ORTHOGONAL POLYNOMIALS ON N-SPHERES: GEGENBAUER, JACOBI AND HEUN

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ABSTRACT. In this expository paper we describe the families of orthogonal and biorthogonal polynomials associated with the Laplace-Beltrami eigenvalue equation $H\Phi = \lambda\Phi$ on the n-sphere, with an added vector potential term motivated by the differential equations for the polynomial Lauricella functions F_A. The operator H is self-adjoint with respect to the natural inner product induced on the sphere and, in certain special coordinates, it admits a spectral decomposition with eigenspaces composed entirely of polynomials. The eigenvalues are degenerate but the degeneracy can be broken through use of the possible separable coordinate systems on the n-sphere. Then a basis for each eigenspace can be selected in terms of the simultaneous eigenfunctions of a family of commuting second order differential operators that also commute with H. The results provide a multiplicity of n-variable orthogonal and biorthogonal families of polynomials that generalize classical results for one and two variable families of Jacobi polynomials on intervals, disks, and paraboloids. We look carefully at the problem of expanding the (product of) Heun polynomials basis for the 2-sphere in terms of the (product of) Jacobi polynomials basis.

1. Introduction. Orthogonal and biorthogonal families of polynomials in several variables which satisfy second order partial differential or difference equations have proven their worth in applications but, in contrast to the case of one variable polynomials, there is as yet no general theory and more examples are needed. In this paper we will study such families which are related to the Laplace-Beltrami eigenvalue equation on the n-sphere. Our procedure provides a uniform setting within which to classify several known examples related to the n-sphere and to generate many new examples. Our approach falls within the theory of Dunkl's differential-difference operators [2, 3]; our main contribution is to point out the power of separation of variable methods in this theory.

We also show how separation of variables methods can be used to expand one basis of orthogonal polynomials in terms of a second basis. The example we present makes a judicious choice of coordinates on the 2-sphere. The expansions that are first derived are for products of Heun polynomials as sums of products of Jacobi polynomials. The coefficients in the expansions obey three term recurrence relations. Corresponding single variable expansions of a Heun polynomial as a sum of Jacobi polynomials are then obtained by allowing one of the variables to take a fixed value.

More of the technical details related to the ideas presented here can be found in the papers [15] and [16].

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2. The n-sphere. It was shown by Lam and Tratnik [20] that the Lauricella functions

$$\Phi = F_A \left[\begin{matrix} M+G-1; & -m_1, \cdots, -m_n \\ & \gamma_1, \cdots, \gamma_n \end{matrix} ; x_1, \cdots, x_n \right] \tag{2.1}$$

and

$$(1-x)^M F_A \left[\begin{matrix} -M-\gamma_{n+1}+1; & -m_1, \cdots, -m_n \\ & \gamma_1, \cdots, \gamma_n \end{matrix} ; -\frac{x_1}{1-x}, \cdots, -\frac{x_n}{1-x} \right] \tag{2.2}$$

form a biorthogonal polynomial family where $m_i = 0, 1, 2, \cdots$, $M = \sum_{k=1}^n m_i$, $G = \sum_{\ell=1}^{n+1} \gamma_\ell$, $x = \sum_{k=1}^n x_i$ and the γ_ℓ are positive real numbers. The inner product is

$$(\Phi_1, \Phi_2) = \int \cdots \int_{x_i>0, x<1} \Phi_1 \overline{\Phi_2}\, d\tilde{\omega}, \tag{2.3}$$

$$d\tilde{\omega} = x_1^{\gamma_1 - 1} \ldots x_n^{\gamma_n - 1} (1-x)^{\gamma_{n+1}-1} dx_1 \ldots dx_n.$$

Here, the Lauricella function F_A is defined by the series

$$\begin{aligned} &F_A \left[\begin{matrix} a; & b_1, \cdots, b_n \\ & c_1, \cdots, c_n \end{matrix} ; x_1, \cdots, x_n \right] \\ &= \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} x_1^{m_1} \cdots x_n^{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n} m_1! \cdots m_n!}, \end{aligned} \tag{2.4}$$

where

$$(a)_m = \begin{cases} 1 & \text{if } m = 0 \\ a(a+1)\ldots(a+m-1) & \text{if } m \geq 1. \end{cases}$$

The standard partial differential equations for the F_A are, [1],

$$\begin{aligned} x_j(1-x_j)\partial_{x_j x_j} F_A - x_j \sum_{k \neq j} x_k \partial_{x_k x_j} F_A + \left[c_j - (a+b_j+1)x_j\right] \partial_{x_j} F_A \\ - b_j \sum_{k \neq j} x_k \partial_{x_k} F_A - a b_j F_A = 0 \end{aligned}$$

for $j = 1, 2, \cdots, n$. Indeed for general $a, b_1, \cdots, b_n, c_1, \cdots, c_n$ the function (2.4) is the only solution of these equations which is analytic in a neighborhood of $(x_1, \cdots, x_n) = (0, \cdots, 0)$. Setting $a = M + G - 1$, $b_k = -m_k$, and $c_k = \gamma_k$, $k = 1, \cdots, n$ in these equations, and adding them together, we see that the polynomial functions Φ, (2.1), satisfy the eigenvalue equation

$$H\Phi = -M(M+G-1)\Phi \tag{2.5}$$

where

$$H = \sum_{i,j=1}^n (x_i \delta_{ij} - x_i x_j)\partial_{x_i x_j} + \sum_{i=1}^n (\gamma_i - G x_i)\partial_{x_i}. \tag{2.6}$$

Here δ_{ij} is the Kronecker delta. Note that the operator H has the property that it maps polynomials of maximum order m_i in x_i to polynomials of the same type. It is easy to see that as the m_i range over all nonnegative integers the functions (2.1) form a basis for the space of all polynomials in variables $x_1, \cdots, x_n$, and that the eigenvalues of H acting on this space are exactly

$$\{-M(M+G-1) : M = 0, 1, 2, \cdots\}.$$

(For $n = 2$ equation (2.5) appears in the classification by Krall and Sheffer [19] of all second order partial differential operators such that the Mth order orthogonal polynomials in two variables, with respect to some weight function, are eigenfunctions of the operator.)

Equation (2.5) is closely related to the Laplace-Beltrami eigenvalue equation on the n-sphere, [4]. To see this consider the contravariant metric determined by the second derivative terms in H:

$$g^{ij} = \delta_{ij} x_i - x_i x_j, \qquad 1 \le i, j \le n. \tag{2.7}$$

Then $\det(g^{ij}) = g^{-1} = x_1 x_2 \cdots x_n (1 - x)$ and

$$g_{ij} = \frac{1}{1-x} + \frac{\delta_{ij}}{x_i}, \tag{2.8}$$

where

$$\sum_{j=1}^{n} g^{ij} g_{jk} = \delta^i_k = \begin{cases} 1 \text{ if } & i = k \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$$

determines a metric on a Riemannian space with associated Laplace-Beltrami operator

$$\Delta_n = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_{x_i} (g^{ij} \sqrt{g}\ \partial_{x_j}). \tag{2.9}$$

A straightforward computation yields

$$H = \Delta_n + \Lambda_n \tag{2.10}$$

where

$$\Lambda_n = \sum_{j=1}^{n} [\gamma_j - \frac{1}{2} + (\frac{n+1}{2} - G) x_j] \partial_{x_j}. \tag{2.11}$$

Thus if $\gamma_1 = \cdots = \gamma_{n+1} = 1/2$ then $H \equiv \Delta_n$, but in general H differs from Δ_n by the first order differential operator Λ_n.

