INTRINSIC CHARACTERISATION OF THE SEPARATION CONSTANT FOR SPIN ONE AND GRAVITATIONAL PERTURBATIONS IN KERR GEOMETRY

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ABSTRACT. In this article the separation constants for the spin one and gravitational perturbations in Kerr geometry are characterised intrinsically. The characterisations are given for the corresponding Hertz potentials and induced for the corresponding vector and metric perturbations using the results of Cohen and Kegeles.

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1. Introduction. One of the remarkable properties of the space time metric of a rotating black hole described by the Kerr metric is that many of the equations of mathematical physics can be exactly solved [1]. This is due in part to the fact that in the Hertz potential formalism as found for instance in Cohen and Kegeles [2], the corresponding potentials can be solved by a separation of variables. Using a Hertz potential formalism, expressions for the spin one and gravitational perturbations can be readily obtained. It is the characterisation of the separation constant appearing in these expressions that we determine here. A thorough treatment of much of the mathematical theory is given in the book by Chandrasekhar [3]. The problem of characterisation of the separation constant has been solved in the case of the spinor form of Maxwell's equations (in a Kerr space time background) as well as the spinor form of the gravitational perturbation equations (in flat space time

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background) in [4, 5, 6]. However this does not solve the problem of characterisation for gravitational perturbations in a Kerr space time. In their article, Cohen and Kegeles have given explicit formulas for the vector potential perturbation and gravitational perturbations. It is for these perturbations that we provide a characterisation. We use the standard notation of Chandrasekhar [3] and Penrose and Rindler [7]. A summary of details about the Kerr metric and the tetrad related to it is given in the second appendix.

2. Intrinsic characterisation of the separation constants. For Maxwell's equations

$$\nabla^\mu F_{\mu\nu} = 0, \quad \text{with } F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{2.1}$$

we can obtain a solution for the vector potential A_μ by deriving it from a Hertz potential $P_{A'B'}$ via the relation

$$A_{CC'} = (\nabla_{CD'} + 2U_{CD'})P^{D'}{}_{C'} \tag{2.2}$$

where $U_{BC'}$ is the vector with components

$$\begin{aligned} U_{00'} =& \rho^*, \quad U_{01'} = -\pi^*, \quad U_{10'} = \tau^*, \quad U_{11'} = -\mu^* \\ U_{AA'} =& \partial_{AA'} \log(1/\bar\rho *). \end{aligned} \tag{2.3}$$

The Hertz potential satisfies

$$\nabla^{C(A'}\nabla_{CC'}P^{B')C'} + 2\nabla^{C(A'}U_{CC'}P^{B')C'} = 0. \tag{2.4}$$

If we choose the natural tetrad given in Appendix B, then in Newman Penrose notation these equations have the form

$$\begin{aligned} [(\Delta + \mu - \gamma + \gamma^*)(D + 2\epsilon^* + \rho^*) - (\delta + \beta - \tau + \alpha^*)(\delta^* + 2\beta^* + \tau^*)]P^{0'0'} = 0 \\ [(\Delta + \mu - \gamma - \gamma^* - \mu^*)D - (\delta + \beta - \tau - \alpha^* - \pi^*)\delta^* \\ -(\delta^* + \pi - \alpha + \beta^* + \tau^*)\delta + (D + \epsilon - \rho + \epsilon^* + \rho^*)\Delta]P^{0'1'} = 0 \\ [(D + \epsilon - \rho - \epsilon^*)(\Delta - 2\gamma^* - \mu^*) - (\delta^* + \pi - \alpha - \beta^*)(\delta - 2\alpha^* - \pi^*)]P^{1'1'} = 0. \end{aligned} \tag{2.5}$$

If we look for solutions of the form $f(r,\theta)e^{i(\sigma t + m\varphi)}$, these decoupled equations become

$$\begin{aligned} (\Delta D_0^+ D_0 + L_0^+ L_1 + 2i\sigma\bar\rho^*)P^{0'0'} = 0 \\ (\Delta D_1^+ D_0 + L_1^+ L_0 - 2i\sigma\bar\rho^* + \frac{2M\bar\rho^*}{\bar\rho^2})\frac{1}{\bar\rho}P^{0'1'} = 0 \\ (\Delta D_1^+ D_1 + L_0 L_1^+ - 2i\sigma\bar\rho^*)\frac{1}{\bar\rho^2}P^{1'1'} = 0. \end{aligned} \tag{2.6}$$

