Homework Problem Set #6

Math 5467

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The first two problems relate to the frequency domain condition that the integer translates of the scaling function form an ON set: $\sum_n |\hat{\phi}(\omega+2\pi n)|^2 \equiv 1$.

Exercise 1 *Let $\phi(t)$ be the Haar scaling function*

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. *Verify that* $\hat{\phi}(\omega) = \frac{1-e^{-i\omega}}{i\omega}$.
2. *Verify that* $|\hat{\phi}(\omega)|^2 = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2$.
3. *From the frequency domain condition that the set $\{\phi(t-k)\}$ is ON, derive the identity*
$$\sum_{k=-\infty}^{\infty} \frac{\sin^2(\frac{\omega}{2}+\pi k)}{(\frac{\omega}{2}+\pi k)^2} = 1.$$
4. *Conclude that*
$$\csc^2 \frac{\omega}{2} = \sum_{k=-\infty}^{\infty} \frac{4}{(\omega+2\pi k)^2}.$$

Exercise 2 *Let $\phi(t)$ be the hat function*

$$\phi(t) = \begin{cases} t+1, & -1 \le t \le 0, \\ 1-t, & 0 < t \le 1, \\ 0, & |t| > 1. \end{cases}$$

We know that the set $\{\phi(t-k)\}$ *is a nonorthogonal base for the space* V_0 *of all square integrable signals* $f(t)$ *that are piecewise linear and continuous with possible corners occuring at only the integers* $0, \pm 1, \pm 2, \cdots$. *It is a so-called Riesz basis. This problem illustrates a method for constructing an orthogonal multiresolution analysis from a nonorthogonal multiresolution analysis. The method proceeds in the frequency domain. In the signal domain it looks messy.*

1. *Show that*
$$\hat{\phi}(\omega) = \frac{\sin^2(\omega/2)}{\omega^2/4}.$$

2. *Show that*
$$\sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^4} = \frac{3 - 2\sin^2(\omega/2)}{48 \sin^4(\omega/2)}.$$
To do this you can differentiate twice one of the results of the previous problem.

3. *Define the function* $\psi(t)$ *by its Fourier transform*
$$\hat{\psi}(\omega) = \frac{4\sin^2(\omega/2)}{\omega^2 \sqrt{1 - \frac{2}{3}\sin^2(\omega/2)}}.$$
Show that $\{\psi(t-k)\}$ *is an ON set. For this, you should use the frequency domain test given above.*

Exercise 3 *(Polynomial Suppression) Sometimes signals have a built-in bias, a polynomial part (say linear or quadratic) added to a bounded and rapidly oscillating part. We would like to remove the bias from the signal. Take the example*
$$f(t) = 2 + 5t + t^2 + \frac{1}{50}\sin(64\pi t)\cos(6\pi t).$$
Suppose we have 1024 samples of $f(t)$ *on the interval* $[-1, 1]$.

1. *Devise a strategy for separating the bias from the remaining signal, via Daubechies wavelet analysis. Hint: How big does* N *have to be to reproduce quadratics exactly in* V_0*?*

2. *(extra credit) If you have access to the Wavelet Toolbox in MATLAB, carry out the decomposition. You will have to pay attention to the fact that the signal is finite. Thus some type of padding is needed at the data boundaries.*

Exercise 4 *Problem 1 in Section 7.1 (page 233) of your text: Find the accuracy p from the sum rules for these filter coefficients. Consider going to the frequency domain to solve some of these problems.*

1. $\mathbf{h} = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$
2. $\mathbf{h} = \frac{1}{16}(1, 4, 6, 4, 1)$
3. $\mathbf{h} = \frac{1}{8}(1 - \sqrt{3}, 3 - \sqrt{3}, 3 + \sqrt{3}, 1 + \sqrt{3})$ *(Daubechies in reverse)*
4. $\mathbf{h} = D_6$
5. $\mathbf{h} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Exercise 5 *Problem 6 in Section 7.2 (page 242) of your text: Show that the filter* $\mathbf{h} = \frac{1}{4}(-1, 3, 3, -1)$ *does not satisfy the eigenvalue condition for* L^2 *convergence of the cascade algorithm, indeed that the matrix* $\mathbf{T}$ *has an eigenvalue* $\lambda = 2.1712\cdots$*, This bad* $\mathbf{h}$ *is dual to the good spline filter* $\mathbf{f} = \frac{1}{8}(1, 3, 3, 1)$*; their product* $\mathbf{h} * \mathbf{f}$ *is Daubechies maxflat halfband. (Note: I suggest that you use MATLAB to construct the matrix* $\mathbf{T}$ *and compute its eigenvalues. It isn't necessary to use the Wavelet Toolbox.)*

Here is another interactive demo and script that you can use to familiarize yourself with the wavelet capabilities of MatLab.

Exercise 6 *Only one user at a time can access the MatLab Wavelet Toolbox on the School of Mathematics linux machines. However MatLab itself is available to all users. Once you login on a Math Department computer type* **matlab** *to open MatLab. Then type* **wavemenu** *to open the wavelet toolbox. If the wavelet toolbox is available the toolbox menu will open. The following interactive demo is directly relevant to this part of the course:*

1. *Wavelet Packet in 1-D. Press the* **Wavelet Packet 1-D** *option. Under the File menu option, Demo Analysis select heavysin.mat, and run through the rest of the wavelet packet exercise 11 on page 466 of your text.*