To identify the Riemannian space we introduce Cartesian coordinates $z_0, z_1, \cdots, z_n$ in $n+1$ dimensional Euclidean space and restrict these coordinates by the conditions

$$
\begin{aligned}
z_0^2 &= 1 - \sum_{i=1}^{n} x_i = 1 - x \\
z_1^2 &= x_1 \\
z_2^2 &= x_2 \\
&\vdots \\
z_n^2 &= x_n.
\end{aligned}
\tag{2.12}
$$

Note that $z_0^2 + z_1^2 + \cdots + z_n^2 = 1$. Defining a metric ds^2 by

$$ds^2 = \sum_{m=0}^{n} (dz_m)^2$$

we find

$$ds^2 = \frac{1}{4} \sum_{i,j=1}^{n} \left(\frac{1}{1-x} + \frac{\delta_{ij}}{x_i}\right) dx_i dx_j, \tag{2.13}$$

in agreement with (2.8) to within a constant factor. Thus the space corresponds to a portion of the n-sphere S^n. We can consider the coordinates $\{x_i\}$ for $0 \le x_i$ and $x \le 1$ as covering the portion of the n-sphere given by $0 \le z_i$, $\sum_{k=1}^{n} z_k^2 = 1$.

The Schrödinger equation (2.5) with vector potential Λ_n can be transformed to a Schrödinger equation with a scalar potential V_n through the use of a multiplier transformation ρ. Setting $\Phi(\mathbf{x}) = \rho(\mathbf{x})\Psi(\mathbf{x})$ for a nonzero scalar function ρ we find

$$
\begin{aligned}
(\Delta_n + \Lambda_n)\Phi &= -M(M+G-1)\Phi \\
&\iff (\Delta_n + V_n(\mathbf{x}))\Psi = -M(M+G-1)\Psi,
\end{aligned}
$$

provided

$$\rho^{-1} = x_1^{\gamma_1/2-1/4} \cdots x_n^{\gamma_n/2-1/4}(1-x)^{\gamma_{n+1}/2-1/4}. \tag{2.14}$$

A straightforward computation gives for the scalar potential:

$$
\begin{aligned}
V_n = &-\frac{1}{4} \sum_{i=1}^{n} \frac{(\gamma_i - \frac{1}{2})(\gamma_i - \frac{3}{2})}{x_i} \\
&- \frac{1}{4} \frac{(\gamma_{n+1} - \frac{1}{2})(\gamma_{n+1} - \frac{3}{2})}{1-x} + \frac{1}{4}\left[(1-G)^2 - 1 - \frac{(n-3)(n+1)}{4}\right].
\end{aligned}
\tag{2.15}
$$

The equation $H'\Psi \equiv (\Delta_n + V_n)\Psi = \lambda\Psi$ has a natural Riemannian metric

$$d\omega = g^{1/2} dx_1 \cdots dx_n = x_1^{-1/2} \cdots x_n^{-1/2}(1-x)^{-1/2} dx_1 \cdots dx_n, \tag{2.16}$$

[4]. Also, the operator $H' = \rho^{-1}H\rho = \Delta_n + V_n$ is formally self-adjoint with respect to the inner product

$$< \Psi_1, \Psi_2 >= \int \cdots \int_{x_i>0, x<1} \Psi_1(\mathbf{x})\overline{\Psi_2}(\mathbf{x})d\omega \tag{2.17}$$

where Ψ_1, Ψ_2 are twice continuously differentiable functions of the x_j which take complex values:

$$< H'\Psi_1, \Psi_2 >=< \Psi_1, H'\Psi_2 > .$$

This induces an inner product on the space of polynomial functions $\Phi(\mathbf{x}) = \rho\Psi$, with respect to which H is self-adjoint:

$$\begin{aligned} (\Phi_1, \Phi_2) &\equiv< \Psi_1, \Psi_2 > \\ &= \int \cdots \int_{x_i>0, x<1} \Phi_1\overline{\Phi_2}\, d\tilde{\omega}, \end{aligned} \tag{2.18}$$

$$d\tilde{\omega} = x_1^{\gamma_1-1} \ldots x_n^{\gamma_n-1}(1-x)^{\gamma_{n+1}-1}dx_1 \ldots dx_n,$$

$$(H\Phi_1, \Phi_2) = (\Phi_1, H\Phi_2).$$

(Indeed, H is formally self-adjoint and the boundary terms obviously vanish for the γ_i sufficiently large. The result can then be extended to all $\gamma_i > 0$ by analytic continuation.) Thus $(\cdot, \cdot)$ is the natural inner product associated with equation (2.5).

A *first order symmetry operator* for the equation $H\Phi = \lambda\Phi$ is a differential operator

$$K = \sum_{i=1}^{n} f_i(\mathbf{x})\partial_{x_i} + g(\mathbf{x})$$

such that

$$[H, K] \equiv HK - KH = 0,$$

[21]. The first order symmetry operators form a real Lie algebra under addition of operators, multiplication of an operator by a real scalar, and the commutator bracket $[A, B] = AB - BA$. If $\gamma_1 = \gamma_2 = \cdots = \gamma_{n+1} = 1/2$ then $H = \Delta_n$ and it is well-known [4, 5] that the Lie algebra of real symmetry operators of Δ_n is so$(n+1)$, with dimension $n(n+1)/2$ and a basis of the form $\{L_{\ell k}\}$ where $0 \le \ell < k \le n$, and $L_{\ell k} = -L_{k\ell}$. Explicitly,

$$L_{\ell k} = z_\ell \partial_{z_k} - z_k \partial_{z_\ell} \tag{2.19}$$

and

$$\begin{aligned} L_{ij} &= 2\sqrt{x_i x_j}(\partial_{x_j} - \partial_{x_i}), \quad 1 \le i, j \le n \\ L_{0i} &= 2\sqrt{x_i(1-x)}\partial_{x_i}, \quad 1 \le i \le n. \end{aligned} \tag{2.20}$$

Furthermore, all real second-order differential operators S that commute with Δ_n can be expressed as linear combinations over $\mathbb{R}$ of real constants, elements $L_{\ell k}$ and elements $L_{\ell k}L_{\ell' k'}$. For $\gamma_1, \ldots, \gamma_{n+1}$ arbitrary, however, we have