The equations for $P^{0'0'}$ and $P^{1'1'}$ admit separable solutions

$$P^{0'0'} = R_{-1}(r)S_1(\theta)e^{i(\sigma t + m\varphi)}, \quad P^{1'1'} = \bar\rho^2 R_1(r)S_{-1}(\theta)e^{i(\sigma t + m\varphi)}$$

where $R_{\pm 1}$, $S_{\pm 1}$ are the so called Teukolsky functions. There is no separable solution for the function $P^{0'1'}$. It is significant that the equations do decouple and the two extreme components decouple via a separation of variables. The Teukolsky functions satisfy the differential equations

(2.7)

$$(\Delta D_0^+ D_0 + 2i\sigma r - \lambda)R_{-1} = 0, \quad (\Delta D_1^+ D_1 - 2i\sigma r - \lambda)R_{+1} = 0$$
$$(L_0^+ L_1 + 2a\sigma\cos\theta + \lambda)S_1 = 0, \quad (L_0 L_1^+ - 2a\sigma\cos\theta + \lambda)S_{-1} = 0.$$

The question naturally arises as to the intrinsic significance of the separation parameter λ. The critical feature of Kerr space time which allows these separation properties appears to be the existence of a Killing Yano tensor $K_{\mu\nu}$, which satisfies

$$\nabla_{(\nu}K_{\mu)\sigma} = 0, \quad K_{\mu\nu} + K_{\nu\mu} = 0. \tag{2.8}$$

In the natural tetrad the nonzero components are $K_{34} = r$, $K_{12} = ia\cos\theta$. This Killing Yano tensor can also be decomposed in terms of symmetric spinors $K_{MN}, K_{M'N'}$ according to

$$K_{\nu\mu} = K_{MM'NN'} = \frac{1}{2}(\epsilon_{M'N'}K_{MN} + \epsilon_{MN}K_{M'N'})$$

where $\nabla_{(AA'}K_{BC)} = 0$, $\nabla_{A(A'}K_{B'C')} = 0$.

For the spinor form of Maxwell's equations, $\nabla^{AA'}\varphi_{AC} = 0$, an intrinsic characterisation has been given in [4]. These ideas have also been extended to the solutions of spinor spin s equation of the form $\nabla^{AA'}\varphi_{AC\ldots D} = 0$, [5]. However the equations are inconsistent for curved space time with $s > 1$. We consider for the moment the Hertz potential equation (2.4). Because of the nonseparability of the equation for $P^{0'1'}$ we modify the equation into a new covariant equation for which all components separate. We choose the equation

(2.9)

$$\frac{1}{2}\nabla^{EE'}\nabla_{EE'}P^{A'B'} + 2\nabla^{B(A'}U_{BC'}P^{C'B')} - \Psi_{E'C'}{}^{A'B'}P^{E'C'}$$
$$+\frac{1}{4}\Psi^{ABCD}K_A{}^E K_{EF}K^F{}_B K_{CD}\left(\frac{1}{\bar\rho^2}P^{A'B'} - \frac{1}{\bar\rho^4}K^{A'}{}_{C'}K^{B'}{}_{D'}P^{C'D'}\right) = 0$$

where $\bar\rho^2 = \frac{1}{2}K^{A'B'}K_{A'B'}$. If one looks for solutions of the form $f(r,\theta)e^{i(\sigma t + m\varphi)}$, the equations for $P^{0'0'}$ and $P^{1'1'}$ are the same as before, but the equation for $P^{0'1'}$ is now

$$(\Delta D_1^+ D_0 + L_1^+ L_0 - 2i\sigma\bar\rho^*)\frac{1}{\bar\rho}P^{0'1'} = 0. \tag{2.10}$$

The modified covariant equation admits separable solutions for all the components. Can we make an invariant characterisation of the corresponding separable solutions of this equation? The answer is yes. Specifically, we establish an intrinsic characterisation of the operator which is diagonal with components

$$[\frac{1}{\rho^2}(r^2 L_0^+ L_1 - a^2\cos^2\theta\Delta D_0^+ D_0) + \frac{2ra\sigma\cos\theta}{\bar\rho}]P^{0'0'},$$
$$\bar\rho[\frac{1}{\rho^2}(r^2 L_1^+ L_0 - a^2\cos^2\theta\Delta D_1^+ D_0) - \frac{2ra\sigma\cos\theta}{\bar\rho}]\frac{1}{\bar\rho}P^{0'1'}, \tag{2.11}$$
$$\bar\rho^2[\frac{1}{\rho^2}(r^2 L_0 L_1^+ - a^2\cos^2\theta\Delta D_1 D_1^+) - \frac{2ra\sigma\cos\theta}{\bar\rho}]\frac{1}{\bar\rho^2}P^{1'1'},$$

to within different additive constants. We demonstrate that this operator can be expressed in covariant form. The operator obtained in this way is unique and can be written as

$$LP^{I'J'} = \hat{K}^{\mu\nu}\nabla_\mu\nabla_\nu P^{I'J'} + L^{I'J'}{}_{K'L'}{}^{EE'}\nabla_{EE'}P^{K'L'} + M^{I'J'}{}_{K'L'}P^{K'L'}. \tag{2.12}$$

where $\hat{K}^{\mu\nu} = K^{\mu\sigma}K^{\nu}_{\sigma}$. The explicit parts of $L^{I'J'}{}_{K'L'}{}^{EE'}$ consist of

