Solutions to Problem Set #1

Math 8600

October 11, 2002

Exercise 1 *For $n > 0$, let P_n be the vector space of all real-valued polynomials $p(t)$ on the real line that are of order $\leq n$. Let $a_0, a_1, \cdots, a_n$ be positive real numbers and choose real numbers $t_0 < t_1 < \cdots < t_n$ to define the real bilinear form $(\cdot, \cdot)$ on P_n such that*

$$(p, q) = \sum_{j=0}^{n} a_j p(t_j) q(t_j), \qquad p, q \in P_n.$$

1. *$(\cdot, \cdot)$ is an inner product on P_n. In particular, it is obvious that $(p, q) = (q, p)$ for all $p, q \in P_n$. Further, it is evident from the definition that*

   $$(\alpha_1 p_1 + \alpha_2 p_2, q) = \alpha_1 (p_1, q) + \alpha_2 (p_2, q)$$

   for all real numbers α_1, α_2 and $p_1, p_2, q \in P_n$. If $q \in P_n$ then

   $$(q, q) = \sum_{j=0}^{n} a_j q^2(t_j) \geq 0.$$

   If $(q, q) = 0$ then we must have $q(t_j) = 0, \qquad j = 0, 1, \cdots, n$. Thus q is a polynomial of order $\leq n$ that vanishes at $n+1$ points. By the fundamental theorem of algebra, this is possible if and only if $q(t) \equiv 0$ for all t.

2. *Show that the* **interpolation polynomials**

   $$\ell_k(t) = \frac{\prod_{j=0, j\neq k}^{n} (t - t_j)}{\prod_{j \neq k} (t_k - t_j)}, \qquad k = 0, \cdots, n$$

form a basis for P_n.

Proof: Note that $\ell_k(t_j) = \delta_{jk}$. Given $p(t) \in P_n$ let

$$\tilde{p}(t) = \sum_{k=0}^{n} p(t_k)\ell_k(t).$$

Clearly, $\tilde{p} \in P_n$, as is the polynomial $q(t) = p(t) - \tilde{p}(t)$. However, $\tilde{p}(t_j) = \sum_{k=0}^{n} p(t_k)\ell_k(t_j) = p(t_j)$ for $j = 0, 1, \cdots, n$, so $q(t_j) = p(t_j) - \tilde{p}(t_j) = 0$. By the argument in the previous proof, this means that $q(t) \equiv 0$. Hence

$$p(t) = \sum_{k=0}^{n} p(t_k)\ell_k(t).$$

and the interpolation polynomials form a basis for P_n.

3. *The ℓ_k, suitably normalized, form an ON basis for P_n.*

   Proof:

$$(\ell_h, \ell_k) = \sum_{j=0}^{n} a_j \ell_h(t_j)\ell_k(t_j) = a_h \delta_{hk}$$

   so the functions

$$\hat{\ell}_k(t) = \frac{1}{\sqrt{a_k}}\ell_k(t), \qquad k = 0, 1, \cdots, n$$

   form an ON basis of P_n.

4. *Expand $p(t) \in P_n$ in terms of the basis, and compute the expansion coefficients: We have*

$$p(t) = \sum_{k=0}^{n} b_k \hat{\ell}_k(t)$$

   where

$$b_k = (p, \hat{\ell}_k) = \sum_{j=0}^{n} a_j p(t_j) \frac{1}{\sqrt{a_k}} \ell_k(t_j) = \sqrt{a_k}\, p(t_k).$$

Exercise 2 *Consider the Gaussian pulse*

$$s(t) = (\frac{2}{\pi T^2})^{\frac{1}{4}} e^{-t^2/T^2 + 2\pi i \omega_0 t},$$

normalized to have unit energy. Verify that the ambiguity function is given by

$$A_s(u,w) = e^{-\frac{1}{2}\left(\frac{u^2}{T^2}+4\pi^2 w^2 T^2\right)} e^{-2\pi i \omega_0 u}.$$

