Problem Set #2

Math 8600

November 12, 2002

Exercise 1 *(Oversampling) In this problem you will show that a positive effect of sampling a band limited signal $f(t)$ at a rate faster than the Nyquist rate is that the expansion for $f(t)$ in terms of the sampled values converges at a faster rate. (The negative effect is that you have to sample more often.)*

1. *Suppose f satisfies the hypotheses of the Shannon sampling theorem proven in the notes; in particular it is a band limited signal with $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \Omega$. Fix $a > 1$ and reprove the theorem to show that*

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-\frac{in\pi\lambda}{a\Omega}}, \qquad c_{-n} = \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}).$$

Solution: *Here we use the definitions*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt, \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda$$

On the interval $[-a\Omega, a\Omega]$ we can expand $\hat{f}(\lambda)$ in a uniformly convergent Fourier series

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-\frac{in\pi\lambda}{a\Omega}}.$$

Since $\hat{f}(\lambda)$ vanishes outside this interval,

$$c_{-n} = \frac{1}{2a\Omega} \int_{-a\Omega}^{a\Omega} \hat{f}(\lambda) e^{\frac{in\pi\lambda}{a\Omega}} d\lambda = \frac{1}{2a\Omega} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{\frac{in\pi\lambda}{a\Omega}} d\lambda = \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}).$$

Now

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{i\lambda t}d\lambda=\frac{1}{2\pi}\int_{-a\Omega}^{a\Omega}\hat{f}(\lambda)e^{i\lambda t}d\lambda$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}c_{-n}\int_{-a\Omega}^{a\Omega}e^{i(-\pi n+at\Omega)\lambda/a\Omega}dt=\sum_{n=-\infty}^{\infty}c_{-n}\frac{\sin[a\Omega t-n\pi]}{\pi(a\Omega t-n\pi)}.$$

2. *Let*

$$\hat{g}_a(\lambda)=\begin{cases}0 & \text{if } |\lambda|>a\Omega\\ \frac{\lambda+a\Omega}{(a-1)\Omega} & \text{if } -a\Omega\le\lambda<-\Omega\\ 1 & \text{if } -\Omega\le\lambda<\Omega\\ -\frac{\lambda-a\Omega}{(a-1)\Omega} & \text{if } \Omega\le\lambda\le a\Omega\end{cases}$$

Show that

$$g_a(t)=\frac{(\cos\Omega t-\cos a\Omega t)}{\pi(a-1)\Omega t^2}.$$

Solution:

$$g_a(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}_a(\lambda)e^{i\lambda t}d\lambda$$

$$=\frac{1}{2\pi(a-1)}\left[\int_{-a\Omega}^{-\Omega}(\lambda+a\Omega)e^{i\lambda t}d\lambda+\int_{\Omega}^{a\Omega}(-\lambda+a\Omega)e^{i\lambda t}d\lambda\right]+\frac{1}{2\pi}\int_{-\Omega}^{\Omega}e^{i\lambda t}d\lambda$$

$$=\frac{1}{2\pi(a-1)}\int_{\Omega}^{a\Omega}\left[-2\lambda\cos\lambda t+2a\Omega\cos\lambda t\right]d\lambda+\frac{\sin\Omega t}{\pi t}$$

$$=\frac{\cos\Omega t-\cos a\Omega t}{\pi(a-1)\Omega t^2}.$$

3. *Since* $\hat{f}(\lambda)=0$ *for* $|\lambda|\ge\Omega$, *we see that* $\hat{f}(\lambda)=\hat{f}(\lambda)\hat{g}_a(\lambda)$. *Prove that*

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{\pi}{a\Omega}f(\frac{n\pi}{a\Omega})g_a(t-\frac{n\pi}{a\Omega}).$$

Solution:

$$\hat{f}(\lambda)=\hat{f}(\lambda)\hat{g}_a(\lambda)=\sum_{n=-\infty}^{\infty}c_{-n}e^{-\frac{in\pi\lambda}{a\Omega}}\hat{g}_a(\lambda).$$

Note that if $\hat{h}(\lambda)=e^{-\frac{in\pi\lambda}{a\Omega}}\hat{g}_a(\lambda)$ *then*

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{in\pi\lambda}{a\Omega}}\hat{g}_a(\lambda)e^{i\lambda t}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}_a(\lambda)e^{i\lambda[t-\frac{n\pi}{a\Omega}]}d\lambda=g_a(t-\frac{n\pi}{a\Omega}).$$

Hence the result follows.