(2.13)
$$\begin{aligned} L^{I'J'}{}_{K'L'}{}^{EE'} &= K^{AB'CD'}(\nabla_{AB'}K^{EE'}{}_{CD'})\epsilon^{I'}{}_{K'}\epsilon^{J'}{}_{L'} \\ &\quad - K^{AA'}{}_{C}{}^{(I'}\nabla_{AA'}K^{C}{}_{(K'}{}^{EE'})\epsilon^{J')}{}_{L')} + L^{(I'}{}_{(K'}{}^{EE'}\epsilon^{J')}{}_{L')} \end{aligned}$$

where

$$\begin{aligned} L^{I'}{}_{K'}{}^{EE'} &= \frac{1}{2}K^{C}{}_{B'CJ'}K^{DB'}{}_{D}{}^{J'}(\epsilon^{I'}{}_{K'}U^{EE'} + \epsilon^{E'}{}_{K'}U^{EI'} - 2\epsilon^{I'E'}U^{E}{}_{K'}) \\ &\quad + \frac{1}{2}K_{B}{}^{C'}{}_{JC'}K^{BD'J}{}_{D'}\epsilon^{I'E'}U^{E}{}_{K'} + 2\epsilon_{K'}{}^{E'}K^{CH'}{}_{C}{}^{I'}K^{EC'S}{}_{C'}U_{SH'}, \end{aligned}$$

and

$$\begin{aligned} M^{I'J'}{}_{K'L'} &= K^{(I'P'}K^{QR}U_{QP'}U_{RK'}\epsilon^{J')}{}_{L'} + K^{PQ}K_{K'L'}U_{P}{}^{I'}U_{Q}{}^{J'} \\ &\quad - 2U^{AA'}U_{AA'}K^{M'P'}K^{QR}U_{QP'}U_{RM'}K^{I'J'}K_{K'L'} \\ &\quad + \frac{1}{8}K^{AB}K_{AB}U^{PI'}U_{PL'}\epsilon^{J'}{}_{K'} + \frac{1}{2}K^{I'J'}K_{K'L'}U^{AA'}U_{AA'} \\ &\quad + (\frac{1}{2} - \frac{M}{2\bar{\rho}^*} + \frac{3M(\bar{\rho}^*)^2}{\bar{\rho}^3})\epsilon^{(I'}{}_{K'}\epsilon^{J')}{}_{L'} \\ &\quad + (\frac{M}{\bar{\rho}} + \frac{M\bar{\rho}^*}{\bar{\rho}^2} - \frac{4M^2}{\bar{\rho}^2} - \frac{2M^2\bar{\rho}^*}{\bar{\rho}^3} + \frac{4Ma^2}{\bar{\rho}^3} \\ &\quad - \frac{2M^2}{\bar{\rho}\bar{\rho}^*} + \frac{4Ma^2}{\bar{\rho}^2\bar{\rho}^*})(\epsilon^{(I'}{}_{K'}\epsilon^{J')}{}_{L'} - \frac{1}{\bar{\rho}^2}K^{(I'}{}_{K'}K^{J')}{}_{L'}). \end{aligned}$$

It is then possible to look for eigenfunctions of the operator L. We note that the eigenvalues need not always be the same. This analysis contains the main ingredient for the solution of the problem for the Hertz potential of the gravitational perturbations. The corresponding Hertz potential equation is

$$\begin{aligned} &\frac{1}{2}\nabla^{EE'}\nabla_{EE'}P^{A'B'C'D'} + 4\nabla^{B(A'}U_{BF'}P^{B'C'D')F'} - 6\Psi_{E'Q'}{}^{(A'B'}P^{C'D')E'Q'} \\ &+\frac{3}{2}\Psi^{ABCD}K_{A}{}^{E}K_{EF}K^{F}{}_{B}K_{CD}(\frac{1}{\bar{\rho}^2}P^{A'B'C'D'} - \frac{1}{\bar{\rho}^4}K^{(A'}{}_{F'}K^{B'}{}_{G'}P^{C'D')F'G'}) \\ &= 0. \end{aligned} \tag{2.14}$$

