Notes on the Solution of Homework 1 of Math 5467- Spring 2004

- **Problem 1**:

  1. Evaluate
  $$\|f_n - f_m\|_2^2 = \int_n^{n+1/n} n dt + \int_m^{m+1/m} m dt = 2.$$
  That implies the distance between any two elements in the sequence is always $\sqrt{2}$. Thus, the sequence is not Cauchy, i.e. the sequence $\{f_n\}$ does not converge in L^2-norm. Recall that a sequence is Cauchy if given $\epsilon > 0$ there exists $N(\epsilon)$ such that $\|f_n - f_m\| < \epsilon \, \forall n, m > N(\epsilon)$.
  2. f_n converges pointwise to 0. Since for any $x \in [0, \infty)$, and given $\epsilon > 0$, $|f_n(x) - 0| < \epsilon \, \forall n \geq [x]$, where $[x]$ is the upper rounding of x.
  3. It does not converge uniformly. The best test for this is to consider the sequence $\{M_n\}$ where
  $$M_n = \sup_{x \in [0,\infty)} (|f_n(x) - 0|) = \sqrt{n}$$
  This sequence diverges since $\lim_{n \to \infty} M_n = \infty$. Thus, It does NOT converge uniformly.
  4. Yes, it is an orthonormal system. It is easy to show that $\langle f_n, f_m \rangle = \delta_{0,|n-m|}$. It is not a basis because it does NOT span $L^2[0, \infty)$. Take for example the gate function $f(t) = 1$ on $[0, 1]$ and 0 otherwise. It is easy to show that the inner product $\langle f_n(t), f(t) \rangle = 0 \, \forall n \geq 1$.

- **Problem 2:** Given
$$f_n(x) = \frac{x}{1 + nx^2}$$
it is easy to show that $f_n(x) \to 0$ for all $x \in \mathbf{R}$. To test its uniform convergence find the sequence
$$M_n = \sup_{x \in \mathbf{R}} |f_n(x) - 0|$$
Notice that this function is differentiable, so the sup $=$ its absolute maximum. Absolute maximum will be at its critical points, since it tends to 0 as $x \to \pm\infty$.
$$\frac{df_n(x)}{dx} = \frac{1 - nx^2}{(1 + nx^2)^2} = 0$$
At $x = 1/\sqrt{n}$ we have a critical point. Thus, the absolute maximum of $|f_n(x)\| = 1/(2\sqrt{n})$. Therefore, $M_n = 1/(2\sqrt{n})$. Since $\lim_{n \to \infty} M_n = 0$, the convergence is uniform.

- **Problem 3b:** Note that h is given as any function in $C[-\pi, \pi]$.
$$\|\chi - h\|_2^2 = \int_0^{\pi} (1 - h) dt + \int_{-\pi}^{0} h^2 dt$$
Since both $(1 - h)^2$ and h^2 are nonnegative, the sum of the two integrals is nonnegative for all $t \in [-\pi, \pi]$. The only possibility for this integral to be 0 is that $h = \chi$, which is not possible since $\chi \notin C[-\pi, \pi]$. This implies that $C[-\pi, \pi]$ is NOT complete, thus <u>not</u> Hilbert.

- **Problem 4:**
$$\int_{-\infty}^{\infty} \phi_{jk} \overline{\phi_{jl}} = \int_{-\infty}^{\infty} 2^{j/2} \phi(2^j t - k) \overline{2^{j/2} \phi(2^j t - l)} dt$$
Let $2^j t - k = x$, so $2^j dt = dx$.
$$\int_{-\infty}^{\infty} \phi_{jk} \overline{\phi_{jl}} = \int_{-\infty}^{\infty} \phi(x) \overline{\phi(x + k - l)} dx = \delta_{0,l-k}$$
When $l = k$ $\int_{-\infty}^{\infty} \phi_{jk} \overline{\phi_{jl}} = 1$, otherwise $\int_{-\infty}^{\infty} \phi_{jk} \overline{\phi_{jl}} = 0$. Thus, this set forms an orthonormal set.

- **Problem 6,7:** The first 4 Hilbert polynomials up to a scalar factor are: $1, 2x, 4x^2 - 2, 8x^3 - 12x$. For your information, the recurrence relation for this polynomial is

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

For problem 7, you need to argue as follows. Since the weight function $w(t)$ is even, i.e. $w(-t) = w(t)$ it follows that the polynomials $K_n(t) = (-1)^n H_n(-t), n = 0, 1, 2, \cdots$ form an orthonormal set on $L^2[-\infty, \infty, w(t)]$. That is

$$\langle K_n, K_m \rangle = \langle H_m, H_n \rangle = \delta_{n,m}$$

They satisfy the following properties:

1. $K_n(t)$ is a polynomial of order n.
2. The coefficients of t^n in $K_n(t)$ is positive.

However, $\{H_n(t)\}$ is the ON set uniquely defined by the previous properties using Gram-Shmidt process. Hence, $(-1)^n H_n(-t) = H_n(t)$ for all $n = 0, 1, 2, \cdots$.

- **Problem 8:** The first function $|t|$ is even, so all coefficients of sine terms are zero.

$$|t| = \frac{\pi^2}{2\sqrt{\pi}} - \frac{4}{\pi}\cos(t) - \frac{4}{9\pi}\cos(3t)$$

The second function is an odd function. All coefficients of cosine terms and the constant term are zero.

$$t = 2\sin(t) - sin(2t) + \frac{2}{3}sin(3t)$$

Notice that by drawing the periodic extension of these two functions on the whole real axis, that the first function is continuous at the boundaries of each interval, while the second function is NOT. Thus, the expansion of the first function using Fourier series converges faster without any Gibbs phenomenon. While for the second one, it converges slower with obvious Gibbs phenomenon (oscillating ripples) at the boundaries. Please, check the graphical illustration by Maple on the website.