}{3}\sum_{s=1}^{3} f^{s,3}G_s, \\
f_3^{1,3} &= \frac{1}{3}\left(-f^{1,2}\tilde{C}^{23} + f^{1,3}(-2\tilde{A}^{13} - \tilde{C}^{33} + \tilde{C}^{11}) + f^{2,3}(-2\tilde{A}^{23} + \tilde{C}^{12})\right) + \frac{1}{3}\sum_{s=1}^{3} f^{s,1}G_s, \\
f_2^{2,3} &= \frac{1}{3}\left(f^{1,2}(2\tilde{C}^{12} - \tilde{B}^{13}) - f^{1,3}\tilde{B}^{12} + f^{2,3}(-\tilde{B}^{22} + \tilde{B}^{33} - 2\tilde{C}^{23})\right) - \frac{1}{3}\sum_{s=1}^{3} f^{s,3}G_s, \\
f_3^{2,3} &= \frac{1}{3}\left(f^{1,2}\tilde{C}^{13} + f^{1,3}(\tilde{C}^{12} - 2\tilde{B}^{13}) + f^{2,3}(-2\tilde{B}^{23} + \tilde{C}^{22} - \tilde{C}^{33})\right) + \frac{1}{3}\sum_{s=1}^{3} f^{s,2}G_s,
\end{aligned}
$$

and

$$
\begin{aligned}
f_1^{2,3} + f_2^{1,3} &= \frac{1}{3}\Big(-f^{1,2}(\tilde{C}^{22} - \tilde{C}^{11}) \qquad (40) \\
&+ f^{1,3}(2\tilde{B}^{33} - 2\tilde{B}^{11} - 3\tilde{C}^{23}) - f^{2,3}(2\tilde{B}^{12} + \tilde{C}^{13})\Big),
\end{aligned}
$$

$$\begin{aligned} -f_1^{2,3}+f_3^{1,2} &= \frac{1}{3}\Big(-f^{1,2}(2\tilde{A}^{13}+\tilde{B}^{23}) \\ &- f^{1,3}(\tilde{B}^{33}-\tilde{B}^{11})+f^{2,3}(\tilde{B}^{12}+2\tilde{C}^{13})\ \Big)\,, \end{aligned}$$

Now, we substitute expressions (39) and (40) into the condition (36) for $2a_3^{123}+a_2^{133}+a_1^{233}=\cdots$ and solve for the derivative $f_3^{1,2}$ as a linear combination of the undifferentiated terms $f^{\ell,j}$. The result takes the form

$$(\tilde{C}^{22}-\tilde{C}^{11}+\tilde{A}^{13}-\tilde{B}^{23})f_3^{1,2}=\cdots$$

where, by assumption, the coefficient of $f_3^{1,2}$ is nonzero. Thus we have 9 equations for the 9 derivatives $f_k^{i,j}$ and the system closes. A solution is determined uniquely by 3 parameters $f^{i,j}(\mathbf{x}_0)$ at a regular point. and these parameters are constrained by at least 8 linearly independent conditions. Thus the solution space must be of dimension ≤ 3. We have still to apply the remaining conditions that the a^{ijk} are third order Killing tensors.

Theorem 6 *For a true 3-parameter system the a^{ijk}, b^ℓ are uniquely determined by the three numbers $f^{1,2},\ f^{1,3},\ f^{2,3}$, at any regular point (x_0,y_0,z_0) of V.*

Let

$$\mathcal{S}_1=\sum a^{kj}_{(1)}p_kp_j+W_{(1)},\quad \mathcal{S}_2=\sum a^{kj}_{(2)}p_kp_j+W_{(2)}$$

be second order constants of the the motion for a superintegrable system with 3-parameter potential and let $\mathcal{A}_{(i)}(x,y,z)=\{a^{kj}_{(i)}(x,y,z)\}$, $i=1,2$ be the corresponding 3×3 matrix functions. Then the Poisson bracket of these symmetries is given by

$$\{\mathcal{S}_1,\mathcal{S}_2\}=\sum_{k,j,i=1}^{3}a^{kji}(x,y,z)p_kp_jp_i+b^\ell(x,y,z)p_\ell$$

where

$$f^{k,\ell}=2\lambda\sum_j(a^{kj}_{(2)}a^{j\ell}_{(1)}-a^{kj}_{(1)}a^{j\ell}_{(2)}). \tag{41}$$

Clearly, $\{\mathcal{S}_1,\mathcal{S}_2\}$ is uniquely determined by the skew-symmetric matrix $[\mathcal{A}_{(2)},\mathcal{A}_{(1)}]\equiv\mathcal{A}_{(2)}\mathcal{A}_{(1)}-\mathcal{A}_{(1)}\mathcal{A}_{(2)}$, hence by the constant matrix $[\mathcal{A}_{(2)}(\mathbf{x}_)),\mathcal{A}_{(1)}(\mathbf{x}_)]$ evaluated at the regular point. Thus $\mathcal{S}_1,\mathcal{S}_2$ are in involution if and only if matrices $\mathcal{A}_{(1)}(\mathbf{x}_0),\mathcal{A}_{(2)}(\mathbf{x}_0)$ commute. It is a straightforward computation to show that the space spanned by all commutators $[\mathcal{A}^{(\ell,m)},\mathcal{A}^{(\ell',m')}]$ of the second order basis symmetries is 3-dimensional. Thus every constant skew-symmetric matrix can be expressed as a linear combination of commutators of second order basis symmetries.

