HYPERGEOMETRIC EXPANSIONS OF HEUN POLYNOMIALS*

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1. Introduction. Any Fuchsian equation of second order with four singularities can be reduced to the form

$$\frac{d^2w}{dx^2}+[\frac{\gamma}{x-e_1}+\frac{\delta}{x-e_2}+\frac{\epsilon}{x-e_3}]\frac{dw}{dx}+\frac{\alpha\beta x-q}{(x-e_1)(x-e_2)(x-e_3)}w=0 \tag{1.1}$$

where $\alpha+\beta-\gamma-\delta-\epsilon+1=0$.

The singularities are located at $x=e_1,e_2,e_3$ and ∞ and have indices depending upon $\alpha,\ldots\epsilon$. The constant q is known as the accessory parameter. This is Heun's equation [1] and solutions may be characterised by the P symbol [2].

$$P\left\{\begin{matrix} e_1 & e_2 & e_3 & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{matrix}\right\} \tag{1.2}$$

Power series expansions for the solutions of Heun's equation have been studied by Heun for various arguments [1], [3]. There turn out to be 96 distinct types of power series. Alternatively, solutions of Heun's equations can be expanded in series of hypergeometric functions. Such expansions were studied by Svartholm [4] and Erdelyi [5]. Typically such expansions have the form

(1.3)
$$P\left\{\begin{matrix} e_1 & e_2 & e_3 & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{matrix}\right\}=\sum_{m\subset 0}^{\infty}A_mP\left\{\begin{matrix} 0 & 1 & \alpha & \\ 0 & 0 & \lambda+m & x \\ 1-\gamma & 1-\delta & \mu-m & \end{matrix}\right\}$$

where $\lambda+\mu=\gamma+\delta-1=\alpha+\beta-\epsilon$. Two types of expansion were given;

(i) Series of type I for which $\lambda=\alpha,\mu=\beta-\epsilon$. These series converge outside an ellipse with foci at e_1,e_2 and which passes through e_3. There are three distinct expansions of this type.

(ii) Series of type II for which $\mu=0$, $\gamma-1$, $\delta-1$ or $\gamma+\delta-2$.

In all these expansions the coefficients A_m satisfy three term recurrence relations

$$\begin{aligned} b_0A_0+c_1A_1&=0 \\ a_rA_{r-1}+b_rA_r+c_{r+1}A_{r+1}&=0, \quad r=1,2,\ldots \end{aligned} \tag{1.4}$$

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where a_r, b_r, c_r are known expressions in r and $c_r \neq 0$. If q is chosen from a number of characteristic values then expansions of this type converge. In this article we derive some of these expansions for the case of Heun polynomials from considerations based on group theory and its connection with separation of variables solutions of the Laplace-Beltrami eigenvalue equation on the n-sphere. The method used makes a judicious choice of coordinates on the n-sphere. The expansions that are first derived are for products of Heun polynomials as sums of products of Jacobi polynomials. The coefficients in the expansions obey three term recurrence relations. The corresponding single variable expansions are then obtained by allowing one of the variables to take a fixed value. This paper is an extension of [8] in which the motivation and background can be found.

Earlier writers on Heun functions who take a somewhat similar point of view are Sleeman [15] and Schmidt and Wolf [14]. These authors use the simultaneous separability of a generalized Schrödinger equation in several cooordinate systems to derive integral relations for Heun functions. In [8] and in the present paper we are making clearer the geometrical setting of these results: polynomial orthogonal bases on the n-sphere characterized as eigenfunctions of commuting sets of self-adjoint symmetry operators.