Proof: We have

$$A_s(u,w) = \int_{-\infty}^{\infty} s(t-\frac{u}{2})\bar{s}(t+\frac{u}{2})e^{4\pi i t w}dt = \left(\frac{2}{\pi T^2}\right)\int_{-\infty}^{\infty} e^{-\frac{2t^2+u^2/2}{T^2}} e^{-2\pi i\omega_0 u + 4\pi i t w}dt$$

$$= \left(\frac{2}{\pi T^2}\right) e^{-2\pi i \omega_0 u} e^{-\frac{1}{2}(\frac{u^2}{T^2}+4\pi^2 w^2 T^2)} \int_{-\infty}^{\infty} e^{-(\frac{\sqrt{2}t}{T}+\sqrt{2}i\pi w T)^2} dt.$$

From complex variable theory, we know that this integral is independent of w. Setting $w = 0$ under the integral sign and using the well known integral $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$, we obtain the result.

Describe the level curves $|A_s(u,w)| = k$ in the $u - w$ plane. Discuss the effect of varying the pulse length T on the problem of estimating the range and velocity of the target.

The level curves are ellipses

$$\frac{u^2}{T^2} + 4\pi^2 w^2 T^2 = \lambda, \qquad \lambda = \ln k^{-2}.$$

For $k = 1$ the ellipse shrinks to the point $(0,0)$. As k decreases, the ellipse grows. As T grows the ellipse becomes elongated along the u-axis and shrinks along the w-axis. Thus we obtain a more accurate estimate of the velocity of the target, but a less accurate estimate of the position. The reverse happens as T grows smaller.

Exercise 3 *Show that the area enclosed by a level curve $|A_s(u,w)| = k$ in problem 2 is independent of T.*

Proof: let A be the area. Then

$$A = 4\int_0^{\frac{\sqrt{\lambda}}{2\pi T}} \sqrt{\lambda T^2 - 4\pi^2 w^2 T^4}\ dw = \frac{2\lambda}{\pi}\int_0^1 \sqrt{1-x^2}\ dx = \frac{\lambda}{2}.$$

Exercise 4 *Consider the rectangular pulse with unit energy*

$$s(t) = \frac{1}{\sqrt{2T}}\chi_T(t)e^{2\pi i \omega_o t}$$

where

$$\chi_T(t) = \begin{cases} 1 & \text{if } -T \le t \le T \\ 0 & \text{otherwise.} \end{cases}$$

Show that the ambiguity function is

$$A_s(u,w) = e^{-2\pi i\omega_0 u}\begin{cases} \frac{\sin[(1-\frac{|u|}{2T})(4\pi wT)]}{4\pi wT} & \text{if } |u| \le 2T \\ 0 & \text{if } |u| > 2T. \end{cases}$$

Proof: Since $A_s(u,w)$ is an even function of u, we can restrict ourselves to the case $u \ge 0$. Then

$$A_s(u,w) = \frac{e^{-2\pi i\omega_0 u}}{2T}\int_{-\infty}^{\infty} \chi_T(t-\frac{u}{2}\chi_T(t-\frac{u}{2})e^{4\pi itw}dt.$$

Note that the integrand is nonzero only in the intersection of the t intervals $(-T+\frac{u}{2}, T+\frac{u}{2})$ and $(-T-\frac{u}{2}, T-\frac{u}{2})$. Thus the integrand is 0 for $u \ge 2T$. Otherwise the integrand is nozero in the overlap $(-T+\frac{u}{2}, T-\frac{u}{2})$ and we have

$$A_s(u,w) = \frac{e^{-2\pi i\omega_0 u}}{2T}\int_{-T+\frac{u}{2}}^{T-\frac{u}{2}} e^{4\pi itw}dt = e^{-2\pi i\omega_0 u}\frac{\sin 4\pi w(T-\frac{u}{2})}{4\pi wT}.$$

Describe the level curves $|A_s(u,w)| = k$ in the $u-w$ plane.