Corollary 3 *Let V be a superintegrable true 3-parameter potential for an FLI superintegrable system on a conformally flat space. Then the space of third order constants of the motion is 3-dimensional and is spanned by Poisson brackets of the second order constants of the motion. The Poisson bracket of two second order constants of the motion is uniquely determined by the matrix commutator of the second order constants at a regular point.*

Theorem 7 *Let V be a superintegrable true 3-parameter potential in a 3D conformally flat space. Then V defines a multiseparable system.*

PROOF: At a regular point the obstruction is determined by the condition trace $(\mathcal{A}J) = 0$ where J is the traceless symmetric matrix (32). According to Theorem 3 of [24], if we can find two commuting matrices $\mathcal{A}_0, \mathcal{B}_0$ at the regular point such that they each satisfy the obstruction condition and $\mathcal{A}_0$ has 3 distinct roots, then the space spanned by $\mathcal{H}_0, \mathcal{A}_0, \mathcal{B}_0$ determines a separable coordinate system for the superintegrable system. (here $\mathcal{H}_0$ is the evaluation of the Hamiltonian at the point, a multiple of the identity matrix.) In this case we can prove more: there exists a 1-parameter family of distinct subspaces satisfying the separability condition. This means that there is a 1-parameter family of distinct separable coordinates, a condition stronger than multiseparability. By performing a suitable complex rotation about the regular point, we can require that the obstruction matrix take exactly one of the canonical forms [25]

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -a-c & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \begin{pmatrix} -2a & 0 & 0 \\ 0 & a+i/2 & 1/2 \\ 0 & 1/2 & a-i/2 \end{pmatrix}, \tag{42}$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1+i & 0 \\ 1+i & 0 & 1-i \\ 0 & 1-i & 0 \end{pmatrix}.$$

For the first canonical form and most instances of the second we have $X_1 + X_2 \neq 0$ in (32) so we can use the basis (33) to represent the matrices. We look for commuting matrices of the form

$$\mathcal{A}_0 = \mathcal{A}^{(11)} + a_{13}\mathcal{A}^{(13)} + a_{23}\mathcal{A}^{(23)}, \quad \mathcal{B}_0 = b_{12}\mathcal{A}^{(12)} + b_{13}\mathcal{A}^{(13)} + b_{23}\mathcal{A}^{(23)}.$$

The requirement $\{\mathcal{A}_0, \mathcal{B}_0\} = 0$ leads to a system of three linear homogeneous equations for b_{12}, b_{13}, b_{23}, and that system has a nonzero solution if and only if the condition

$$(X_2^2+X_1X_2)a_{13}^2-([X_1Y_1+X_2Y_1]a_{23}+X_2Y_2)a_{13}+([X_1^2+X_1X_2]a_{23}^2+X_1Y_3a_{23}-X_1X_2) = 0.$$

This is a quadratic equation for a_{13}, in terms of the parameter $\mu = a_{23}$, and (to be definite) we choose the solution $a_{13}(\mu)$ with the positive sign in front of the square root term. At this point we have shown the existence of a 1-parameter family of commuting symmetries with μ as the parameter. Each such pair will necessarily define a separable coordinate system if the eigenvalues of the matrix $\mathcal{A}_0$ are pairwise distinct. The characteristic equation for the eigenvalues is

$$-\xi^3 + M_1\xi^2 + M_2\xi + M_3 = 0 \tag{43}$$

where $M_3 = -\mu^2$ and

$$M_1 = \frac{a_{13}Y_2 + \mu Y_3 + 2X_1 + X_2}{X_1 + X_2}, \quad M_2 = a_{13}^2 + \mu^2 - \frac{a_{13}Y_2 + \mu Y_3 + X_1}{X_1 + X_2}.$$

This equation will have a repeated root if and only if (43) and the derivative equation

$$-3\xi^2 + 2M_1\xi + M_2 = 0 \tag{44}$$

have a common solution, which works out to be $\xi_0 = -\frac{1}{2}(9M_3+M_1M_2)/(3M_3+M_1^2)$. If we require that $\xi = \xi_0$ be a solution of (43) for all μ we find that there is no solution. Thus the eigenvalues of the matrix $\mathcal{A}_0(\mu)$ are pairwise distinct for a continuum range of values of μ.