For this modified equation each component separates and satisfies an equation of the form

$$(\Delta D^{+}_{p-1}D_0 + L^{+}_{p-1}L_{2-p} - 2i\sigma(3-2p)\bar{\rho}^*)\frac{1}{(\bar{\rho})^p}P^p = 0, \tag{2.15}$$

where $P^p = P^{0'\ldots 0'1'\ldots 1'}$ is the component having p $1'$ entries and $4-p$ $0'$ entries, $(p=0,..,4)$. The operator in this case can be sought just as before and taken to be of the form

$$
\begin{aligned}
LP^{I'J'K'L'} =&\hat{K}^{\mu\nu}\nabla_\mu\nabla_\nu P^{I'J'K'L'} + L^{I'J'K'L'}{}_{M'N'O'P'}{}^{EE'}\nabla_{EE'}P^{M'N'O'P'} \\
&+ M^{I'J'K'L'}{}_{M'N'O'P'}P^{M'N'O'P'},
\end{aligned} \tag{2.16}
$$

or

$$
\begin{aligned}
LP^p =&(\bar{\rho})^p[\frac{1}{\rho^2}(r^2L^+_{p-1}L_{2-p} - a^2\cos^2\theta\ \Delta D^+_{p-1}D_0) \\
&+ \frac{2}{\bar{\rho}}(3-2p)ra\sigma\cos\theta\ ]\frac{1}{\bar{\rho}^p}P^p.
\end{aligned}
$$

The contributing components are

$$
\begin{aligned}
L^{I'J'K'L'}{}_{M'N'O'P'}{}^{EE'} =& K^{AB'CD'}(\nabla_{AB'}K^{EE'}{}_{CD'})\epsilon^{I'}{}_{M'}\epsilon^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L'}{}_{P'} \\
&- \frac{1}{6}K^{AA'}{}_C{}^{(I'}(\nabla_{AA'}K^C{}_{M'}{}^{EE'})\epsilon^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&+ \frac{1}{12}L^{(I'}{}_{M'}{}^{EE'}\epsilon^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'}
\end{aligned}
$$

where $L^{I'}{}_{K'}{}^{EE'}$ is as given previously, and

$$
\begin{aligned}
M^{I'J'K'L'}{}_{M'N'O'P'} =& \frac{1}{6}K^{(I'P'}K^{QR}U_{QP'}U_{RM'}\epsilon^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&+ \frac{1}{4}K^{PQ}K_{M'N'}U_P{}^{(I'}U_Q{}^{J'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&- \frac{1}{2}U^{AA'}U_{AA'}K^{E'S'}K^{QR}U_{QS'}U_{RE'}K^{(I'J'}K_{M'N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&+ \frac{1}{12}(\bar{\rho}^*)^2U^{P(I'}U_{PM'}\epsilon^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&+ \frac{1}{12}K^{(I'J'}K_{M'N'}U^{AA'}U_{AA'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&+ \frac{1}{24}(-\frac{5}{8} + \frac{12Ma^2}{\bar{\rho}^3} + \frac{12Ma^2}{\bar{\rho}^2\bar{\rho}^*} + \frac{5\bar{\rho}^{*2}}{\bar{\rho}^3} + \frac{5M}{\bar{\rho}} + \frac{3M\bar{\rho}^*}{\bar{\rho}^2} \\
&+ \frac{3M}{\bar{\rho}^*} - \frac{6M^2}{\bar{\rho}^*\bar{\rho}} - \frac{12M^2}{\bar{\rho}^2} - \frac{6M^2\bar{\rho}^*}{\bar{\rho}^3})\epsilon^{(I'}{}_{M'}\epsilon^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&+ \frac{1}{24}(-\frac{5}{2\bar{\rho}}(K^{(I'}{}_{M'}\epsilon^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} \\
&+ \frac{1}{\bar{\rho}^2}(\frac{1}{4} - \frac{4M}{\bar{\rho}^*} - \frac{7M}{\bar{\rho}} + \frac{6M^2}{\bar{\rho}^*\bar{\rho}} + \frac{12M^2}{\bar{\rho}^2} + \frac{6M^2\bar{\rho}^*}{\bar{\rho}^2} \\
&- \frac{12Ma^2}{\bar{\rho}^2\bar{\rho}^*} - \frac{12Ma^2}{\bar{\rho}^3} - \frac{3M\bar{\rho}*}{\bar{\rho}^2})K^{(I'}{}_{M'}K^{J'}{}_{N'}\epsilon^{K'}{}_{O'}\epsilon^{L')}{}_{P'} + \\
&\frac{3}{2\bar{\rho}^3}K^{(I'}{}_{M'}K^{J'}{}_{N'}K^{K'}{}_{O'}\epsilon^{L')}{}_{P'} + \frac{3}{8\bar{\rho}^4}K^{(I'}{}_{M'}K^{J'}{}_{N'}K^{K'}{}_{O'}K^{L')}{}_{P'}).
\end{aligned}
$$