Lemma 1. *If K is a first order operator such that $[K,H]=0$ then $K=c$, multiplication by the real constant c. The second order operators*

$$\begin{aligned} S_{ij} &\equiv 4x_ix_j(\partial_{x_i}-\partial_{x_j})^2+4(\gamma_ix_j-\gamma_jx_i)(\partial_{x_i}-\partial_{x_j})\\ &= L_{ij}^2+4[(\gamma_i-\frac{1}{2})x_j-(\gamma_j-\frac{1}{2})x_i](\partial_{x_i}-\partial_{x_j})\\ &= S_{ji}, \qquad 1\le i<j\le n, \end{aligned} \tag{2.21}$$

$$\begin{aligned} S_{0i} &\equiv 4x_i(1-x)\partial_{x_i}^2+4[\gamma_i(1-x)-\gamma_{n+1}x_i]\partial_{x_i}\\ &= L_{0i}^2+4[(\gamma_i-\frac{1}{2})(1-x)-(\gamma_{n+1}-\frac{1}{2})x_i]\partial_{x_i}\\ &= S_{i0}, \qquad 1\le i\le n, \end{aligned} \tag{2.22}$$

do commute with H: $[S_{ij},H]=[S_{0i},H]=0$. Also

$$8H \equiv \sum_{i,j=1}^n S_{ij}+2\sum_{i=1}^n S_{0i}. \tag{2.23}$$

If S is a second order symmetry operator for H then $S'=\rho^{-1}S\rho$ is a second order symmetry for $H'=\Delta_n+V_n$ and, necessarily, $S'=\Upsilon+f$ where Υ is a second order symmetry for Δ_n and f is a real-valued function. Thus S' is a formally self-adjoint operator with respect to the inner product $<\cdot,\cdot>$ and S is formally self-adjoint with respect to $(\cdot,\cdot)$.

3. Orthogonal bases of separable solutions. In the paper [14] and in the book [12] all separable coordinates for the equation $\Delta_n\Psi=\lambda\Psi$ are constructed, where Δ_n is the Laplace-Beltrami operator on S^n. It is shown that all separable coordinates are orthogonal and that for each separable coordinate system the corresponding separated solutions are characterized as simultaneous eigenfunctions of a set of n second order commuting symmetry operators for Δ_n. These operators are real linear combinations of the symmetries L_{ij}^2, $1\le i<j\le n+1$, where L_{ij} is a rotational generator in so$(n+1)$. For $n=2$ there are two separable systems (ellipsoidal and spherical coordinates), while for $n=3$ there are 6 systems. The number of separable systems grows rapidly with n, but all systems can be constructed through a simple graphical procedure. (In general, the possible separable systems are the various polyspherical coordinates [25], the basic ellipsoidal coordinates, and combinations of polyspherical and ellipsoidal coordinates.) Moreover, the equation $(\Delta_n+V_n)\Psi=\lambda\Psi$ where the scalar potential takes the special form

$$V_n=\sum_{i=1}^n\frac{\alpha_i}{z_i^2}+\frac{\alpha_0}{z_0^2}, \quad \alpha_0,\alpha_1,\dots,\alpha_n \text{ const.}, \tag{3.1}$$

is separable in *all* the coordinate systems in which the Laplace-Beltrami eigenvalue equation is separable. Indeed, the equation with potential (3.1) is separable in general ellipsoidal coordinates. Since all other coordinates are limiting cases of ellipsoidal coordinates, the conclusion follows.

The results of the authors, characterizing separable systems by symmetry operators, can easily be translated to the present case. In those references (for $V_n = 0$) the symmetry operators are given explicitly as linear combinations of the symmetries L_{ij}^2. The results for the potential (2.15) are similar: one replaces L_{ij}^2 by $S'_{ij} = \rho^{-1} S_{ij} \rho$ and takes the same linear combinations. Moreover, since the defining symmetry operators for a separable system are real linear combinations of the L_{ij}^2 plus scalar functions, they are formally self-adjoint with respect to the inner product $< \cdot, \cdot >$.

These results can now easily be extended to results for solutions of

$$(\Delta_n + \Lambda_n)\Phi = \lambda\Phi \tag{3.2}$$

through the mappings

$$\begin{aligned} \Delta_n + \Lambda_n &= \rho(\Delta_n + V_n)\rho^{-1} \\ S_{ij} &= \rho S'_{ij} \rho^{-1} \\ \Phi &= \rho\Psi. \end{aligned} \tag{3.3}$$

Thus all separable solutions Ψ map to R-separable solutions Φ of (3.2), [21]. The R-separable coordinates and solutions are determined by commuting symmetry operators S of $\Delta_n + \Lambda_n$ which are obtained from expressions in [12, 14] where each occurrence of L_{ij}^2 is replaced by S_{ij}. The defining symmetry operators are all formally self-adjoint with respect to the inner product $(\cdot, \cdot)$. Finally, since each S_{ij} maps polynomials of maximum order m_k in x_k to polynomials of the same type, it follows that a basis of separated solutions can be expressed as *polynomials* in the x_i. Since the symmetry operators are self-adjoint, the basis of simultaneous eigenfunctions can be chosen to be *orthogonal*.

We conclude from this argument that every separable coordinate system for the Laplace-Beltrami eigenvalue equation on the n-sphere yields an orthogonal basis of polynomial solutions of equation (2.5), hence an orthogonal basis for all n-variable polynomials with inner product (2.18).

As an example we work out the separation equations for spherical coordinates $\{u_i\}$ on S^n:

$$\begin{aligned} z_0^2 &= 1 - x = 1 - u_n \\ z_1^2 &= x_1 = u_1 u_2 \ldots u_n \\ z_2^2 &= x_2 = (1 - u_1) u_2 \ldots u_n \\ &\vdots \\ z_{n-1}^2 &= x_{n-1} = (1 - u_{n-2}) u_{n-1} u_n \\ z_n^2 &= x_n = (1 - u_{n-1}) u_n. \end{aligned} \tag{3.4}$$

(Note that in terms of angles $\{\theta_i\}$ one usually sets $u_i = \sin^2 \theta_i$.) It follows that

$$u_j = \begin{cases} w_j / w_{j+1}, & j = 1, \cdots, n-1 \\ w_n, & j = n \end{cases} \tag{3.5}$$

where

$$w_\ell = \sum_{i=1}^{\ell} x_i.$$

In terms of the $\{u_i\}$, the operator (2.6) becomes

$$H = \sum_{i=1}^{n} \frac{1}{u_{i+1}\cdots u_n}\left[u_i(1-u_i)\partial^2_{u_i} + \left(\sum_{j=1}^{i}\gamma_j - (\sum_{p=1}^{i+1}\gamma_p)u_i\right)\partial_{u_i}\right]. \tag{3.6}$$

Equation (2.5) is separable in these coordinates with separation equations

$$\begin{gathered} u_1(1-u_1)\partial^2_{u_1}\Theta_1 + [\gamma_1 - (\gamma_1+\gamma_2)u_1]\,\partial_{u_1}\Theta_1 = c_1\Theta_1, \\ \left[\frac{c_{k-1}}{u_k} + u_k(1-u_k)\partial^2_{u_k}\right]\Theta_k + \left[\sum_{j=1}^{k}\gamma_j - (\sum_{p=1}^{k+1}\gamma_p)u_k\right]\partial_{u_k}\Theta_k = c_k\Theta_k, \\ k = 2,3,\cdots,n. \end{gathered} \tag{3.7}$$

Here $\Theta = \prod_{k=1}^{n}\Theta_k(u_k)$ and the c_i are the separation constants, with $c_n = -M(M+G-1)$.