2. Derivation of the expansion formula. The graphical calculus of separable coordinates for the Laplace-Beltrami eigenvalue equation on the n-sphere has been completely worked out by Kalnins and Miller [6], [7]. To derive an expansion for Heun polynomials we consider coordinate systems corresponding to graphs of the type

$$\begin{array}{|c|c|c|}\hline e_1 & e_2 & e_3 \\ \hline \end{array}$$
$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ S_{n_1} & S_{n_2} & S_{n_3} \end{array}$$

on the n sphere, $n = n_1 + n_2 + n_3 + 2$. A suitable choice of coordinates is

$$\begin{aligned} s_i &= u_1 w_i, \quad && i = 1, \ldots, n_1 + 1 \\ s_{j+n_1+1} &= u_2 t_j, \quad && j = 1, \ldots, n_2 + 1 \\ s_{k+n_1+n_2+2} &= u_3 z_k, \quad && k = 1, \ldots, n_3 + 1 \end{aligned} \tag{2.1}$$

where

$$\sum_{i=1}^{n_1+1} w_i^2 = 1, \quad \sum_{j=1}^{n_2+1} t_j^2 = 1, \quad \sum_{k=1}^{n_3+1} z_k^2 = 1$$

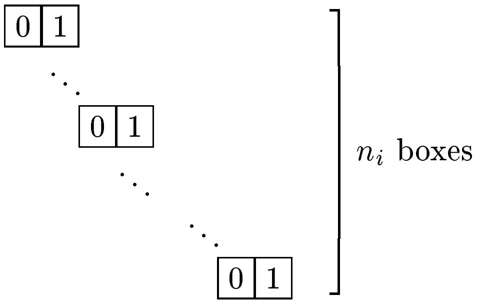
and

$$u_i^2 = \frac{(x - e_i)(y - e_i)}{(e_j - e_i)(e_k - e_i)}, \quad i = 1, 2, 3, \quad i, j, k \text{ pairwise distinct.} \tag{2.2}$$

The metric on the n sphere is

(2.3)
$$ds^2 = -\frac{(x-y)}{4}\left[\frac{dx^2}{(x-e_1)(x-e_2)(x-e_3)} - \frac{dy^2}{(y-e_1)(y-e_2)(y-e_3)}\right]$$
$$+ \frac{(x-e_1)(y-e_1)}{(e_2-e_1)(e_3-e_1)}\sum_{i=1}^{n_1+1} dw_i^2 + \frac{(x-e_2)(y-e_2)}{(e_3-e_2)(e_1-e_2)}\sum_{j=1}^{n_2+1} dt_j^2 +$$
$$\frac{(x-e_3)(y-e_3)}{(e_2-e_3)(e_1-e_3)}\sum_{k=1}^{n_3+1} dz_k^2.$$

The coordinate systems chosen for w_i, t_j, z_k can be taken to be, say, spherical coordinates in each case, corresponding to the graph [6].



We then seek eigenfunctions ψ of the Laplacian satisfying

$$\Delta\psi = -J(J+n_1+n_2+n_3+1)\psi, \tag{2.4}$$

where J is a non-negative integer. In the coordinates we have chosen, this equation has the form

(2.5)
$$\Delta\psi = -\frac{4}{(x-y)}[(x-e_1)(x-e_2)(x-e_3)\left[\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left[\frac{n_1+1}{x-e_1} + \frac{n_2+1}{x-e_2} + \frac{n_3+1}{x-e_3}\right]\frac{\partial}{\partial x}\right]\psi$$
$$-(y-e_1)(y-e_2)(y-e_3)\left[\frac{\partial^2}{\partial y^2} + \frac{1}{2}\left[\frac{n_1+1}{y-e_1} + \frac{n_2+1}{y-e_2} + \frac{n_3+1}{y-e_3}\right]\frac{\partial}{\partial y}\right]\psi]$$
$$+\left[\frac{(e_1-e_2)(e_1-e_3)}{(x-e_1)(y-e_1)}\Delta_1 + \frac{(e_2-e_1)(e_2-e_3)}{(x-e_2)(y-e_2)}\Delta_2 + \frac{(e_3-e_1)(e_3-e_2)}{(x-e_3)(y-e_3)}\Delta_3\right]\psi$$
$$= -J(J+n_1+n_2+n_3+1)\psi,$$

where Δ_k is the Laplacian on the sphere S_{n_k}.