To complete the characterisation of λ, the Teukolsky equation constant, we ask the question "what operators act on the vector potential $A_{BB'}$ and the perturbed metric tensor $h_{ABA'B'}$ to give the corresponding eigenfunctions when these quantities are expressed in terms of the corresponding Hertz potentials?" We answer this by inducing the operators from a corresponding set of coupled equations. This then provides a covariant description of the perturbation process due to Cohen and Kegeles. The corresponding coupled equations are

(2.17)
$$\nabla^{AX'}(\nabla_{(AS'}A_{B)}{}^{S'}) =$$
$$-\nabla_{BS'}\left(-\frac{1}{4}\Psi^{AKCD}K_A{}^E K_{EQ}K^Q{}_B K_{CD}(\frac{1}{\bar\rho^2}P^{X'S'}-\frac{1}{\bar\rho^4}K^{X'}{}_{C'}K^{S'}{}_{D'}P^{C'D'})\right)$$

and

$$\begin{aligned}\nabla^{AX'}\nabla_{AX'}h_{CD}{}^{M'N'} &- \nabla_{AX'}\nabla_D{}^{N'}h^A{}_C{}^{M'X'} - \nabla_{AX'}\nabla_C{}^{M'}h^A{}_D{}^{N'X'}\\ &= \nabla_{(CP'}\nabla_{D)Q'}(\frac{3}{2}\Psi^{ABCD}K_A{}^E K_{EF}K^F{}_B K_{CD}\\ &\quad\times(\frac{1}{\bar\rho^2}P^{M'N'Q'P'}-\frac{1}{\bar\rho^4}K^{(M'}{}_{F'}K^{N'}{}_{G'}P^{Q'D')F'G'})).\end{aligned} \tag{2.18}$$

We illustrate this procedure for the case of spin 1 perturbations. Indeed we look for an operator $\hat L$ such that

$$\begin{aligned}\hat L A_{BB'} &= \hat K^{\mu\nu}\nabla_\nu\nabla_\mu A_{BB'} + Q_{BB'}{}^{EE'RR'}\nabla_{RR'}A_{EE'} + P_{BB'}{}^{EE'}A_{EE'}\\ &= (\nabla_{BC'}+2U_{BC'})Q^{C'}{}_{B'}\end{aligned} \tag{2.19}$$

where
$$A_{BB'} = (\nabla_{BC'}+2U_{BC'})P^{C'}{}_{B'}$$

and

$$Q^{I'J'} = \hat K^{\mu\nu}\nabla_\mu\nabla_\nu P^{I'J'} + L^{I'J'}{}_{K'L'}{}^{EE'}\nabla_{EE'}P^{K'L'} + M^{I'J'}{}_{K'L'}P^{K'L'}.$$

In order to more easily illustrate how the procedure works, we write down the equations obtained from this identification for flat space. The procedure works the same, in principle, in a curved Kerr background space time. (The only difference in the resulting equations is the addition of curvature terms that make the corresponding equations more complicated.) Equating coefficients of both sides of this identity modulo the equation satisfied by $P^{A'B'}$, we obtain the following expression

$$Q_{BB'}{}^{HJ'KK'}\epsilon^{H'I'} + \nabla_B{}^{I'}K^{KK'HH'}\epsilon^{J'}{}_{B'} - L^{H'}{}_{B'}{}^{I'J'KK'}\epsilon_B{}^H = 0$$

which is the coefficient of $\nabla_{KK'}\nabla_{HH'}P_{I'J'}$ and must accordingly be symmetrised on the pairs of indices $KH, K'H'$ and $I'J'$. This equation enables one to solve for all the components of $Q_{BB'}{}^{(HI'K)J'}$ by contraction on suitable indices. The coefficient of $\nabla_{LL'}P_{M'N'}$ is

$$\begin{aligned}&2Q_{BB'}{}^{EM'LL'}U_E{}^{K'} + P_{BB'}{}^{LM'}\epsilon^{L'K'} + 2K^{EE'LL'}(\nabla_{EE'}U_B{}^{K'})\epsilon_{B'}{}^{M'} +\\ &Q_{BB'}{}^{L'M'}U^{LK'} + Q_{BB'}{}^{M'L'}U^{LK'} + 2U_{BC'}L^{C'}{}_{B'}{}^{LL'K'M'} + M^{L'}{}_{B'}{}^{K'M'}\epsilon_B{}^L = 0,\end{aligned}$$