Noting that the hypergeometric equation

$$u(1-u)\frac{d^2g}{du^2} + [c - (a+b+1)u]\,\frac{dg}{du} - abg = 0$$

admits the solution

$$g = {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix};u\right) = \sum_{m=1}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}u^m,$$

a polynomial for $a = 0,-1,-2,\cdots$, and requiring that Θ be a polynomial in the $\{x_i\}$ we obtain the solutions

$$\begin{aligned} \Theta_1(u_1) &= {}_2F_1\left(\begin{matrix} -\ell_1, & \ell_1+\gamma_1+\gamma_2-1 \\ & \gamma_1 \end{matrix};u_1\right) \\ &c_1 = -\ell_1(\ell_1+\gamma_1+\gamma_2-1), \\ \Theta_k(u_k) &= u_k^{\ell_1+\ell_2+\cdots+\ell_{k-1}}\,{}_2F_1\left(\begin{matrix} -\ell_k, & 2(\ell_1+\cdots+\ell_{k-1})+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1 \\ & 2(\ell_1+\cdots+\ell_{k-1})+\gamma_1+\cdots+\gamma_k \end{matrix};u_k\right), \\ &c_k = -(\ell_1+\cdots+\ell_k)(\ell_1+\cdots+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1), \\ &k = 2,3,\cdots,n, \end{aligned} \tag{3.8}$$

where $\sum_{i=1}^{n}\ell_i = M$ and $\ell_i = 0,1,2\cdots$. This determines Θ to within a normalization factor.

In the special case $n = 2$ we have the result of Proriol [23] and of Karlin and McGregor [17]:

$$\begin{aligned}\Theta_{\ell_1,\ell_2}(x_1,x_2) = & {}_2F_1\left(\begin{matrix}-\ell_1, & \ell_1+\gamma_1+\gamma_2-1\\ & \gamma_1\end{matrix};\frac{x_1}{x_1+x_2}\right)(x_1+x_2)^{\ell_1}\times\\ & {}_2F_1\left(\begin{matrix}-\ell_2, & 2\ell_1+\ell_2+\gamma_1+\gamma_2+\gamma_3-1\\ & 2\ell_1+\gamma_1+\gamma_2\end{matrix};x_1+x_2\right)\\ \sim & P_{\ell_2}^{(\gamma_3-1,\gamma_1+\gamma_2+2\ell_1-1)}(2x_1+2x_2-1)\ (x_1+x_2)^{\ell_1}\times\\ & P_{\ell_1}^{(\gamma_2-1,\gamma_1-1)}(\frac{2x_1}{x_1+x_2}-1).\end{aligned} \tag{3.9}$$

where $P_k^{(\alpha,\beta)}(x)$ is a Jacobi polynomial [7]. (See [18] for a review of this and other 2-variable orthogonal polynomial bases.)

Returning to the general case, we have the eigenvalue equations

$$S_\ell\Theta_\ell = c_\ell\Theta_\ell, \quad \ell = 1,\cdots,n \tag{3.10}$$

where

$$\begin{aligned}S_1 &= u_1(1-u_1)\partial^2_{u_1} + [\gamma_1-(\gamma_1+\gamma_2)u_1]\,\partial_{u_1},\\ S_k &= \frac{1}{u_k}S_{k-1} + u_k(1-u_k)\partial^2_{u_k} + [\gamma_1+\cdots+\gamma_k-(\gamma_1+\cdots+\gamma_{k+1})u_k]\,\partial_{u_k},\\ k &= 2,3,\cdots,n,\end{aligned} \tag{3.11}$$

and $S_n = H$. Furthermore, $[S_i, S_j] = 0$ and the S_i are self-adjoint with respect to the inner product $(\cdot,\cdot)$. It follows immediately that

$$(\Theta_\ell, \Theta_{\mathbf{m}}) = 0$$

unless $\ell_1 = m_1$, $\ell_2 = m_2$, $\ldots$, $\ell_n = m_n$. The measure $d\tilde{\omega}$ becomes, in these coordinates,

$$d\tilde{\omega} = u_1^{\gamma_1-1}u_2^{\gamma_1+\gamma_2-1}\ldots u_n^{\gamma_1+\cdots+\gamma_n-1}(1-u_1)^{\gamma_2-1}(1-u_2)^{\gamma_3-1}\ldots(1-u_n)^{\gamma_{n+1}-1}du_1\ldots du_n,$$

where $0 < u_i < 1$. In terms of the symmetries S_{ij}, S_{0i}, (2.21-22), we have:

$$\begin{aligned}S_k &= \frac{1}{8}\sum_{i,j=1}^{k+1} S_{ij}, \quad k = 1,\ldots,n-1\\ S_n &= H = \frac{1}{8}(\sum_{h,p=0}^{n} S_{hp}),\end{aligned} \tag{3.12}$$

where we set $S_{hh} = 0$.

4. An extension. Next we consider a more general case with variables $x_1, \ldots, x_{n_1}$, $y_1, \ldots, y_{n_2}$, $n_1 + n_2 = n$, where

$$
\begin{aligned}
z_0^2 &= 1 - \sum_{i=1}^{n_1} x_i - \sum_{a=1}^{n_2} y_a^2, \\
z_1^2 &= x_1, \\
&\vdots \\
z_{n_1}^2 &= x_{n_1}, \\
z_{n_1+1}^2 &= y_1^2, \\
&\vdots \\
z_n^2 &= y_{n_2}^2,
\end{aligned}
\tag{4.1}
$$

and look for polynomial solutions in x_i, y_a of the equation

$$
H\Phi(\mathbf{x}, \mathbf{y}) = -M(M + G - 1)\Phi(\mathbf{x}, \mathbf{y}),
\tag{4.2}
$$

where $\gamma_{n_1+1} = \gamma_{n_1+2} = \cdots = \gamma_n = 1/2$ and

$$
\begin{aligned}
H &= \frac{1}{4}\sum_{a,b}(\delta_{ab} - y_a y_b)\partial_{y_a y_b} + \sum_{i,j}(\delta_{ij}x_i - x_i x_j)\partial_{x_i x_j} \\
&\quad - \sum_{a,i} y_a x_i \partial_{y_a x_i} + \sum_i (\gamma_i - G x_i)\partial_{x_i} + \frac{1}{2}\sum_a (\frac{1}{2} - G) y_a \partial_{y_a}, \\
G &= \frac{n_2 + 1}{2} + \sum_i \gamma_i + s/2, \quad 2M \text{ a nonnegative integer.}
\end{aligned}
\tag{4.3}
$$