If we seek eigenfunctions such that

$$\Delta_i\psi = -\ell_i(\ell_i+n_i-1)\psi, \qquad i=1,2,3, \tag{2.6}$$

where the ℓ_i are non-negative integers, then writing

$$\psi = \prod_{i=1}^{3}[(x-e_i)(y-e_i)]^{\ell_i/2}\phi, \tag{2.7}$$

we find (2.5) has the form

(2.8)
$$
\begin{aligned}
-\frac{4}{(x-y)}\Bigg\{&(x-e_1)(x-e_2)(x-e_3)\left(\frac{\partial^2}{\partial x^2}+\left[\frac{\ell_1+\frac{1}{2}(n_1+1)}{x-e_1}+\frac{\ell_2+\frac{1}{2}(n_2+1)}{x-e_2}\right.\right.\\
&\left.\left.+\frac{\ell_3+\frac{1}{2}(n_3+1)}{x-e_3}\right]\frac{\partial}{\partial x}\right)\phi+Ax\phi\\
&\quad-(y-e_1)(y-e_2)(y-e_3)\left(\frac{\partial^2}{\partial y^2}+\left[\frac{\ell_1+\frac{1}{2}(n_1+1)}{y-e_1}+\frac{\ell_2+\frac{1}{2}(n_2+1)}{y-e_2}\right.\right.\\
&\quad\left.\left.+\frac{\ell_3+\frac{1}{2}(n_3+1)}{y-e_3}\right]\frac{\partial}{\partial y}\right)\phi-Ay\phi\Bigg\}=-J(J+N+1)\phi
\end{aligned}
$$

where
$$A=\frac{1}{4}(L+N+1)L,\quad L=\ell_1+\ell_2+\ell_3,\quad N=n_1+n_2+n_3.$$
The corresponding separable solutions have the form

$$\psi=u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3}\Phi^1_{J\ell_1\ell_2\ell_3q}(x)\Phi^2_{J\ell_1\ell_2\ell_3q}(y)\Theta_{\ell_1\ell_2\ell_3}(\mathbf{w},\mathbf{t},\mathbf{z}) \tag{2.9}$$

where a complete set of functions $\Theta_{\ell_1\ell_2\ell_3}(\mathbf{w},\mathbf{t},\mathbf{z})$ can be taken as

$$\Theta_{\ell_1\ell_2\ell_3}(\mathbf{w},\mathbf{t},\mathbf{z})=\Theta_{\ell_1}(\mathbf{w})\Theta_{\ell_2}(\mathbf{t})\Theta_{\ell_3}(\mathbf{z}) \tag{2.10}$$

and typically,

$$\Theta_{\ell_1}(\mathbf{w})=\prod_{j=0}^{n_1-2}C^{\frac{1}{2}(n_1-j-1)+K_{j+1}}_{K_j-K_{j+1}}(\cos(\theta_{n_1-j}))\,(\sin\theta_{n_1-j})^{K_j+1}e^{\pm iK_{n_1-1}\theta_1}, \tag{2.11}$$

for $\ell_1=K_0\geq K_1\geq\cdots\geq K_{n_1-1}\geq 0$, and

$$\Delta_{(k)}\Theta_{\ell_1}(\mathbf{w})=-K_k(K_k+n_1-k-1)\Theta_{\ell_1}(\mathbf{w}) \tag{2.12}$$

where $C^v_\mu(z)$ is a Gegenbauer polynomial. The coordinates on S_{n_1} are

$$
\begin{aligned}
w_1&=\sin\theta_{n_1}\ldots\sin\theta_2\sin\theta_1\\
w_2&=\sin\theta_{n_1}\ldots\sin\theta_2\cos\theta_1\\
&\vdots\\
w_{n_1}&=\sin\theta_{n_1}\cos\theta_{n_1-1}\\
w_{n_1+1}&=\cos\theta_{n_1}
\end{aligned} \tag{2.13}
$$

and the operator $\Delta_{(k)}$ is given by

$$\Delta_{(k)}=\sum_{r<\ell\leq n_1+1-k}I^2_{r\ell},\quad I_{r\ell}=w_r\frac{\partial}{\partial w_\ell}-w_\ell\frac{\partial}{\partial w_r},\quad k=0,\ldots,n_1-1. \tag{2.14}$$