where the indices $M'N'$ are symmetrised and $Q_{BB'}{}^{L'M'} = Q_{BB'}{}^{LM'}{}_{L}{}^{L'}$. For the coefficients of $P_{M'L'}$ we have

$$
\begin{aligned}
&2Q_{BB'}{}^{EL'RR'}(\nabla_{RR'}U_E{}^{M'}) + 2P_{BB'}{}^{EL'}U_E{}^{M'} + 2K^{EE'FF'}(\nabla_{EE'}\nabla_{FF'}U_B{}^{L'})\epsilon_{B'}{}^{M'}\\
&+ 2Q_{BB'E'}{}^{L'}(\nabla^{CE'}U_C{}^{M'}) + 2Q_{BB'}{}^{L'}{}_{F'}(\nabla^{CF'}U_C{}^{M'}) - (\nabla_{BC'} + 2U_{BC'})M^{C'}{}_{B'}{}^{L'M'}\\
&- 2L_{BB'E'}{}^{M'}(\nabla^{QE'}U_Q{}^{L'}) = 0
\end{aligned}
$$

and $L_{BB'E'}{}^{M'} = L^{C'}{}_{B'E'}{}^{M'}{}_{BC'}$. Symmetrisation over primed indices in these last two equations enables one to find $Q^{M'K'}_{BB'}$ and $Q_{BB'} = Q^{M'}_{BB'M'}$ in terms of known quantities. The remaining equations enable one to find all the components of $P_{BB'}{}^{CC'}$. Crucial in these determinations are the relations

$$\nabla^{C(C'}U_C^{K')} = 0, \quad U_{CM'}V^C_{P'} = \epsilon_{M'P'},$$

where $V_{AA'}$ is the spinor with components

$$
\begin{aligned}
V_{00'} &= \frac{2\bar\rho^2\bar\rho^*}{\bar\rho\bar\rho^* - 2Mr}, \quad V_{11'} = \frac{\Delta\bar\rho}{\bar\rho\bar\rho^* - 2Mr}\\
V_{01'} &= \frac{-2i\sqrt{2}a\bar\rho\bar\rho^*\sin\theta}{\bar\rho\bar\rho^* - 2Mr}, \quad V_{10'} = \frac{2i\sqrt{2}a\bar\rho^2\sin\theta}{\bar\rho\bar\rho^* - 2Mr}.
\end{aligned}
$$

Appendix A. Operator characterisations of Hertz potentials. The general form of the modified Hertz potential equation for arbitrary s is

$$
\begin{aligned}
&\frac{1}{2}\nabla^{EE'}\nabla_{EE'}P^{A'B'C'\cdots D'} + 4\nabla^{B(A'}U_{BF'}P^{B'C'\cdots D')F'} - s(2s-1)\Psi^{\cdots(A'B'}_{E'C'\cdots}P^{C'D')E'C'}\\
&\text{(A.1)}\\
&\quad + \frac{s(2s-1)}{4}\Psi^{ABCD}K_A{}^E K_{EF}K^F{}_B K_{CD}(\frac{1}{\bar\rho^2}P^{A'B'C'\cdots D'}\\
&\quad - \frac{1}{\bar\rho^4}K^{(A'\cdots}_{\cdots F'}K^{B'\cdots}_{\cdots G'}P^{C'\cdots D')F'G'}) = 0.
\end{aligned}
$$

This equation separates into individual equations for each component; the corresponding separation equations are

$$\text{(A.2)} \qquad \bar\rho^p\left(\Delta D^+_{p-1}D_0 + L^+_{p-1}L_{2-p} - 2i\sigma(2s-2p-1)\bar\rho^*\right)\frac{1}{\bar\rho^p}P^p = 0$$

where the spinor $P^{A'B'\cdots D'}$ contains $2s$ indices consisting of p $1'$ entries and $2s-p$ $0'$ entries. Here we give the operators that describe the spin 1/2 and 3/2 Hertz potential operator characterisations. The operator for general s looks like

$$
\begin{aligned}
\text{(A.3)} \qquad LP^p =&\bar\rho^p[(\frac{1}{\rho^2}(r^2L^+_{p-1}L_{2-p} - a^2\cos^2\theta\ \Delta D^+_{p-1}D_0)\\
&- \frac{2}{\bar\rho}(2s-1-2p)ra\sigma\cos\theta]\frac{1}{\bar\rho^p}P^p
\end{aligned}
$$

for $s = 1/2, 3/2$ and $p = 0, \cdots, 2s$.