For reference,

$$
\begin{aligned}
\Delta_n &= \frac{1}{4}\sum_{a,b}(\delta_{ab} - y_a y_b)\partial_{y_a y_b} + \sum_{i,j}(\delta_{ij}x_i - x_i x_j)\partial_{x_i x_j} \\
&\quad - \sum_{a,i} y_a x_i \partial_{y_a x_i} + \frac{1}{2}\sum_i \left(1 - (n+1)x_i\right)\partial_{x_i} \\
&\quad - \frac{n}{4}\sum_a y_a \partial_{y_a}, \\
\Lambda_n &= \sum_i \left[\gamma_i - \frac{1}{2} + (\frac{n+1}{2} - G)x_i\right]\partial_{x_i} + \frac{1}{2}\sum_a (\frac{n+1}{2} - G) y_a \partial_{y_a}.
\end{aligned}
\tag{4.4}
$$

Note that H maps polynomials in x_i, y_a to polynomials of at most the same order. (Unless the conditions $\gamma_{n_1+1} = \gamma_{n_1+2} = \cdots = \gamma_n = 1/2$ hold, the operator H will fail to map polynomials in x_i, y_a to polynomials.) The induced measure is

$$
\begin{aligned}
d\tilde{\omega} &= x_1^{\gamma_1 - 1} \cdots x_{n_1}^{\gamma_{n_1} - 1}(1 - \sum x_i - \sum y_a^2)^{s/2 - 1/2} dx_1 \cdots dx_{n_1} dy_1 \cdots dy_{n_2}, \\
& 0 < x_i, \ -1 < y_a < 1, \ \sum_i x_i + \sum_a y_a^2 < 1,
\end{aligned}
\tag{4.5}
$$

and

$$\rho^{-1} = x_1^{\gamma_1/2-1/4}\cdots x_{n_1}^{\gamma_{n_1}/2-1/4}(1-\sum_i x_i - \sum_a y_a^2)^{s/4}.$$

Equation (4.2) admits the symmetry algebra so(n_2) with basis

$$L_{ab} = -L_{ba} = y_a\partial_{y_b} - y_b\partial_{y_a}, \quad 1\le a<b\le n_2.$$

The operators H and S_{mk} are formally self-adjoint on the space of polynomials in x_i, y_a with respect to the inner product

$$(\Phi_1,\Phi_2) = \int\cdots\int_{0<x_i,\sum_i x_i+\sum_a y_a^2<1} \Phi_1(\mathbf{x},\mathbf{y})\overline{\Phi_2}(\mathbf{x},\mathbf{y})\ d\tilde{\omega}.$$

It follows that all separable coordinate systems for the n-sphere yield bases of orthogonal polynomials in this case, (indeed multiple sets of such bases, depending on the ordering of the variables x_i, y_a). For example, if we choose spherical coordinates $u_\ell = \sin^2\theta_\ell$ in the form

$$u_\ell = \frac{w_\ell}{w_{\ell+1}}$$

where

$$w_\ell = \begin{cases} \sum_{i=1}^{\ell} x_i, & \ell = 1,\cdots,n_1 \\ \sum_{i=1}^{n_1} x_i + \sum_{a=1}^{\ell-n_1} y_a^2, & \ell = n_1+1,\cdots,n_1+n_2 \\ 1, & \ell = n_1+n_2+1 \end{cases} \tag{4.6}$$

we find the orthogonal basis of polynomials:

$$\Theta = \prod_{k=1}^{n}\Theta_k(u_k),$$

where

$$\Theta_k(u_k) = u_k^{\ell_1+\cdots+\ell_{k-1}}\,{}_2F_1\left(\begin{matrix}-\ell_k, & 2(\ell_1+\cdots+\ell_{k-1})+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1\\ & 2(\ell_1+\cdots+\ell_{k-1})+\gamma_1+\cdots+\gamma_k\end{matrix};u_k\right),$$

$$c_k = -(\ell_1+\cdots+\ell_k)(\ell_1+\cdots+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1), \quad k=1,\cdots,n_1, \tag{4.7}$$

$$\Theta_k(u_k) = [\sin\theta_k]^{2(\ell_1+\cdots+\ell_{k-1})}C_{2\ell_k}^{2(\ell_1+\cdots+\ell_{k-1})+\sum_{i=1}^k\gamma_i-1/2}(\cos\theta_k),$$

$$c_k = -(\ell_1+\cdots+\ell_k)(\ell_1+\cdots+\ell_k+\gamma_1+\cdots+\gamma_{k+1}-1), \quad k=n_1+1,\cdots,n_1+n_2-1,$$

$$\Theta_n(u_n) = u_n^{\ell_1+\cdots+\ell_{n-1}}\,{}_2F_1\left(\begin{matrix}-\ell_n, & 2(\ell_1+\cdots+\ell_{n-1})+\ell_n+G-1\\ & 2(\ell_1+\cdots+\ell_{n-1})+G-s/2-1/2\end{matrix};u_n\right).$$

Here $\ell_1,\cdots,\ell_{n_1}$, ℓ_n and $2\ell_{n_1+1},\cdots,2\ell_{n_1+n_2-1}$ are nonnegative integers. (Recall that $\gamma_{n_1+1}=\gamma_{n_1+2}=\cdots=\gamma_{n_1+n_2}=1/2$.) The $C_k^\alpha(\cos\theta)$ are Gegenbauer polynomials [7].

5. Biorthogonal families of polynomials on S^n. We begin this section with a simplified proof of the biorthogonality of the polynomials (2.1) and (2.2) with respect to the inner product $(\cdot,\cdot)_\gamma$, see (2.18). Let $\mathcal{S}_\gamma$ be the space of all polynomials in $x_1,\cdots,x_n$ with respect to this inner product and let $\mathcal{H}_{\gamma,M}$ be the subspace of $\mathcal{S}_\gamma$ consisting of solutions Φ to the eigenvalue equation

$$H\Phi = -M(M+G-1)\Phi, \tag{5.1}$$

where H is given by (1.5). Since the functions

$$D^\gamma_{\mathbf{m}}(\mathbf{x}) = F_A\left[\begin{matrix} M+G-1; & -m_1,\cdots,-m_n \\ & \gamma_1,\cdots,\gamma_n \end{matrix}; x_1,\cdots,x_n\right] \tag{5.2}$$

clearly satisfy (5.1) for $M=\sum_{i=1}^n m_i$ and since the highest order monomial in these solutions is $x_1^{m_1}\dots x_n^{m_n}$ it follows that the $D^\gamma_{\mathbf{m}}$ for $m_1+\cdots+m_n \equiv m = M$ form a basis for $\mathcal{H}_{\gamma,M}$ and, as the m_i range over all nonnegative integers, a basis for $\mathcal{S}_\gamma$. [Note: $\dim \mathcal{H}_{\gamma,M} = \binom{M+n-1}{n-1}$.] Since H is self-adjoint we have $\mathcal{H}_{\gamma,M} \perp \mathcal{H}_{\gamma,M'}$ for $M' \neq M$. Thus

$$(D^\gamma_{\mathbf{m}'}, D^\gamma_{\mathbf{m}})_\gamma = 0 \quad \text{for } m' \neq m. \tag{5.3}$$