Since $g_a(t)$ has a factor of t^2 in the denominator, this expression for $f(t)$ converges faster than the expression in the Shannon theorem. Note that the nth term behaves like $1/n^2$, rather than $1/n$.

Exercise 2 *(Filtering with the FFT) Let*

$$f(t) = e^{-\frac{t^2}{10}} \left(\sin 2t + 2\cos 4t + 0.4 \sin t \sin 50t \right).$$

Discretize f by setting $y_k = f(2k\pi/256), k = 1, \cdots, 256$. Use MATLAB's FFT to compute $\hat{y}_k$ for $0 \leq k \leq 256$. (Note that $y_{n-k} = \overline{y_k}$. Thus the low frequency coefficients are $\hat{y}_0, \cdots, \hat{y}_m$ and $\hat{y}_{256-m}, \cdots, \hat{y}_{256}$ for some small integer m. Filter out the high-frequency terms by setting $\hat{y}_k = 0$ for $m \leq k \leq 255 - m$ with $m = 6$. Then apply the inverse FFT to these filtered $\hat{y}_k$ to compute the filtered y_k. Plot the results and compare with the original unfiltered signal. Experiment with several different values of m.

Exercise 3 *(Compression with the FFT) Consider the signal $f(t)$ as given in the previous problem. Let $tol = 1.0$ In the previous problem compress the transformed signal by setting $\hat{y}_k = 0$ whenever $|\hat{y}_k| < tol$. Apply the inverse FFT to the compressed transfirmed signal to get a compressed signal y_k. Plot the results and compare with the original uncompressed signal. Experiment with several different values of tol. Keep track of the percentage of Fourier coefficients that have been filtered out.*

Exercise 4 *Construct the unitary rep $\tilde{\mathbf{T}}_N$ of H_R induced by the one-dimensional rep $\tilde{\mathbf{T}}_0^N(a_1, a_2, y_3) = e^{2\pi i N y_3}$ of the subgroup H^1, where N is an integer (not necessarily positive). Determine the action of H_R on the rep space. Under what conditions on N is $\tilde{\mathbf{T}}_N$ irred?*

Solution Here $\tilde{\mathbf{T}}_N$ is defined on the space V of functions $\mathbf{f}$ on H_R such that $\mathbf{f}(BA) = \mathbf{T}_0(B)\mathbf{f}(A)$ for all $B \in H', A \in H_R$, i.e.,

$$\mathbf{f}(a_1 + x_1,\ a_2 + x_2,\ y_3 + x_3 + a_1 x_2) = e^{2\pi i N y_3} \mathbf{f}(x_1, x_2, x_3). \qquad (1)$$

The operators $\tilde{\mathbf{T}}_N(A), A \in H_R$ act on V according to

$$[\tilde{\mathbf{T}}_N(A)\mathbf{f}](A') = \mathbf{f}(A'A). \qquad (2)$$

We see that for any $A(x_1,x_2,x_3)$ we can always choose $B(a_1,a_2,y_3)$ such that $BA = A'(x_1',x_2',0)$ where $0 \le x_1' < 1, 0 \le x_2' < 1$. Thus $\mathbf{f}$ can be restricted to $X = H'\backslash H_R$ with coordinates $(x_1',x_2',0)$. Moreover, setting $x_3 = 0, y_3 = -a_1x_2$ in we have the periodicity condition

$$\varphi(a_1+x_1,a_2+x_2) = e^{-2\pi i N a_1 x_2}\varphi(x_1,x_2) \tag{3}$$

where $\varphi(x_1,x_2) = \mathbf{f}(x_1,x_2,0)$. Conversely, given φ we can define a unique $\mathbf{f}$ satisfying by

$$\mathbf{f}(x_1,x_2,x_3) = \varphi(x_1,x_2)e^{2\pi i N x_3}.$$

The H_R-invariant inner product on X is dx_1dx_2:

$$\langle\varphi_1,\varphi_2\rangle = \int_0^1\int_0^1 \varphi_1(x_1,x_2)\overline{\varphi_2}(x_1,x_2)dx_1dx_2, \tag{4}$$

and the operator $\tilde{\mathbf{T}}_N[\mathbf{y}] \equiv \tilde{\mathbf{T}}_N(A(y_1,y_2,y_3))$ acts on these functions by

$$(\tilde{\mathbf{T}}_N[\mathbf{y}]\varphi)(x_1,x_2) = \exp[2\pi i N(y_3+x_1y_2)]\varphi(x_1+y_1,x_2+y_2). \tag{5}$$