Here we give the operators that describe the spin 1/2 and 3/2 Hertz potential operator characterisations.

Spin 1/2. The operator in this case looks like

$$LP^{I'} = \hat{K}^{\mu\nu}\nabla_\mu\nabla_\nu P^{I'} + \hat{L}^{I'}{}_{K'}{}^{EE'}\nabla_{EE'}P^{K'} + M^{I'}{}_{K'}P^{K'}, \tag{A.4}$$

where

$$\begin{aligned}
\hat{L}^{I'}{}_{K'}{}^{EE'} &= K^{AB'CD'}(\nabla_{AB'}K_{CD'}{}^{EE'}) - K^{AA'}{}_{C}{}^{I'}(\nabla_{AA'}K^{C}{}_{K'}{}^{EE'}) - (1/2)L^{I'}{}_{K'}{}^{EE'},\\
M^{I'}_{K'} &= \frac{2}{5}(K^{I'P'}K^{QR}U_{QP'}U_{RK'} + K^{S'P'}K^{QR}U_{QP'}U_{RS'}\epsilon^{I'}_{K'})+\\
&\quad \frac{1}{5}(K^{PQ}U^{I'}_{P}U^{J'}_{Q}K_{J'K'}) + (\bar{\rho}^*)^2(-\frac{1}{3}U^{PI'}U_{PK'} + \frac{1}{12}\epsilon^{I'}_{K'}U^{PI'}U_{PI'})\\
&\quad -\frac{1}{4}K^{I'S'}K_{S'K'} + \frac{1}{2}(-1 + \frac{M}{\bar{\rho}} + \frac{M(\bar{\rho}^*)^2}{\bar{\rho}^3})\epsilon^{I'}_{K'} - \frac{1}{2}K^{I'}_{K'}.
\end{aligned}$$

Spin 3/2. For this case the operator is

$$\begin{aligned}
LP^{I'J'K'} &= \hat{K}^{\mu\nu}\nabla_\mu\nabla_\nu P^{I'J'K'} + L^{I'J'K'}{}_{M'N'O'}{}^{EE'}\nabla_{EE'}P^{M'N'O'}\\
&\quad + M^{I'J'K'}{}_{M'N'O'}P^{M'N'O'}.
\end{aligned} \tag{A.5}$$

The contributing components are

$$
\begin{aligned}
L^{I'J'K'}{}_{M'N'O'}{}^{EE'} &= K^{AB'CD'}(\nabla_{AB'}K^{EE'}{}_{CD'})\epsilon^{(I'}{}_{M'}\epsilon^{J'}{}_{N'}\epsilon^{K')}{}_{O'} \\
&\quad - \frac{1}{2}K^{AA'}{}_{C}{}^{(I'}(\nabla_{AA'}K^{C}{}_{M'}{}^{EE'})\epsilon^{J'}{}_{N'}\epsilon^{K')}{}_{O'} + \frac{1}{4}L^{(I'}{}_{M'}{}^{EE'}\epsilon^{J'}{}_{N'}\epsilon^{K')}{}_{O'}, \\
M^{I'J'K'L'}{}_{M'N'O'P'} &= \frac{1}{2}K^{(I'P'}K^{QR}U_{QP'}U_{RM'}\epsilon^{J'}{}_{N'}\epsilon^{K')}{}_{O'} \\
&\quad + \frac{1}{2}K^{PQ}K_{M'N'}U_{P}{}^{(I'}U_{Q}{}^{J'}\epsilon^{K')}{}_{O'} \\
&\quad - \frac{1}{2}U^{AA'}U_{AA'}K^{E'S'}K^{QR}U_{QS'}U_{RE'}K^{(I'J'}K_{M'N'}\epsilon^{K')}{}_{O'} \\
&\quad + \frac{1}{16}(\bar\rho^*)^2U^{P(I'}U_{PM'}\epsilon^{J'}{}_{N'}\epsilon^{K')}{}_{O'} + \frac{7}{32}K^{(I'J'}K_{M'N'}U^{AA'}U_{AA'}\epsilon^{K')}{}_{O'}\times \\
&\quad (-\frac{27}{32} - \frac{3M\bar\rho^*}{\bar\rho^3} - \frac{9M}{16\bar\rho^*} + \frac{3M\bar\rho^*}{2\bar\rho^2} + \frac{6Ma^2}{\bar\rho^3} \\
&\quad + \frac{6Ma^2}{\bar\rho^2\bar\rho^*} + \frac{3M(\bar\rho^*)^2}{\bar\rho^3} + \frac{15M}{16\bar\rho} - \frac{6M^2}{\bar\rho^2} - \frac{3M^2}{\bar\rho\bar\rho^*}) \\
&\quad + \frac{15}{32\bar\rho^3} + \frac{9}{32\bar\rho^2}\epsilon^{(I'}{}_{M'}\epsilon^{J'}{}_{N'}\epsilon^{K')}{}_{O'} \\
&\quad + (\frac{45}{32(\bar\rho)^4} - \frac{57}{32\bar\rho} + \frac{9}{32\bar\rho^3})K^{(I'}{}_{M'}\epsilon^{J'}{}_{N'}\epsilon^{K')}{}_{O'} \\
&\quad + (-\frac{33}{32\bar\rho^2} + \frac{3M^2\bar\rho^*}{\bar\rho^4} - \frac{3M}{16\bar\rho^2\bar\rho^*} - \frac{3M\bar\rho^*}{2\bar\rho^4} \\
&\quad - \frac{6Ma^2}{\bar\rho^5} - \frac{6Ma^2}{\bar\rho^5\bar\rho^*} - \frac{27M}{16\bar\rho^3} + \frac{6M^2}{\bar\rho^4} + \frac{3M^2}{\bar\rho^3\bar\rho^*} \\
&\quad + \frac{45}{32\bar\rho^5} - \frac{9}{32\bar\rho^4})K^{(I'}{}_{M'}K^{J'}{}_{N'}\epsilon^{K')}{}_{O'} + \\
&\quad (-\frac{9}{32\bar\rho^4} - \frac{15}{32\bar\rho^5} + \frac{3}{32\bar\rho^3})K^{(I'}{}_{M'}K^{J'}{}_{N'}K^{K')}{}_{O'}.
\end{aligned}
$$