It is simple to verify the recurrence relation

$$\partial_{x_i} D^\gamma_{\mathbf{m}}(\mathbf{x}) = \frac{(M+G-1)(-m_i)}{\gamma_i} D^{\hat\gamma}_{\hat{\mathbf{m}}}(\mathbf{x}), \tag{5.4}$$

where

$$\hat\gamma_j = \begin{cases} \gamma_j & \text{for } j\neq i,\ 1\le j\le n \\ \gamma_i+1 & \text{for } j=i \\ \gamma_{n+1}+1 & \text{for } j=n+1 \end{cases}$$

$$\hat m_j = \begin{cases} m_j & \text{for } j\neq i,\ 1\le j\le n \\ m_i-1 & \text{for } j=i \end{cases} \tag{5.5}$$

$$\hat M = M-1, \quad \hat G = G+2.$$

We can consider $P_i = \partial_{x_i}$ as an operator

$$P_i : \mathcal{S}_\gamma \to \mathcal{S}_{\hat\gamma}.$$

Indeed we have

Lemma 2. *P_i, $(1\le i\le n)$, maps $\mathcal{H}_{\gamma,M}$ onto $\mathcal{H}_{\hat\gamma,\hat M}$.*

The operator P_i induces an adjoint operator $P_i^*: \mathcal{S}_{\hat\gamma} \to \mathcal{S}_\gamma$, defined by

$$(P_i^*\Phi, \Phi')_\gamma = (\Phi, P_i\Phi')_{\hat\gamma}$$

for all $\Phi\in\mathcal{S}_{\hat\gamma}$, $\Phi'\in\mathcal{S}_\gamma$. A straightforward computation yields

$$P_i^* = -x_i(1-x)\partial_{x_i} - \gamma_i(1-x) + \gamma_{n+1}x_i. \tag{5.6}$$

Theorem 1. *P_i^* is a 1-1 map of $\mathcal{H}_{\hat\gamma,\hat M}$ into $\mathcal{H}_{\gamma,M}$.*

Proof. Taking the adjoint of the relation (5.4) we obtain

$$P_i^* \hat H = GP_i^* + HP_i^*.$$

Furthermore, P_i^* is 1-1 since P_i is onto.

□

Let

$$C^\gamma_{\mathbf m}(\mathbf x) = (P_1^*)^{m_1} \cdots (P_n^*)^{m_n} 1 \in \mathcal{S}_\gamma$$

be the result of applying m_n operators $P_n^*, \cdots,$ m_1 operators P_1^*, one at a time, to the function $1 \in \mathcal{S}_{\gamma'}$ where

$$\gamma_i' = \gamma_i + m_i, \quad 1 \le i \le n,$$
$$\gamma_{n+1}' = \gamma_{n+1} + m.$$

(Each time an operator P_j^* is applied it lowers γ_j and γ_{n+1} by 1 and leaves the other γ_k's unchanged. The order in which these operators are applied makes no difference in the result.) It follows from the recurrence relation

$$\left(x_i \sum_{j=1}^n x_j \partial_{x_j} - x_i\partial_{x_i} + x_i(-M - \gamma_{n+1} + 1) - \gamma_i + 1\right) \times$$

$$F_A \left[\begin{matrix} -M - \gamma_{n+1} + 1; & -m_1, \cdots, -m_n \\ & \gamma_1, \cdots, \gamma_n \end{matrix} ; x_1, \cdots, x_n \right]$$

(5.7)

$$= (1-\gamma_i) F_A \left[\begin{matrix} -(M+1) - (\gamma_{n+1} - 1) + 1; & -m_1, \cdots, -(m_i+1), \cdots, -m_n \\ & \gamma_1, \cdots, \gamma_i - 1, \cdots, \gamma_n \end{matrix} ; x_1, \cdots, x_n \right]$$

and a simple induction argument that

(5.8)

$$C^\gamma_{\mathbf m}(\mathbf x) = c_{\gamma,m}(1-x)^M F_A \left[\begin{matrix} -M - \gamma_{n+1} + 1; & -m_1, \cdots, -m_n \\ & \gamma_1, \cdots, \gamma_n \end{matrix} ; -\frac{x_1}{1-x}, \cdots, -\frac{x_n}{1-x} \right]$$

where $c_{\gamma,m}$ is a nonzero constant. By Theorem 1 the $C^\gamma_{\mathbf m}$ belong to $\mathcal{H}_{\gamma,M}$ for $M = m_1 + \cdots + m_n$. Since there are $\binom{M+n-1}{n-1}$ of these functions for fixed M and since they are clearly linearly independent, they form a basis for $\mathcal{H}_{\gamma,M}$.

Now consider the inner product

$$(C^\gamma_{\mathbf m}, D^\gamma_{\mathbf m'})_\gamma.$$

If $m = M \ne m' = M'$ the inner product vanishes, since $\mathcal{H}_{\gamma,M} \perp \mathcal{H}_{\gamma,M'}$. If $m = m'$ but $\mathbf m \not\equiv \mathbf m'$ then $m_i > m_i'$ for some i. Thus

$$(C^\gamma_{\mathbf m}, D^\gamma_{\mathbf m'})_\gamma = \kappa\ (1, P_1^{m_1} \cdots P_n^{m_n} D^\gamma_{\mathbf m'})_{\gamma'} = 0$$

since $P_i^{m_i} D^\gamma_{\mathbf{m}'} = 0$. (Here, κ is a nonzero constant.) We conclude that the set $\{C^\gamma_{\mathbf{m}}, D^\gamma_{\mathbf{m}'}\}$ is biorthogonal. (This family is a generalization of biorthogonal polynomals in two variables studied by Appell and Kampé de Fériet [1] and extended by Fackerell and Littler [9].)

Note that the norm of the weight function is

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-1}} dx_n \, [\prod_{k=1}^{n} x_k^{\gamma_k-1}](1-x)^{\gamma_{n+1}-1}$$

$$= (1,1)_\gamma = \frac{\left[\prod_{k=1}^{n+1}\Gamma(\gamma_k)\right]}{\Gamma(G)}. \tag{5.9}$$

The relation

$$(P_i^* C^{\hat\gamma}_{\hat{\mathbf{m}}}, D^\gamma_{\mathbf{m}'})_\gamma = (C^{\hat\gamma}_{\hat{\mathbf{m}}}, P_i D^\gamma_{\mathbf{m}'})_{\hat\gamma}$$

yields (for $\mathbf{m} = \mathbf{m}'$) the recurrence relation

$$(C^\gamma_{\mathbf{m}}, D^\gamma_{\mathbf{m}})_\gamma = -\frac{m_i(M+G-1)}{\gamma_i}(C^{\hat\gamma}_{\hat{\mathbf{m}}}, D^{\hat\gamma}_{\hat{\mathbf{m}}})_{\hat\gamma}.$$

The normalization of the biorthogonal basis can be obtained from this result and (5.9).