Demonstration: In the $u-w$ plane the curves have the equation

$$\left|\frac{\sin\left[4\pi w(T-\frac{|u|}{2})\right]}{4\pi wT}\right| = k, \qquad |u| \le 2T.$$

Since the left-hand side is an even function of u and an even function of w, we can, without loss of generality, limit ourselves to the quadrant $u \ge 0, w \ge 0$. Now we rewrite the equation in the form

$$\frac{\sin\left[4\pi wT(1-\frac{u}{2T})\right]}{4\pi wT(1-\frac{u}{2T})} = \frac{k}{1-\frac{u}{2T}}, \qquad 0 \le u \le 2T. \tag{1}$$

Then the left-hand side takes the form

$$f(x) = \frac{\sin x}{x}, \qquad \text{for } x > 0$$

and $f(0) = 1$. *Here,*

$$x = 4\pi w T(1 - \frac{u}{2T}).$$

From elementary calculus (also from the treatment of the Gibb's phenomenon example in the online wavelets notes) we know that $f(x)$ *has the following properties*

1. f *takes on its maximum value at* $x = 0$, *and* $f(0) = 1$.

2. f *is continous and differentiable for all* x.

3. f *is monotonically decreasing from* 1 *to* 0 *on the interval* $[0, \pi]$ *and the has a unique, continuous, inverse on this interval.*

4. *Thus, for each* z, $0 \leq z \leq 1$ *there is a unique* $x(z) \in [0, \pi]$ *such that* $z = f(x)$. *Further* $x(z) \to 0$ *as* $z \to 1$.

For fixed k, $0 \leq k \leq 1$ *we see that the left-hand side of (1) can take values only in the range* $0 \leq f(x) \leq 1$, *whereas the right-hand side* $z = \frac{k}{1-\frac{u}{2T}}$ *can take values only in the range* $k \leq z$. *Since the two sides are equal, for a solution* z *we must have* $k \leq z \leq 1$. *Thus* $u = 2T(1 - \frac{k}{z})$ *is constrained to lie in the range* $0 \leq u \leq 2T(1-k)$. *Once* u *is chosen in this range then* x, *hence* $w = x/4\pi T(1 - \frac{u}{2T}) = \frac{xz}{4\pi kT}$ *is uniquely determined by the equation* $f(x) = z$. *As* $k \to 1$ *we must have* $z \to 1$, *since* $k \leq z \leq 1$. *Thus* $x \to 0$ *as* $k \to 1$. *We see from this that the solutions* (u, w) *of (1) go smoothly to* $(0, 0)$ *as* $k \to 1$.

Now let us fix k *and consider the effect of varying* T. *Choose a* z *in the range* $k \leq z \leq 1$ *and let* x *be the unique value in the range* $0 \leq x \leq \pi$ *such that* $z = f(x)$. *Then*

$$u = 2T(1 - \frac{k}{z}), \qquad w = \frac{xz}{4\pi kT},$$

so the determination of position becomes less accurate and the determination of velocity more accurate as T *grows.*

Show that for $k = 1 - c^2$ *with* c *very close to zero, the level curves can be approximated by* $\frac{|u|}{2T} + \frac{8}{3}\pi^2 w^2 T^2 = c^2$.

Proof: For k *very close to* 1, (u, w) *is constrained to lie very close to* $(0, 0)$, *so, expanding the* $\sin$ *in a power series and dropping terms of order* ≥ 3 *in* *u, w, we have*

$$\frac{\sin[4\pi wT(1-\frac{|u|}{2T})]}{4\pi wT} \sim 1-\frac{|u|}{2T}-\frac{1}{6}(1-\frac{|u|}{2T})^3(16\pi^2w^2T^2) = 1-c^2$$

or

$$\frac{|u|}{2T}+\frac{8}{3}\pi^2w^2T^2 = c^2.$$

Exercise 5 *Assuming that $\chi(t)$ is a continuously differentiable function of t, use a differential equations argument to show that the only nonzero solutions of the functional equation*

$$\chi(t_1+t_2) = \chi(t_1)\chi(t_2), \quad t_1, t_2 \in R$$

are $\chi(t) = e^{at}$, where a is a constant.

