Problem Set #2

Math 8600

November 12, 2002

Exercise 1 *(Oversampling) In this problem you will show that a positive effect of sampling a band limited signal $f(t)$ at a rate faster than the Nyquist rate is that the expansion for $f(t)$ in terms of the sampled values converges at a faster rate. (The negative effect is that you have to sample more often.)*

1. *Suppose f satisfies the hypotheses of the Shannon sampling theorem proven in the notes; in particular it is a band limited signal with $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \Omega$. Fix $a > 1$ and reprove the theorem to show that*

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-\frac{in\pi\lambda}{a\Omega}}, \qquad c_{-n} = \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}).$$

2. *Let*

$$\hat{g}_a(\lambda) = \begin{cases} 0 & \text{if } |\lambda| > a\Omega \\ \frac{\lambda + a\Omega}{(a-1)\Omega} & \text{if } -a\Omega \leq \lambda < -\Omega \\ 1 & \text{if } -\Omega \leq \lambda < \Omega \\ -\frac{\lambda - a\Omega}{(a-1)\Omega} & \text{if } \Omega \leq \lambda \leq a\Omega \end{cases}$$

Show that

$$g_a(t) = \frac{\cos \Omega t - \cos a\Omega t}{\pi(a-1)\Omega t^2}.$$

3. *Since $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \Omega$, we see that $\hat{f}(\lambda) = \hat{f}(\lambda)\hat{g}_a(\lambda)$. Prove that*

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}) g_a(t - \frac{n\pi}{a\Omega}).$$

Since $g_a(t)$ has a factor of t^2 in the denominator, this expression for $f(t)$ converges faster than the expression in the Shannon theorem. Note that the nth term behaves like $1/n^2$, rather than $1/n$.

Exercise 2 *(Filtering with the FFT) Let*

$$f(t) = e^{-\frac{t^2}{10}} \left(\sin 2t + 2\cos 4t + 0.4 \sin t \sin 50t \right).$$

Discretize f by setting $y_k = f(2k\pi/256), k = 1, \cdots, 256$. Use MATLAB's FFT to compute $\hat{y}_k$ for $0 \le k \le 255$. (Note that $y_{n-k} = \overline{y_k}$. Thus the low frequency coefficients are $\hat{y}_0, \cdots, \hat{y}_m$ and $\hat{y}_{256-m}, \cdots, \hat{y}_{256}$ for some small integer m. Filter out the high-frequency terms by setting $\hat{y}_k = 0$ for $m \le k \le 255 - m$ with $m = 6$. Then apply the inverse FFT to these filtered $\hat{y}_k$ to compute the filtered y_k. Plot the results and compare with the original unfiltered signal. Experiment with several different values of m.

Exercise 3 *(Compression with the FFT) Consider the signal $f(t)$ as given in the previous problem. Let $tol = 1.0$ In the previous problem compress the transformed signal by setting $\hat{y}_k = 0$ whenever $|\hat{y}_k| < tol$. Apply the inverse FFT to the compressed transfirmed signal to get a compressed signal y_k. Plot the results and compare with the original uncompressed signal. Experiment with several different values of tol. Keep track of the percentage of Fourier coefficients that have been filtered out.*

Exercise 4 *Construct the unitary rep $\tilde{\mathbf{T}}_n$ of H_R induced by the one-dimensional rep $\tilde{\mathbf{T}}_0^n(a_1, a_2, y_3) = e^{2\pi i n y_3}$ of the subgroup H^1, where n is an integer (not necessarily positive). Determine the action of H_R on the rep space. Under what conditions on n is $\tilde{\mathbf{T}}_n$ irred?*

Exercise 5 *Suppose $f \in L_2(R)$ such that $f_{\mathbf{P}} \ne 0$ almost everywhere. Prove that the set $\{e^{2\pi i(m_1 x_1 + m_2 x_2)} f_{\mathbf{P}}/|f_{\mathbf{P}}|,\ m_1, m_2 = \pm 1, \pm 2, \cdots\}$ is an ON basis for the lattice Hilbert space V'. Find an explicit expression for the corresponding ON basis of $L_2(R)$ obtained from the mapping $\mathbf{P}^{-1}$.*

The following MATLAB routines should be useful:

```
function fc=compress( f, r)
% Input the vector f and ratio r: 0<= r <=1.
```

```
% The output is the vector fc in which the smallest
% 100r% of the terms f_k, in absolute value, are set
% equal to zero.
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
N=length(f); Nr=floor(N*r);
ff=sort(abs(w));
tol=abs(ww(Nr+1);
fc=(abs(w)>=tol).*w;
```

You can discretize the interval $[0, 2\pi]$ and read in the signal as a vector by using the commands

```
t=linspace(0,2*pi,2^8);
f=exp(-t.^2/10).*(sin(2*t)+2*cos(4*t)+0.4*sin(t).*sin(50*t));
```

If $hatf$ is the FFT of f, you can filter out high frequency components from$hatf$ with a command such as

```
filterhatf=[ hatf(1: m) zeros(1, 2^8-2*m) hatf(2^8-m+1:2^8)]
```

```
function L2error =fftcomp(t,f,r)
% Input: time vector t, signal vector f, compression rate r, (between
% 0 and 1)
Output: graph of f, graph of the compression of f, and the relative L2
error
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
hatf=fft(f);
hatfc=compress(hatf,r);
fc=ifft(hatfc);
plot(t,f,t,fc)
L2error=norm(f-fc,2)/norm(f)
```