6. Derivation of the Heun expansion formula. Now we restrict our attention to the case $n = 2$, i.e., to the 2-sphere. Then equations (2.5), (2.6) become

$$H\Phi = -j(j+G-1)\Phi, \tag{6.1}$$

where

$$H = \sum_{i,k=1}^{2}(x_i\delta_{ik} - x_i x_k)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{2}(\gamma_i - Gx_i)\frac{\partial}{\partial x_i}. \tag{6.2}$$

Here $G = \gamma_1 + \gamma_2 + \gamma_3$. Recall that H maps polynomials of maximum degree m_i in x_i to polynomials of the same type. The polynomial eigenfunctions of H form a basis for the space of all polynomials $f(x_1, x_2)$ and the spectrum of H acting on this space is exactly $\{-j(j+G-1) : j = 0, 1, \ldots\}$. Furthermore $H = \Delta_2 + \Lambda_2$ where Δ_2 is the Laplace Beltrami operator on S^2 and

$$\Lambda_2 = \sum_{i=1}^{2}[\gamma_i - \frac{1}{2} + (\frac{3}{2} - G)x_i]\frac{\partial}{\partial x_i}. \tag{6.3}$$

Moreover, H is self-adjoint with respect to the inner product

$$(f_1, f_2) = \iint_{x_1,x_2>0,1-x_1-x_2>0} f_1(\mathbf{x})\overline{f_2(\mathbf{x})}d\omega \tag{6.4}$$

where

$$dw = x_1^{\gamma_1-1}x_2^{\gamma_2-1}(1-x_1-x_2)^{\gamma_3-1}dx_1 dx_2, \tag{6.5}$$
$$(Hf_1, f_2) = (f_1, Hf_2),$$

and $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers. Here f_1, f_2 are polynomials in $\mathbf{x} = (x_1, x_2)$. There are exactly two separable coordinate systems for equation (6.1): spherical coordinates and ellipsoidal coordinates [12, 14] .

We have already worked out the basis corresponding to spherical coordinates. For fixed j the polynomials

$$\begin{aligned}\psi_{jm}(x_1,x_2) &= (x_1+x_2)^m P_{j-m}^{\gamma_1+\gamma_2+2m-1,\gamma_3-1}(2x_1+2x_2-1) \\ &\quad\times P_m^{\gamma_2-1,\ \gamma_1-1}\left(\frac{2x_1}{x_1+x_2}-1\right),\quad m=0,1,\ldots,j\end{aligned} \tag{6.6}$$

form an orthogonal basis for the eigenspace corresponding to eigenvalue $-j(j+G-1)$. In terms of spherical coordinates

$$z_1 = \sin\theta\cos\phi, \quad z_2 = \sin\theta\sin\phi, \quad z_3 = \cos\theta, \tag{6.7}$$

and with $\psi_{jm}(x_1,x_2) \equiv \psi_{jm}[\theta,\phi]$, this basis reads

$$\psi_{jm}[\theta,\phi] = (\sin\theta)^{2m} P_{j-m}^{\gamma_1+\gamma_2+2m-1,\gamma_3-1}(\cos 2\theta) P_m^{\gamma_2-1,\ \gamma_1-1}(\cos 2\phi). \tag{6.8}$$

For the case of ellipsoidal coordinates $\{x,y\}$ we have

$$z_i^2 = \frac{(x-e_i)(y-e_i)}{(e_j-e_i)(e_k-e_i)}, \quad i=1,2,3, \quad i,j,k \text{ pairwise distinct.} \tag{6.9}$$

The metric on the 2-sphere is, in terms of these coordinates,

$$ds^2 = -\frac{(x-y)}{4}\left[\frac{dx^2}{(x-e_1)(x-e_2)(x-e_3)} - \frac{dy^2}{(y-e_1)(y-e_2)(y-e_3)}\right]. \tag{6.10}$$

The corresponding separation equations are

$$\begin{aligned}&[-(\lambda-e_1)(\lambda-e_2)(\lambda-e_3)\left[\frac{d^2}{d\lambda^2}+\left(\frac{\gamma_1}{\lambda-e_1}+\frac{\gamma_2}{\lambda-e_2}\right.\right. \\ &\quad\left.\left.+\frac{\gamma_3}{\lambda-e_3}\right)\frac{d}{d\lambda}\right]+j(j+G-1)\lambda+q]\Phi_{jq}^{\epsilon}(\lambda)=0\end{aligned} \tag{6.11}$$

where $\lambda = x, y$ according as $\epsilon = 1, 2$, respectively.

Recall that any Fuchsian equation of second order with four singularities can be reduced to the form

$$\frac{d^2w}{dx^2} + [\frac{\gamma}{x-e_1} + \frac{\delta}{x-e_2} + \frac{\epsilon}{x-e_3}]\frac{dw}{dx} + \frac{\alpha\beta x - q}{(x-e_1)(x-e_2)(x-e_3)}w = 0 \tag{6.12}$$

where $\alpha+\beta-\gamma-\delta-\epsilon+1=0$. The singularities are located at $x = e_1, e_2, e_3$ and ∞ and have indices depending upon $\alpha, \ldots \epsilon$. The constant q is known as the accessory parameter. This is Heun's equation [6, 10, 11] and solutions may be characterised by the P symbol [8]

$$P\left\{\begin{array}{ccccc} e_1 & e_2 & e_3 & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{array}\right\}. \tag{6.13}$$

Thus each separation equation (6.11) is Heun's equation of the form (6.12) with $\gamma = \gamma_1$, $\delta = \gamma_2$, $\epsilon = \gamma_3$, $\alpha = -j$, $\beta = j + G - 1$. The solutions for the functions $\Phi^\epsilon_{jq}(\lambda)$ are Heun polynomials which for fixed j will form a complete set of basis functions once the eigenvalues q have been calculated. To calculate the eigenvalues it is convenient to observe that in the coordinate system x_1, x_2 the operator $\mathcal{M}$ whose eigenvalues χ are

$$\chi = -4q - (e_1 + e_2 + e_3)j(j + G - 1) \tag{6.14}$$

is given by [12, 14]

$$\mathcal{M} = (e_1 + e_2)S_{12} + (e_2 + e_3)S_{23} + (e_1 + e_3)S_{13} \tag{6.15}$$

where the S_{ik} are the symmetry operators (2.21). That is, $\mathcal{M}$ is the second order symmetry operator for the Laplacian ($[\mathcal{M}, \Delta] = 0$) which corresponds to the separable coordinates x, y, and the Heun basis $\psi = \Phi^1_{jq}(x)\Phi^2_{jq}(y)$ for solutions of (6.1) is characterized as the set of eigenfunctions $\mathcal{M}\psi = \chi\psi$. Expression (6.14) gives the relationship between the eigenvalues χ and q.