(The $\Delta_{(k)}$ are the second order symmetry operators for Δ_1 whose eigenvalue equations (2.12) characterize the separable coordinates (2.13), see [6], [7].) The corresponding separation equations are

(2.15)
$$[-4(\lambda-e_1)(\lambda-e_2)(\lambda-e_3)\left[\frac{d^2}{d\lambda^2}+\left(\frac{\ell_1+\frac{1}{2}(n_1+1)}{\lambda-e_1}+\frac{\ell_2+\frac{1}{2}(n_2+1)}{\lambda-e_2}\right.\right.$$
$$\left.\left.+\frac{\ell_3+\frac{1}{2}(n_3+1)}{\lambda-e_3}\right)\frac{d}{d\lambda}\right]+(J-L)(J+L+N+1)\lambda+4q]\Phi^{\epsilon}_{J\ell_1\ell_2\ell_3 q}(\lambda)=0$$

where $\lambda = x, y$ according as $\epsilon = 1, 2$, respectively. This is Heun's equation of the form (1.1) with $\gamma = \ell_1 + \frac{1}{2}(n_1+1)$, $\delta = \ell_2 + \frac{1}{2}(n_2+1)$, $\epsilon = \ell_3 + \frac{1}{2}(n_3+1)$, $\alpha = \frac{1}{2}(L-J)$, $\beta = \frac{1}{2}(L+J+N+1)$. The solutions for the functions $\Phi^{\epsilon}_{J\ell_1\ell_2\ell_3 q}(\lambda)$ are Heun polynomials which for fixed J will form a complete set of basis functions once the eigenvalues q have been calculated. To calculate the eigenvalues it is convenient to observe that in the coordinate system (2.1) the operator M whose eigenvalue χ is

(2.16)
$$\begin{aligned}\chi = {} & (e_1+e_2+e_3)[\ell_1^2+\ell_2^2+\ell_3^2+\ell_1 n_1+\ell_2 n_2+\ell_3 n_3-J(J+N+1)]\\ & +2\ell_1\ell_2 e_3+2\ell_1\ell_3 e_2+2\ell_2\ell_3 e_1-\ell_1 e_1-\ell_2 e_2-\ell_3 e_3\\ & +\ell_1 n_2 e_3+\ell_1 n_3 e_2+\ell_2 n_1 e_3+\ell_2 n_3 e_1+\ell_3 n_1 e_2+\ell_3 n_2 e_1-4q\end{aligned}$$

is given by [6], [7]

(2.17)
$$\begin{gathered}M = (e_1+e_2)\sum_{p\in P}\sum_{q\in Q} I_{pq}^2+(e_2+e_3)\sum_{q\in Q}\sum_{r\in R} I_{rq}^2\\ +(e_1+e_3)\sum_{p\in P}\sum_{r\in R} I_{pr}^2\\ P=\{1,\ldots,n_1+1\}, Q=\{n_1+2,\ldots,n_1+n_2+2\},\\ R=\{n_1+n_2+3,\ldots,n_1+n_2+n_3+3\}\end{gathered}$$

That is, M is the second order symmetry operator for the Laplacian ($[M,\Delta]=0$) which corresponds to the separable coordinates x, y. (The separable solutions (2.9) are eigenfunctions of M with eigenvalues χ.) Expression (2.16) gives the relationship between the eigenvalues χ and q. (The terms involving the ℓ_j result from consideration of the factor $u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3}$.)

