Homework Problem Set #3

Willard Miller

February 25, 2002

Exercise 1 *If the continuous-time band-limited signal is* $x(t) = \cos t$*, what is the period* T *that gives sampling exactly at the Nyquist rate? What samples* $\mathbf{x}(nT)$ *do you get at this rate? What samples do you get from* $\mathbf{x}(t) = \sin t$*?*

Exercise 2 *Suppose the only nonzero components of the input vector* $\mathbf{x}$ *and the impulse response vector* $\mathbf{h}$ *are* $\mathbf{x}(0) = 1$*,* $\mathbf{x}(1) = 3$*, and* $\mathbf{h}(0) = \frac{1}{2}$*,* $\mathbf{h}(1) = \frac{1}{2}$*,. Compute the outputs* $\mathbf{y}(n)$*. Verify in the frequency domain that* $Y(\omega) = H(\omega)X(\omega)$*.*

Exercise 3 *Iterate the averaging filter* $\mathbf{H}$ *of Exercise 2 four times to get* $\mathbf{K} = \mathbf{H}^4$*. What is* $K(\omega)$ *and what is the impulse response* $\mathbf{k}(n)$*?*

Exercise 4 *A direct approach to the convolution rule* $Y = HX$*. What is the coefficient of* z^{-n} *in* $\left(\sum \mathbf{h}(k)z^{-k}\right)\left(\mathbf{x}(\ell)z^{-\ell}\right)$*? Show that your answer agrees with* $\sum \mathbf{h}(k)\mathbf{x}(n-k)$*.*

Exercise 5

- *Write down the infinite matrix* $(\downarrow 2)$ *that executes downsampling:* $(\downarrow 2)\mathbf{x}(n) = \mathbf{x}(2n)$*.*
- *Write down the transpose matrix* $(\uparrow 2) = (\downarrow 2)^{\rm tr}$*. Multiply the matrices* $(\uparrow 2)(\downarrow 2)$ *and* $(\downarrow 2)(\uparrow 2)$*. Describe the output from* $(\uparrow 2)\mathbf{y}(n)$*.*

Exercise 6 *Supose a real infinite matrix has the property* $\mathbf{Q}^{\rm tr}\mathbf{Q} = \mathbf{I}$*. Show that the columns of* $\mathbf{Q}$ *are mutually orthogonal unit vectors. Does it follow that* $\mathbf{Q}\mathbf{Q}^{\rm tr} = \mathbf{I}$*?*

Exercise 7 *An $n \times n$ matrix A is called a* **circulant** *if all of its diagonals (main, sub and super) are constant and the indices are interpreted mod n. EXAMPLE:*

$$\begin{pmatrix} 1 & 5 & 3 & 2 \\ 2 & 1 & 5 & 3 \\ 3 & 2 & 1 & 5 \\ 5 & 3 & 2 & 1 \end{pmatrix}.$$

- *Look at the n-periodic sequence a where $a_\ell = A_{\ell+1,1}$, $\ell = 0, 1 \cdots, n-1$. Write the entries of A in terms of the sequence a.*

- *Let X be an $n \times 1$ column vector. Show that $Y = AX$ is equivalent to $y = a * x$ if x, y are n-periodic sequences for which $x_\ell = X_{\ell+1,1}$ and similarly for $y_\ell = Y_{\ell+1,1}$, $\ell = 0, \cdots, n-1$.*

- *Prove that the DFT diagonalizes all circulant matrices. That is, that $\frac{1}{n}\overline{\mathcal{F}}^{\rm tr} A\mathcal{F} = D$, where D is diagonal. What are the diagonal entries of D? (i.e., what are the eigenvalues of A)?*

Exercise 8 *Let* **C**, **H** *be real low pass filters, each satisfying the double-shift row orthogonality condition. Does the product* **CH** *satisfy the double- shift row orthogonality condition?*