Homework Problem Set #2

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Exercise 1 *Expand* $f(t) = t^2$ *in a Fourier series on the interval* $-\pi \le t \le \pi$. *Plot both* f *and the partial sums*

$$S_k(t) = \frac{a_0}{2} + \sum_{n=0}^{k} (a_n \cos nt + b_n \sin nt)$$

for $k = 1, 2, 5, 7$. *Observe how the partial sums approximate* f.

Exercise 2 *Expand*

$$f(t) = \begin{cases} 0 & -\pi < t \le -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} < t \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t \le \pi \end{cases}$$

in a Fourier series on the interval $-\pi \le t \le \pi$. *Plot both* f *and the partial sums* S_k *for* $k = 5, 10, 20, 40$. *Observe how the partial sums approximate* f. *What accounts for the slow rate of convergence?*

Exercise 3 *Let* $a > 0$. *Use the Fourier transforms of* $\mathrm{sinc}(x)$ *and* $\mathrm{sinc}^2(x)$ *derived in the notes, together with the basic tools of Fourier transform theory, such as Parseval's equation, substitution,* $\cdots$ *to show the following. (Use only rules from Fourier transform theory. You shouldn't do any detailed computation such as integration by parts.)*

- $\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^3 dx = \frac{3a^2\pi}{4}$

- $\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^4 dx = \frac{2a^3\pi}{3}$

Exercise 4 *Show that the n-translates of sinc are orthonormal:*

$$\int_{-\infty}^{\infty} \mathrm{sinc}(x-n) \cdot \mathrm{sinc}(x-m)\ dx = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{otherwise,} \end{cases} \quad n, m = 0, \pm, 1, \cdots$$

Exercise 5 *Let*

$$f(x) = \begin{cases} 1 & -2 \le t \le -1 \\ 1 & 1 \le t \le 2 \\ 0 & \text{otherwise,} \end{cases}$$

- *Compute the Fourier transform $\hat{f}(\lambda)$ and sketch the graphs of f and $\hat{f}$.*
- *Compute and sketch the graph of the function with Fourier transform $\hat{f}(-\lambda)$*
- *Compute and sketch the graph of the function with Fourier transform $\hat{f}'(\lambda)$*
- *Compute and sketch the graph of the function with Fourier transform $\hat{f} * \hat{f}(\lambda)$*
- *Compute and sketch the graph of the function with Fourier transform $\hat{f}(\frac{\lambda}{2})$*

Exercise 6 *Deduce what you can about the Fourier transform $\hat{f}(\lambda)$ if you know that $f(t)$ satisfies the dilation equation*

$$f(t) = f(2t) + f(2t-1).$$

Exercise 7 *Let $f(t) = \frac{a}{t^2+a^2}$ for $a > 0$.*

- *Show that $\hat{f}(t) = \pi e^{-a|\lambda|}$. Hint: It is easier to work backwards.*
- *Use the Poisson summation formula to derive the identity*

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}.$$

  What happens as $a \to 0+$? Can you obtain the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ from this?

Exercise 8 *The Butterworth filter is a causal filter, used for noise reduction. It is defined by*

$$h(t) = \begin{cases} Ae^{-\alpha t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where A, α are positive parameters.

- *Compute the Fourier transform $\hat{h}(\lambda)$ and verify that it decays as $\lambda \to \infty$ thus diminishing the high-frequency components of the filered signal $\hat{h}(\lambda)\hat{f}(\lambda)$*

- *Consider the signal*

$$f(t) = e^{-t}(\sin 5t + \sin 3t + \sin t + \sin 40t), \quad \text{for } 0 \leq t \leq \pi,$$

  and zero elsewhere. Filter this signal with the Butterworth filter: compute $(f * h)(t)$ for $0 \leq t \leq \pi$. Starting with $A = a = 10$, try various values of $A = a$. Compare the original signal with the filtered signal.