The basis functions on the sphere S_n corresponding to coordinates of the graph can also be expanded in terms of the basis functions of the coordinate system corresponding to the graph [6],

$$
\begin{array}{ccc}
 & \boxed{0}\boxed{1} & \\
 & \vdots \quad \vdots & \\
 & \vdots \quad S_{n_3} & \\
 \boxed{0}\boxed{1} & & \\
 \vdots \quad \ddots & & \\
 S_{n_1} \qquad S_{n_2} & &
\end{array}
$$

i.e., the coordinates (2.1) with

$$u_1 = \sin\theta\cos\phi, \quad u_2 = \sin\theta\sin\phi, \quad u_3 = \cos\theta \tag{2.18}$$

and the infinitesimal distance

$$
\begin{aligned}
ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\cos^2\phi \sum_{i=1}^{n_1+1} dw_i^2 \\
+ \sin^2\theta\sin^2\phi \sum_{j=1}^{n_2+1} dt_j^2 + \cos^2\theta \sum_{k=1}^{n_3+1} dz_k^2.
\end{aligned} \tag{2.19}
$$

Eigenfunction solutions of (2.4) in these coordinates are

$$
\begin{aligned}
\psi &= (\sin\theta)^M (\cos\theta)^{\ell_3} (\sin\phi)^{\ell_2} (\cos\phi)^{\ell_1} \\
&\quad \times P_{(J-M-\ell_3)/2}^{M+\frac{1}{2}(n_1+n_2),\ell_3+\frac{1}{2}(n_3-1)}(\cos 2\theta) \\
&\quad \times P_{(M-\ell_1-\ell_2)/2}^{\ell_2+\frac{1}{2}(n_2-1),\ \ell_1+\frac{1}{2}(n_1-1)}(\cos 2\phi)\ \Theta_{\ell_1\ell_2\ell_3}(\mathbf{w},\mathbf{t},\mathbf{z}) \\
&= \psi_{JM}\Theta_{\ell_1\ell_2\ell_3}
\end{aligned} \tag{2.20}
$$

where $P_n^{\alpha,\beta}(z)$ are Jacobi polynomials. Here $J = L + 2j$ and $M = L + 2m$ where $j = 0, 1, \ldots,\ m = 0, 1, \ldots j-1, j$. The eigenfunctions satisfy

$$\Delta'\psi = -M(M + n_1 + n_2)\psi, \tag{2.21}$$

where

$$\Delta' = \sum_{i>j} I_{ij}^2. \tag{2.22}$$

and i, j range from 1 to $n_1 + n_2 + 2$.

Note that in terms of the Cartesian coordinates u_1, u_2, u_3 on the 2-sphere ($u_1^2 + u_2^2 + u_3^2 = 1$) these eigenfunctions take the form

$$\begin{aligned}
\psi_{JM} &= u_1^{\ell_1} u_2^{\ell_2} u_e^{\ell_e} (u_1^2 + u_2^2)^{(M-\ell_1-\ell_2)/2} \\
&\quad \times P_{(J-M-\ell_3)/2}^{M+\frac{1}{2}(n_1+n_2),\ \ell_3+\frac{1}{2}(n_3-1)} (1 - 2u_1^2 - 2u_2^2) \\
&\quad \times P_{(M-\ell_1-\ell_2)/2}^{\ell_2+\frac{1}{2}(n_2-1),\ \ell_1+\frac{1}{2}(n_1-1)} \left(\frac{2u_1^2}{u_1^2+u_2^2} - 1 \right) \\
&= u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} \Phi_{jm},
\end{aligned} \tag{2.23}$$

i.e., the form $u_1^{\ell} u_2^{\ell_2} u_3^{\ell_3} \Phi(u_1^2, u_2^2)$ where Φ is a polynomial.