Proof: Differentiating the functional equation with respect to t_1 we have $\chi'(t_1+t_2) = \chi'(t-1)\chi(t_2)$. Setting $t_1 = 0, t_2 = t$ we have

$$\chi'(t) = a\chi(t), \quad a = \chi'(0).$$

The general solution of this differential equation is $\chi(t) = \alpha e^{at}$ where α is a constant. Substituting this result into the functional equation we have $\alpha e^{t_1+t_2} = \alpha^2 e^{t_1+t_2}$, or $\alpha = \alpha^2$. Thiis means $\alpha = 0, 1$. Since $\chi(t) \neq 0$ we must have $\alpha = 1$.

Exercise 6 *(Haar wavelets on [0,1]) Let $\phi(t)$ be the Haar scaling function*

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let n be a positive integer and $h_k(t) = \sqrt{n}\phi(nt-k), \quad k = 0, 1, \cdots, n-1$

1. *Show that $\{h_0, \cdots, h_{n-1}\}$ is an ON set in $L^2[0,1]$.*

   Proof: Note that

$$h_k(t) = \begin{cases} \sqrt{n} & \text{if } \frac{k}{n} \le t < \frac{k+1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

   Thus $\int_0^1 h_k(t)h_\ell(t)dt = 0$ if $k \neq \ell$ because $h_k(t)h_\ell(t) = 0$ for all $t \in (0,1)$. However,

$$\int_0^1 h_k^2(t)dt = n\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt = 1.$$

2. *Let $f(t)$ be a continuous function on $[0,1]$ and form the projection $f_n(t)$ on the subspace S_n of $L^2[0,1]$ spanned by $\{h_0, \cdots, h_{n-1}\}$:*

$$f_n = \sum_{k=0}^{n-1} (f, h_k) h_k.$$

Show that $f_n(t) \to f(t)$ pointwise uniformly in t as $n \to \infty$.

Proof: We have

$$f_n(t) = \sum_{k=0}^{n-1} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\tau) d\tau \phi(nt - k).$$

Thus,

$$f_n(t) = n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\tau) d\tau \quad \text{for } \frac{k}{n} \le t < \frac{k+1}{n}.$$

By the mean value theorem of calculus, there is a point $t_{n,k}$ in the interval $(\frac{k}{n}, \frac{k+1}{n})$ such that the integral expression on the right is equal to $f(t_{n,k})$. Thus

$$f_n(t) = f(t_{n,k}) \quad \text{for } \frac{k}{n} \le t < \frac{k+1}{n}.$$

Since f is continuous on the closed bounded set $[0,1]$, it is uniformly continuous on this set. Thus for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $|f(t) - f(t')| < \epsilon$ whenever $|t - t'| < \delta(\epsilon)$. Now given $t \in [0,1)$ and $\epsilon > 0$, choose $n > \frac{1}{\delta(\epsilon)}$. Then

$$|f(t) - f_n(t)| = |f(t) - f(t_{n,k})| < \epsilon, \text{ because } |t - t_{n.k}| < \frac{1}{n} < \delta(\epsilon)$$

so $f_n(t) \to f(t)$, uniformly in t as $n \to \infty$.

3. *For $f(t) = 1 - t^2$, use MATLAB, Maple, or $\cdots$ to compute explicitly the Haar wavelet decomposition for $n = 4, 8,$ and 16. Plot the results.*

A solution using MATLAB. The following MATLAB code will give a simultaneous plot of the three functions.

```
t= 0:1/(2^7):1-1/(2^7);
f4= zeros(1,2^7);
N=4;
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
f4(number)=1- (Integer*(Integer+1)/(N)^2)-(1/(3*N^2));
end
end
f8= zeros(1,2^7);
N=8;
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
f8(number)=1- (Integer*(Integer+1)/(N)^2)-(1/(3*N^2));
end
end
f16= zeros(1,2^7);
N=16;
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
f16(number)=1- (Integer*(Integer+1)/(N)^2)-(1/(3*N^2));
end
end
plot(t,f4,t,f8,t,f16,t , 1-t.^2)
title('Plot of f(t)=1-t^2 and Haar approximations for N=4,8,16')
```