In the following our expansion formulas are valid for all real $\gamma_i > 0$. In the special case $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{2}$ we have $H = \Delta_2$, the Laplace-Beltrami operator on S^2. In this case the eigenvalue equation $\Delta_2\Phi = -j(j + \frac{1}{2})\Phi$ admits the Lie algebra $so(3)$ as a symmetry algebra. A basis for $so(3)$ is $\{u_1\partial_{u_2} - u_2\partial_{u_1}, u_3\partial_{u_1}, u_3\partial_{u_2}\}$ where $u_3 = \pm(1 - u_1^2 - u_2^2)^{\frac{1}{2}}$. This extra symmetry is associated with the fact that there are additional polynomial solutions of the eigenvalue equation (see §3 of reference [15]). In particular the equation admits polynomial solutions of the form $f(u_1, u_2)$ and the spectrum of Δ_2 acting on the space of all such polynomials is $-j(j + \frac{1}{2})$ where now $2j = 0, 1, 2 \ldots$. Furthermore there exist solutions of the form $u_3 g(u_1, u_2)$ with g a polynomial and with the same eigenvalues. The dimension of each eigenspace is $2j + 1$ rather than $j + 1$ for the general case. In this special case the eigenfunctions corresponding to spherical coordinates are just the spherical harmonics whereas those corresponding to ellipsoidal coordinates are products of Lamé polynomials. For the solution of the problem of expanding the Lamé basis in terms of a spherical harmonic basis see [13, 21, 22].

Now we consider the problem of expanding the Heun basis $\Phi^1_{jq}(x)\Phi^2_{jq}(y)$ in terms of the Jacobi polynomial basis (6.6), (6.8):

$$\psi = \Phi^1_{jq}(x)\Phi^2_{jq}(y) = \sum_{m=0}^{j} \xi_m \psi_{jm}[\theta, \phi]. \tag{6.16}$$

Three term recurrence relations for the expansion coefficients ξ_m can be deduced by requiring that

$$\mathcal{M}\psi = \chi\psi. \tag{6.17}$$

To carry out the computation we need the action of the various pieces S_{ik} of $\mathcal{M}$ on the Jacobi bases $\psi_{jm}[\theta, \phi]$. Since $\mathcal{M}$ commutes with H there must exist an expansion of the form $\mathcal{M}\psi_{jm} = \sum_r X_r \psi_{j,m+r}$. From the known recurrence relations of the form $xP_n^{\alpha,\beta}=\mathcal{A}P_{n-1}^{\alpha,\beta}+\mathcal{B}P_n^{\alpha,\beta}+\mathcal{C}P_{n+1}^{\alpha,\beta}$ and $(1-x^2)\frac{d}{dx}P_n^{\alpha,\beta}=AP_{n-1}^{\alpha,\beta}+BP_n^{\alpha,\beta}+CP_{n+1}^{\alpha,\beta}$ for Jacobi polynomials we can obtain the result

$$\mathcal{M}\psi_{jm}[\theta,\phi]=\sum_{r=-1}^{+1}X_r\psi_{j,m+r}[\theta,\phi] \tag{6.18}$$

where

$$\begin{aligned}
X_1(m,j)&\\
&=\frac{4(e_1-e_2)(\gamma_1+\gamma_2+\gamma_3+m+j-1)(\gamma_3-m+j-1)(m+1)(\gamma_1+\gamma_2+m-1)}{(\gamma_1+\gamma_2+2m-1)(\gamma_1+\gamma_2+2m)}, \qquad (6.19)\\
X_{-1}(m,j)&\\
&=\frac{4(e_1-e_2)(\gamma_1+\gamma_2+m+j-1)(-m+j+1)(\gamma_2-1)(\gamma_1-1)}{(\gamma_1+\gamma_2+2m-1)(\gamma_1+\gamma_2+2m-2)},\\
X_0(m,j)-\chi&\\
&=\frac{2(e_1-e_2)[m^2+m(\gamma_1+\gamma_2-1)-j^2-j(\gamma_1+\gamma_2+\gamma_3-1)](\gamma_1+\gamma_2-2)(\gamma_1-\gamma_2)}{(\gamma_1+\gamma_2+2m-2)(\gamma_1+\gamma_2+2m)}\\
&\quad+4\frac{(e_1-e_2)m\gamma_3(\gamma_1-\gamma_2)(m+\gamma_2)}{(\gamma_1+\gamma_2+2m-2)(\gamma_1+\gamma_2+2m)}\\
&\quad+2(e_1+e_2)[-m^2-m(\gamma_1+\gamma_2-1)+j^2+j(\gamma_1+\gamma_2+\gamma_3-1)]\\
&\quad+4e_3[m^2+m(\gamma_1+\gamma_2-1)]+4q.
\end{aligned}$$

See [16] for more of the details. Now substituting the expansion (6.16) into the eigenvalue equation $\mathcal{M}\psi=\chi\psi$ and using (6.18) we find the three term recurrence relation

$$X_1(m-1,j)\xi_{m-1}+(X_0(m,j)-\chi)\,\xi_m+X_{-1}(m+1,j)\xi_{m+1}=0 \tag{6.20}$$

where $m=0,1,\dots j$. Consequently the $j+1$ independent eigenvalues q are calculated from the determinant

$$\begin{vmatrix}
X_0(j,j)-\chi & X_1(j-1,j) & & & \\
X_{-1}(j,j) & X_0(j-1,j)-\chi & X_1(j-2,j) & & \\
\ddots & \ddots & \ddots & & \\
 & & & & \\
 & & & X_{-1}(1,j) & X_0(0,j)-\chi
\end{vmatrix}=0. \tag{6.21}$$

To obtain the expansions in terms of one variable from (6.16) we proceed as follows. For the two choices of $z_i,\ i=1,2,3$ given by (6.9) and (6.7) take $y=e_3,\ \theta=\frac{\pi}{2}$. Then the expansion has the form

$$\Phi^1_{jq}(x)=\sum_{m=0}^{j}\hat{\gamma}_mP_m^{\gamma_2-1,\gamma_1-1}(\cos 2\phi) \tag{6.22}$$

where

$$\cos 2\phi = 2\frac{(x-e_1)}{(e_2-e_1)} - 1,$$

see [8, 24]. A different type of expansion can be obtained by taking $\phi = \pi/2$ and $y = e_1$. The resulting expression has the form

$$\Phi^1_{jq}(x) = \sum_{m=0}^{j} \tilde{\gamma}_m (\sin\theta)^{2m} P_{j-m}^{2m+\gamma_1+\gamma_2-1,\ \gamma_3-1}(\cos 2\theta) \tag{6.24}$$

where

$$\cos 2\theta = -2\frac{(x-e_2)}{(e_2-e_3)} - 1.$$

This second type of one variable expansion of a Heun polynomial appears to be new. The convergence of series of this type will be discussed elsewhere.

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