This remark leads to another way of viewing the Heun and Jacobi bases. In the equation $\Delta\psi = -J(J+N+1)\psi$ with $\Delta\psi$ given by (2.5) and Δ_k replaced by the values $-\ell_k(\ell_k + n_k - 1)$, $k = 1, 2, 3$ we set $\psi = u_1^{\ell} u_2^{\ell_2} u_3^{\ell_3} \Phi(x_1, x_2)$ and introduce the new coordinates $x_1 = u_1^2, x_2 = u_2^2$. The eigenvalue equation for Φ reads

$$H\Phi = -j(j + G - 1)\Phi \tag{2.24}$$

where

$$H = \sum_{i,j=1}^{2} (x_i \delta_{ij} - x_i x_j) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} (\gamma_i - G x_i) \frac{\partial}{\partial x_i}. \tag{2.25}$$

Here $G = \gamma_1 + \gamma_2 + \gamma_3$ and in this particular case

$$\begin{aligned}
\gamma_i &= \ell_i + \frac{1}{2}(n_i + 1), \quad i = 1, 2, 3 \\
j &= \frac{1}{2}(J - L) = 0, 1, 2, \ldots.
\end{aligned} \tag{2.26}$$

This coincides with equation (1.4) in [8]. In particular H maps polynomials of maximum degree m_i in x_i to polynomials of the same type. Furthermore, it is easy to see that the polynomial eigenfunctions of H form a basis for the space of all polynomials $f(x_1, x_2)$ and that the spectrum of H acting on this space is exactly $\{-j(j + G - 1) : j = 0, 1, \ldots\}$. It is also shown in [8] that $H = \Delta_2 + \Lambda_2$ where Δ_2 is the Laplace Beltrami operator on S_2 and

$$\Lambda_2 = \sum_{i=1}^{2} [\gamma_i - \frac{1}{2} + (\frac{3}{2} - G)x_i] \frac{\partial}{\partial x_i}. \tag{2.27}$$

Moreover, H is self-adjoint with respect to the inner product

$$(f_1, f_2) = \iint_{x_1, x_2 > 0, 1 - x_1 - x_2 > 0} f_1(\mathbf{x}) \overline{f_2(\mathbf{x})} d\omega \tag{2.28}$$

where

$$\begin{aligned}
dw &= x_1^{\gamma_1 - 1} x_2^{\gamma_2 - 1} (1 - x_1 - x_2)^{\gamma_3 - 1} dx_1 dx_2 : \\
&(Hf_1, f_2) = (f_1, Hf_2).
\end{aligned} \tag{2.29}$$

Here f_1, f_2 are polynomials in $\mathbf{x} = (x_1, x_2)$. For fixed j the polynomials

(2.30)
$$\Phi_{jm}(x_1,x_2) = (x_1+x_2)^m P_{j-m}^{\gamma_1+\gamma_2+2m-1,\gamma_3-1}(2x_1+2x_2-1) \times P_m^{\gamma_2-1,\ \gamma_1-1}\left(\frac{2x_1}{x_1+x_2}-1\right), \quad m = 0,1,\ldots,j$$

form an orthogonal basis for the eigenspace corresponding to eigenvalue $-j(j+G-1)$. (This is the orthogonal basis of Proriol [9] and of Karlin and McGregor [10]). Similarly the Heun polynomials $\Phi^1_{J\ell_1\ell_2\ell_3q}(x)\Phi^2_{J\ell_1\ell_2\ell_3q}(y)$ where q runs over the possible eigenvalues, form an alternate orthogonal basis for this same space. Moreover as pointed out in [11] these bases correspond to spherical and ellipsoidal coordinates on the 2-sphere and are the only coordinates in which Δ_2 separates.

With this point of view we are operating on the sphere S_2 rather than S_n and our two distinguished orthogonal bases are the only ones possible rather than two out of a multiplicity of separable systems on S_n for large n. The principal advantage of this new point of view is that the eigenfunctions are obviously polynomials in x_1, x_2 and that the only requirement on the constants $\gamma_1, \gamma_2, \gamma_3$ to ensure orthogonality is that they be strictly positive. Thus the ℓ_i and n_i need not be integers; it is only required that $2\ell_i + n_i + 1 > 0$.