Appendix B. Summary of properties of the Kerr metric. The Kerr metric describes the space time region exterior to a rotating black hole. In empty space time the line element is

$$
\begin{aligned}
ds^2 &= (1 - \frac{2M}{\bar\rho\bar\rho^*})dt^2 - \frac{\bar\rho\bar\rho^*}{\Delta}dr^2 - \bar\rho\bar\rho^* d\theta^2 \\
&\quad - (r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\bar\rho\bar\rho^*})\sin^2\theta d\varphi^2 + \frac{4aMr\sin^2\theta}{\bar\rho\bar\rho^*}dt\, d\varphi,
\end{aligned}
\tag{B.1}
$$

where

$$\bar\rho = r + ia\cos\theta, \quad \Delta = r^2 + a^2 - 2Mr.$$

A convenient tetrad basis is that described by the vectors

$$
\begin{aligned}
\ell^\mu &= \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \quad n^\mu = \frac{1}{2\bar\rho\bar\rho^*}(r^2 + a^2, -\Delta, 0, a) \\
m^\mu &= \frac{1}{\sqrt{2}\bar\rho}(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}), \quad \bar m^\mu = \frac{1}{\sqrt{2}\bar\rho^*}(-ia\sin\theta, 0, 1, -\frac{i}{\sin\theta})
\end{aligned}
$$

and the nonzero spin coefficients are

$$\rho = -\frac{1}{\bar\rho^*},\ \beta = \frac{\cot\theta}{2\sqrt{2}\bar\rho},\ \pi = \frac{ia\sin\theta}{\sqrt{2}(\bar\rho^*)^2},\ \alpha = \pi - \beta^*$$
$$\tau = -\frac{ia\sin\theta}{\sqrt{2}\bar\rho^*\bar\rho},\ \mu = -\frac{\Delta}{2\bar\rho(\bar\rho^*)^2},\ \gamma = \mu + \frac{r-M}{2\bar\rho^*\bar\rho}.$$

The directional derivatives are defined as

$$\begin{aligned}
\partial_{00'} &= D = D_0 = \partial_r + \frac{iK}{\Delta}\\
\partial_{11'} &= \Delta = -\frac{\Delta}{2\bar\rho\bar\rho^*}D_0^+ = -\frac{\Delta}{2\bar\rho\bar\rho^*}(\partial_r - \frac{iK}{\Delta})\\
\partial_{01'} &= \delta = \frac{1}{\sqrt{2}\bar\rho}L_0^+ = \frac{1}{\sqrt{2}\bar\rho}(\partial_\theta + Q)\\
\partial_{10'} &= \delta^* = \frac{1}{\sqrt{2}\bar\rho^*}L_0 = \frac{1}{\sqrt{2}\bar\rho^*}(\partial_\theta - Q),
\end{aligned}$$

where

$$K = (r^2+a^2)\sigma + am, \quad Q = a\sigma\sin\theta + m\csc\theta.$$

Here notation used throughout is that of Penrose and Rindler [7], and Chandrasekhar [3].

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