Problem Set #1

Math 8600

October 10, 2002

Exercise 1 *For $n > 0$, let P_n be the vector space of all real-valued polynomials $p(t)$ on the real line that are of order $\leq n$. Let $a_0, a_1, \cdots, a_n$ be positive real numbers and choose real numbers $t_0 < t_1 < \cdots < t_n$ to define the real bilinear form $(\cdot, \cdot)$ on P_n such that*

$$(p, q) = \sum_{j=0}^{n} a_j p(t_j) q(t_j), \qquad p, q \in P_n.$$

1. *Show that $(\cdot, \cdot)$ is an inner product on P_n. In particular, only the zero function has zero length.*

2. *Show that the* **interpolation polynomials**

$$\ell_k(t) = \frac{\prod_{j=0, j\neq k}^{n} (t - t_j)}{\prod_{j \neq k} (t_k - t_j)}, \qquad k = 0, \cdots, n$$

   form a basis for P_n.

3. *Show that the ℓ_k, suitably normalized, form an ON basis for P_n.*

4. *Expand $p(t) \in P_n$ in terms of the basis, and compute the expansion coefficients.*

Exercise 2 *Consider the Gaussian pulse*

$$s(t) = (\frac{2}{\pi T^2})^{\frac{1}{4}} e^{-t^2/T^2 + 2\pi i \omega_0 t},$$

normalized to have unit energy. Verify that the ambiguity function is given by

$$A_s(u,w) = e^{-\frac{1}{2}\left(\frac{u^2}{T^2}+4\pi^2 w^2 T^2\right)} e^{-2\pi i \omega_0 u}.$$

Describe the level curves $|A_s(u,w)| = k$ *in the* $u - w$ *plane. Discuss the effect of varying the pulse length* T *on the problem of estimating the range and velocity of the target.*

Exercise 3 *Show that the area enclosed by a level curve* $|A_s(u,w)| = k$ *in problem 2 is independent of* T*.*

Exercise 4 *Consider the rectangular pulse with unit energy*

$$s(t) = \frac{1}{\sqrt{2T}} \chi_T(t) e^{2\pi i \omega_o t}$$

where

$$\chi_T(t) = \begin{cases} 1 & \text{if } -T \le t \le T \\ 0 & \text{otherwise.} \end{cases}$$

Show that the ambiguity function is

$$A_s(u,w) = e^{-2\pi i \omega_0 u} \begin{cases} \frac{\sin[(1-\frac{|u|}{2T})(4\pi w T)]}{4\pi w T} & \text{if } |u| \le 2T \\ 0 & \text{if } |u| > 2T. \end{cases}$$

Describe the level curves $|A_s(u,w)| = k$ *in the* $u - w$ *plane. Show that for* $k = 1 - c^2$ *with c very close to zero, the level curves can be approximated by* $\frac{|u|}{2T} + \frac{8}{3}\pi^2 w^2 T^2 = c^2$.

Exercise 5 *Assuming that* $\chi(t)$ *is a continuously differentiable function of* t*, use a differential equations argument to show that the only nonzero solutions of the functional equation*

$$\chi(t_1 + t_2) = \chi(t_1)\chi(t_2), \quad t_1, t_2 \in R$$

are $\chi(t) = e^{at}$*, where* a *is a constant.*

Exercise 6 *(Haar wavelets on [0,1]) Let* $\phi(t)$ *be the Haar scaling function*

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let n *be a positive integer and* $h_k(t) = \sqrt{n}\phi(nt - k), \quad k = 0, 1, \cdots, n-1$

1. *Show that* $\{h_0, \cdots, h_{n-1}\}$ *is an ON set in* $L^2[0,1]$.

2. *Let* $f(t)$ *be a continuous function on* $[0,1]$ *and form the projection* $f_n(t)$ *on the subspace* S_n *of* $L^2[0,1]$ *spanned by* $\{h_0, \cdots, h_{n-1}\}$:

$$f_n = \sum_{k=0}^{n-1} (f, h_k) h_k.$$

   Show that $f_n(t) \to f(t)$ *pointwise uniformly in* t *as* $n \to \infty$.

3. *For* $f(t) = 1 - t^2$, *use MATLAB, Maple, or* $\cdots$ *to compute explicitly the Haar wavelet decomposition for* $n = 4, 8,$ *and* 16. *Plot the results.*

Exercise 7 *Let* G *be a finite group and* T *an irreducible matrix representation of* G. *Choose* $h \in G$ *and let*

$$C_h = \{k \in G : k = g^{-1}hg \text{ for some } g \in G\}$$

be the **conjugacy class** *containing* h. *Show that*

$$\sum_{k \in C_h} T(k)$$

is a multiple of the identity matrix.