In the following our expansion formulas are valid for all real $\gamma_i > 0$. In the special case $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{2}$ we have $H = \Delta_2$, the Laplace-Beltrami operator on S_2. In this case the eigenvalue equation $\Delta_2\Phi = -j(j+\frac{1}{2})\Phi$ admits the Lie algebra $so(3)$ as a symmetry algebra. A basis for $so(3)$ is $\{u_1\partial_{u_2} - u_2\partial_{u_1}, u_3\partial_{u_1}, u_3\partial_{u_2}\}$ where $u_3 = \pm(1-u_1^2-u_2^2)^{\frac{1}{2}}$. This extra symmetry is associated with the fact that there are additional polynomial solutions of the eigenvalue equation (see §3 of reference [8]). In particular the equation admits polynomial solutions of the form $f(u_1,u_2)$ and the spectrum of Δ_2 acting on the space of all such polynomials is $-j(j+\frac{1}{2})$ where now $2j = 0,1,2\ldots$. Furthermore there exist solutions of the form $u_3 g(u_1,u_2)$ with g a polynomial and with the same eigenvalues. The dimension of each eigenspace is $2j+1$ rather than $j+1$ for the general case. In this special case the eigenfunctions corresponding to spherical coordinates are just the spherical harmonics whereas those corresponding to ellipsoidal coordinates are products of Lamé polynomials. For the group theoretic solution of the problem of expanding the Lamé basis in terms of a spherical harmonic basis see [11], [12], [13].

Returning to the case of general ℓ_i, n_i we consider the problem of expanding the Heun basis (2.9) in terms of the Jacobi polynomial basis (2.20), (2.23), (2.30):

(2.31)
$$\psi = u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3}\Phi^1_{J\ell_1\ell_2\ell_3q}(x)\Phi^2_{J\ell_1\ell_2\ell_3q}(y) = \sum_{m=0}^{j}\xi_m\psi_{J\ell_1\ell_2\ell_3M}(\theta,\phi).$$

Three term recurrence relations for the expansion coefficients ξ_m (where $M = \ell_1+\ell_2+2m$) can be deduced by requiring that

(2.32)
$$M\psi = \chi\psi.$$

To obtain the recurrence relations we need the action of the various pieces of M on the Jacobi bases $\psi_{J\ell_1\ell_2\ell_3 M}(\theta,\phi)$. Since M commutes with H there must exist an expansion of the form $M\psi_{jm} = \sum_r X_r \psi_{j,m+r}$. Indeed, we have

$$M\psi_{J\ell_1\ell_2\ell_3 M}(\theta,\phi) = \sum_{r=-1}^{+1} X_r \psi_{J\ell_1\ell_2\ell_3,M+2r}(\theta,\phi) \tag{2.33}$$

where

$$\begin{aligned}
&X_1(m,j)\\
&= \frac{4(e_1-e_2)(\gamma_1+\gamma_2+\gamma_3+m+j-1)(\gamma_3-m+j-1)(m+1)(\gamma_1+\gamma_2+m-1)}{(\gamma_1+\gamma_2+2m-1)(\gamma_1+\gamma_2+2m)},\\
&X_{-1}(m,j)\\
&= \frac{4(e_1-e_2)(\gamma_1+\gamma_2+m+j-1)(-m+j+1)(\gamma_2-1)(\gamma_1-1)}{(\gamma_1+\gamma_2+2m-1)(\gamma_1+\gamma_2+2m-2)},\\
&X_0(m,j)-\chi\\
&= \frac{2(e_1-e_2)[m^2+m(\gamma_1+\gamma_2-1)-j^2-j(\gamma_1+\gamma_2+\gamma_3-1)](\gamma_1+\gamma_2-2)(\gamma_1-\gamma_2)}{(\gamma_1+\gamma_2+2m-2)(\gamma_1+\gamma_2+2m)}\\
&\quad + 4\frac{(e_1-e_2)m\gamma_3(\gamma_1-\gamma_2)(m+\gamma_2)}{(\gamma_1+\gamma_2+2m-2)(\gamma_1+\gamma_2+2m)}\\
&\quad + 2(e_1+e_2)[-m^2-m(\gamma_1+\gamma_2-1)+j^2+j(\gamma_1+\gamma_2+\gamma_3-1)]\\
&\quad + 4e_3[m^2+m(\gamma_1+\gamma_2-1)]+4q.
\end{aligned} \tag{2.34}$$

