Homework Problem Set #1

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Exercise 1 *For* $n > 0$, *let*

$$f_n(t) = \begin{cases} 1, & \text{for } 0 \le t \le \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_n \to 0$ *in the norm of* $L^2[0,1]$ *as* $n \to \infty$. *Show that* f_n *does not converge to 0 pointwise uniformly on* $[0,1]$.

Exercise 2 *For* $n > 0$, *let*

$$f_n(t) = \begin{cases} \sqrt{n}, & \text{for } 0 \le t \le \frac{1}{n^2} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_n \to 0$ *in the norm of* $L^2[0,1]$ *as* $n \to \infty$, *but* $f_n(0) \to \infty$ *as* $n \to \infty$.

Exercise 3 *For* $n > 0$, *let*

$$f_n(t) = \begin{cases} \frac{1}{n}, & \text{for } 0 \le t \le n \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_n(t) \to 0$ *pointwise uniformly on* $[0,\infty]$, *as* $n \to \infty$ *but that* f_n *doesn't converge to 0 in the norm of* $L^1[0,\infty]$.

Exercise 4 *Suppose the function* $w(t) \in L^2[-\infty,\infty]$ *has unit norm. Show that the functions* $w_{jk}(t) = 2^{j/2}w(2^jt - k)$, $(j,k = 0,\pm1,\pm2,\cdots)$ *also have unit norm.*

Exercise 5 *Project the function $f(t) = t$ onto the subspace of $L^2[0,1]$ spanned by the functions $\phi(t), \psi(t), \psi(2t), \psi(2t-1)$, where*

$$\phi(t) = \begin{cases} 1, & \text{for } 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad \psi(t) = \begin{cases} 1, & \text{for } 0 \le t \le \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(This is related to the Haar wavelet expansion for f.)

Exercise 6 *The Legendre polynomials $P_n(t)$, $n = 0, 1, \cdots$ are the ON set of polynomials on $L^2[-1,1]$, obtained by applying the Gram-Schmidt process to the monomials $1, t, t^2, t^3, \cdots$ and defined uniquely by the requirement that the coefficient of t^n in $P_n(t)$ is positive. (In fact they form an ON basis for $L^2[-1,1]$.) Compute the first 4 of these polynomials.*

Exercise 7 *Using the facts from the preceding exercise, show that the Legendre polynomials must satisfy a three-term recurrence relation*

$$tP_n(t) = a_n P_{n+1}(t) + b_n P_n(t) + c_n P_{n-1}(t), \qquad n = 0, 1, 2, \cdots$$

where we take $P_{-1}(t) \equiv 0$. (Note: If you had a general sequence of polynomials $\{p_n(t)\}$, where the highest order term on $p_n(t)$ was a nonzero multiple of t^n, then the best you could say was that

$$tp_n(t) = \sum_{j=0}^{n+1} \alpha_j p_j(t).$$

What is special about orthogonal polynomials that leads to three-term relations?) What can you say about a_n, b_n, c_n without doing any detailed computations?

Exercise 8 *Find the $L^2[-\pi,\pi]$ projection of the function $f_1(t) = t^2$ onto the $(2n+1)$-dimensional subspace spanned by the ON set*

$$\{\frac{1}{\sqrt{2\pi}}, \frac{\cos kt}{\sqrt{\pi}}, \frac{\sin kt}{\sqrt{\pi}} : \ k = 1, \cdots, n\}$$

for $n = 1$. Repeat for $n = 2, 3$. Plot these projections along with f_1. (You can use MATLAB, a computer algebra system, a calculator, etc.) repeat the whole exercise for $f_2(t) = t^3$. Do you see any marked differences between the graphs in the two cases?

Exercise 9 *Prove the Fredholm alternative for finite dimensional inner product spaces: Let $\mathbf{T} : U \to V$ be a linear map from the space U to the space V. Then either*

- *for any $v \in V$ we can solve the equation $\mathbf{T}u = v$ for some $u \in U$, or*
- *there is a nonzero $v_0 \in V$ such that $\mathbf{T}^* v_0 = \Theta$.*

Exercise 10 *Use least squares to fit an exponential equation of the form $y = ae^{bx}$ to the data*

$$\begin{array}{rcccccccc} x & = & 0.6 & 1.0 & 2.3 & 3.1 & 4.4 & 5.8 & 7.2 \\ y & = & 3.6 & 2.5 & 2.2 & 1.5 & 1.1 & 1.3 & 0.9 \end{array}$$

in order to estimate the value of y when $x = 2.0$. Hint: Write the equation in linear form.