If we needed to use many different values of N, it would be easier to define a function haar(N) to compute $f_N(t)$ for us, save it in the file haar.m, and then write a simple code to plot haar(N) for different values of N. (You would need to open MATLAB in the directory containing haar.m, or make sure that this directory is in the startup path of MATLAB. For example, the file haar.m could take the form

```
function x = haar(N)
% This function computes the Haar approximation f_N(t)
% to f(t)=1-t^2 at 128 equally spaced points on the
% unit interval, for any positive integer N.
x= zeros(1,2^7);
```

```
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
x(number)=1- (Integer*(Integer+1)/(N)^2)-(1/(3*N^2));
end;
end;
```

Then the main program would be very short (even if we add the graph for N=32):

```
t= 0:1/(2^7):1-1/(2^7);
plot(t,haar(4),t,haar(8),t,haar(16),t,haar(32),t , 1-t.^2)
title('Plot of f(t)=1-t^2 and Haar approximations for N=4,8,16,32')
```

A more flexible (but less accurate) approach, useful for treating several different funtions $f(t)$ and values of N, would be easier to define a function $Haar(f, N)$ to compute $f_N(t)$ for us (using the MATLAB function $quad(f, a, b)$, Simpson's rule for evaluation of $\int_a^b f(t)dt$), save the definition in the file Haar.m, and then write a simple code to plot $Haar(f, N)$ for different choices of f, N. (You would need to open MATLAB in the directory containing Haar.m, or make sure that this directory is in the startup path of MATLAB. For example, the file Haar.m could take the form

```
function x = Haar(f,N)
% This function computes the Haar approximation f_N(t)
% to the function f(t) at 128 equally spaced points
%on the unit interval
x= zeros(1,2^7);
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
x(number)=N*quad(f,Integer/N,(Integer+1)/N);
end;
end;
```

Then the main program would be very short:

```
t= 0:1/(2^7):1-1/(2^7);
plot (t,Haar('1-t.^2',4),t,Haar('1-t.^2',8),t,Haar('1-t.^2',16),t,1-t.^2)
title('Plot of f(t)=1-t^2 and Haar approximations for N=4,8,16')
```

or

```
plot (t, Haar('sin(2*pi*t)',8),t, Haar('sin(2*pi*t)',16),t,sin(2*pi*t))
title('Plot of f(t)=sin(2*pi*t) and Haar approximations for N=8,16')
```

Exercise 7 *Let G be a finite group and T an irreducible matrix representation of G. Choose $h \in G$ and let*

$$C_h = \{k \in G : k = g^{-1}hg \text{ for some } g \in G\}$$

be the **conjugacy class** *containing h. Show that*

$$\sum_{k \in C_h} T(k)$$

is a multiple of the identity matrix.

Proof: Let $g_0 \in G$ and consider the set

$$g_0^{-1}C_h g_0 = \{g_0^{-1}kg_0 : k \in C_h\}.$$

1) If $k \in C_h$ then $k \in g_0^{-1}C_h g_0$. Indeed, then $k = g^{-1}hg$ for some $g \in G$ and we have

$$k = (g_0^{-1}g_0)g^{-1}hg(g_0^{-1}g_0) = g_0^{-1}\left((gg_0^{-1})^{-1}h(gg_0^{-1})\right)g_0.$$

2) If $k \in g_0^{-1}C_h g_0$ then $k \in C_h$. Indeed, then $k = g_0^{-1}\left(g^{-1}hg\right)g_0$ for some $g \in G$ and we have

$$k = (gg_0)^{-1}h(gg_0).$$

Thus

$$g_0^{-1}C_h g_0 \equiv C_h$$

for all $h, g_0 \in G$.

Now set $A = \sum_{k \in C_h} T(k)$. Then for any $g_0 \in G$ we have

$$T(g_0^{-1})AT(g_0)T(g_0) = \sum_{k \in C_h} T(g_0^{-1})T(k)T(g_0) = \sum_{k \in C_h} T(g_0^{-1}kg_0) = \sum_{k \in C_h} T(k) = A.$$

Thus $AT(g_0) = T(g_0)A$ for all $g_0 \in G$. Since T is irreducible, A is a multiple of the identity matrix.