∎

Keys to deriving this result are the following recurrence formulas for Jacobi polynomials $P_n^{\alpha,\beta}(x)$

$$xP_n^{\alpha,\beta} = AP_{n-1}^{\alpha,\beta} + BP_n^{\alpha,\beta} + CP_{n+1}^{\alpha,\beta} \tag{2.35}$$

$$A = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)},\quad B = \frac{(\beta-\alpha)(\beta+\alpha)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}$$

$$C = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)},$$

$$(1-x^2)\frac{d}{dx}P_n^{\alpha,\beta} = AP_{n-1}^{\alpha,\beta} + BP_n^{\alpha,\beta} + CP_{n+1}^{\alpha,\beta}$$

$$A = \frac{2(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)},\quad B = \frac{2n(\alpha-\beta)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}$$

$$C = -\frac{2n(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}.$$

(To prove (2.33) it is enough to use relations (2.35) to evaluate both sides of (2.33) for a fixed choice of the variable θ. Thus a one-variable expansion in ϕ leads to a two-variable expansion in θ and ϕ.) Now substituting the expansion (2.31) into the eigenvalue equation $M\psi = \chi\psi$ and using (2.33) we find the three term recurrence relation

$$X_1(m-1,j)\xi_{m-1} + (X_0(m,j)-\chi)\xi_m + X_{-1}(m+1,j)\xi_{m+1} = 0 \tag{2.36}$$

where $m = 0, 1, \ldots j$. Consequently the $j+1$ independent eigenvalues q are calculated from the determinant

(2.37)
$$\begin{vmatrix} X_0(j,j)-\chi & X_1(j-1,j) & & & \\ X_{-1}(j,j) & X_0(j-1,j)-\chi & X_1(j-2,j) & & \\ \ddots & \ddots & \ddots & & \\ & & & & \\ & & & X_{-1}(1,j) & X_0(0,j)-\chi \end{vmatrix} = 0.$$

To obtain the expansions in terms of one variable from (2.31) we proceed as follows. For the two choices of $u_i,\ i = 1, 2, 3$ given by (2.2) and (2.18) take $y = e_3,\ \theta = \frac{\pi}{2}$. Then the expression has the form

$$\Phi^1_{jq}(x) \equiv \Phi^1_{J\ell_1\ell_2\ell_3 q}(x) = \sum_{m=0}^{j} \hat{\gamma}_m P_m^{\gamma_2-1,\gamma_1-1}(\cos 2\phi) \tag{2.38}$$

where

$$\cos 2\phi = 2\frac{(x-e_1)}{(e_2-e_1)} - 1.$$

This is an expansion of type 2 with $\mu = 0$. A different type of expansion can be obtained by taking $\phi = \pi/2$ and $y = e_1$. The resulting expression has the form

$$\Phi^1_{jq}(x) = \sum_{m=0}^{j} \tilde{\gamma}_m (\sin\theta)^{2m} P_{j-m}^{2m+\gamma_1+\gamma_2-1,\ \gamma_3-1}(\cos 2\theta) \tag{2.39}$$

where

$$\cos 2\theta = -2\frac{(x-e_2)}{(e_2-e_3)} - 1.$$

In both these examples the dependence of the $\hat{\gamma}_m$ and $\tilde{\gamma}_m$ coefficients on the indices $\ell_1, \ell_2, \ell_3, q$ has been suppressed.

This second type of expansion of a Heun polynomial appears to be new, at least in this explicit form. Everything that was done in the derivation of expansions (except the limits of summation on r) could be extended to the representation of Heun functions when $J, \ell_1, \ell_2, \ell_3$ are complex. Consequently representations of such functions in terms of expansions whose coefficients obey three term recurrence relations can be derived. The convergence of series of this type will be discussed elsewhere.

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