An analogous computation for the remaining cases of the canonical forms (42), where there are fewer parameters and we assume that $Y_3 \neq 0$ rather than $X_1+X_2 \neq 0$, leads to the same conclusion: the eigenvalues of the matrix $\mathcal{A}_0(\mu)$ are pairwise distinct for a continuum range of values of μ. Q.E.D.

Using the result of the preceding theorem we can mimic the corresponding proof in [4] to obtain the following.

Theorem 8 *Every FLI superintegrable system with true 3-parameter potential on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.*

Although the spaces of higher order symmetries for true 3-parameter systems have an interesting structure, the algebra generated by the second order symmetries and their commutators doesn't close at level six.

Theorem 9 *For an FLI superintegrable system with true 3-parameter potential on a 3D conformally flat space there exist two second order constants of the motion $\mathcal{S}_1, \mathcal{S}_2$ such that $\{\mathcal{S}_1, \mathcal{S}_2\}^2$ is not expressible as a cubic polynomial in the second order constants of the motion.*

PROOF: According to Corollary 3 the space of third order constants of the motion is spanned by commutators of second order constants of the motion.

Thus it is sufficient to show that there exists a third order constant of the motion

$$\mathcal{K}=\sum_{k,j,i=1}^{3} a^{kji}p_kp_jp_i+b^\ell(x,y,z)p_\ell,\ \ b^\ell=\sum_{j=1}^{3} f^{\ell,j}V_j$$

such that, at a regular point $\mathbf{x}_0$, $\mathcal{K}^2$ is not expressible as a cubic polynomial in the basis symmetries

$$\mathcal{S}^{(\ell,m)}=\sum_{ij}\mathcal{A}^{(\ell,m)}_{ij}p_ip_j+W^{(\ell,m)}.$$

Recall that $W^{(\ell,m)}(\mathbf{x}_0)\equiv 0$. We will also choose the orientation of the Cartesian-like coordinates around $\mathbf{x}_0$ such that the elements of $\mathcal{J}$ are all nonvanishing. Thus a basis for the matrices $\mathcal{A}$ is given by (33). Since the space of third order symmetries is 3-dimensional we can find a symmetry $\mathcal{K}$ for any choice of the skew-symmetric elements $f^{\ell,j}$ at $\mathbf{x}_0$. The sixth order symmetry $\mathcal{K}^2$ contains terms of order 6, 4 and 2 in the p_j. The terms of order 2 are $k=\sum_{\ell,m,i,j} f^{\ell,j}f^{m,i}V_jV_ip_\ell p_m$. Now suppose that we could expand K^2 as a polynomial of order 3 in the basis symmetries. The contribution of a third order monomial $\mathcal{S}^{(\ell_1,m_1)}\mathcal{S}^{(\ell_2,m_2)}\mathcal{S}^{(\ell_3,m_2)}$ and a second order monomial $\mathcal{S}^{(\ell_1,m_1)}\mathcal{S}^{(\ell_2,m_2)}$ to the quadratic terms p_ip_j will vanish at the regular point because each contribution will contain at least one factor $W^{(\ell,m)}$ that vanishes at the point. The only way that we can match a nonvanishing quadratic expression in the momenta is via a parameter-dependent linear combination of second order symmetries $\mathcal{S}^{(\ell,m)}$, and we must be able to do so for all choices of the V_j. Now choose $V_1=1, V_2=V_3=0, f^{1,3}=1, f^{1,2}=f^{2,3}=0$. Then k reduces to the quadratic form with matrix

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

However, it is not possible to expand this matrix as a linear combination of the standard basis elements (33). Q.E.D.

6 Outlook

The basic structure and classification problems for 2D second order superintegrable systems systems have been solved, and the methods of this paper provide tools to complete the more difficult fine structure analysis for 3D systems. The theory for nondegenerate (4-parameter) potentials is nearly complete and this paper makes major progress for 3-parameter potentials.

More details about the dimensions and structure of the spaces of higher order symmetries, and classification of the possible systems still remain to do.

The 3D fine structure analysis can be extended to analyze 2-parameter and 1-parameter potentials with 5 functionally linearly independent second order symmetries. Here first order PDEs for the potential appear, as well as second order, and Killing vectors may occur. A separate class of 3D superintegrable systems is that for which the 5 functionally independent symmetries are functionally linearly dependent. This class contains the Calogero potential [19, 20, 21] and necessarily leads to first order PDEs for the potential, as well as second order [5]. However, the integrability condition methods discussed here should be able to handle this class with no special difficulties.

Whereas 2D superintegrable systems are very special, the 3D systems seem to be good guides to the structure of general nD systems, and we intend to proceed with this analysis. Finally, our ultimate goal is to understand the structure of superintegrable systems in general. We have started with second order systems because of their historical connection to the Kepler-Coulomb problem and to separation of variables. However, since most of the methods that we have developed make use of integrability conditions alone, not separation of variables (a purely second order phenomenon) they show promise of being extendable to higher order superintegrable systems.

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