If $N = 0$, it is obvious that this representation is reducible. Indeed, each exponential $e^{1(m_1x_1+m_2x_2)}$ for m_1,m_2 integers is mapped to a multiple of itself by the group action. Now assume that the integer N is nonzero.

Consider the periodizing operators

$$\begin{array}{ll} \mathbf{P}_j\psi(x_1,x_2) & = \frac{e^{2\pi i j x_2}}{\sqrt{N}}\sum_{n=-\infty}^{\infty}(\mathbf{T}^N[x_1,x_2,0]\psi)(n) \\ & = \frac{1}{\sqrt{N}}\sum_{n=-\infty}^{\infty} e^{2\pi i(j+nNx_2}\psi(n+x_1) \end{array}, \quad j = 0,1,\cdots,|N|-1, \tag{6}$$

which is well defined for any $\psi \in L_2(R)$ which belongs to the Schwartz space. It is straightforward to verify that $\mathbf{f}_j = \mathbf{P}_j\psi$ satisfies the required periodicity condition, hence $\mathbf{f}$ belongs to V. Further $\mathbf{f}_j = e^{2\pi i j x_2}F(x_1,X_2)$ where F is periodic in x_2 with period $1/N$. We say that V_j' is the space of those functions in V' with this form. Also

$$\begin{array}{ll} & \langle\mathbf{P}_j\psi(\cdot,\cdot,0),\mathbf{P}_k\psi'(\cdot,\cdot,0)\rangle \\ = & \frac{1}{N}\int_0^1 dx_1\int_0^1 dx_2\sum_{m,n=-\infty}^{\infty} e^{2\pi i(j-k+N(n-m))x_2}\psi(n+x_1)\overline{\psi'(m+x_1)} \\ = & \int_0^1 dx_1\sum_{n=-\infty}^{\infty}\psi(n+x_1)\overline{\psi'(n+x_1)}\frac{\delta_{jk}}{N} = \int_{-\infty}^{\infty}\psi(t_1)\overline{\psi'}(t)\ dt\frac{\delta_{jk}}{N} \\ = & (\psi,\psi')\frac{\delta_{jk}}{N}, \quad 0 \le j,k \le |N|-1, \end{array}$$

so $\mathbf{P}=\sum_{j=0}^{|N|-1}\mathbf{P}_j$ can be extended to an inner product preserving mapping of $L_2(R)$ into V. That is, $\mathbf{P}$ maps ψ,ψ' to $\sum_j \mathbf{f}_j, \sum_j \mathbf{f}'_j$ respectively, and we have

$$\langle \sum_j \mathbf{f}_j, \sum_k \mathbf{f}'_k\rangle = \sum_j \langle \mathbf{f}_j, \mathbf{f}'_j\rangle = \frac{1}{N}\sum_{j=0}^{|N|-1}(\psi,\psi') = (\psi,\psi').$$

It is clear from that if $\varphi_j(x_1,x_2) = \mathbf{P}_j\psi(x_1,x_2,0)$ then we can recover $\psi(x_1)$ by integrating with respect to x_2 : $\psi(x_1) = \int_0^1 \varphi_j(x_1,y)e^{-2\pi i j x_2}dy$. Thus we define the mapping $\mathbf{P}_j^*$ of V'_j into $L_2(R)$ by

$$\mathbf{P}_j^*\varphi_j(t) = \frac{1}{\sqrt{N}}\int_0^1 \varphi(t,y)e^{-2\pi i j x_2}dy, \quad \varphi\in V'_j. \tag{7}$$

Since $\varphi_j \in V'_j$ we have

$$\mathbf{P}^*\varphi_j(t+a) = \frac{1}{\sqrt{N}}\int_0^1 \varphi(t,y)e^{-2\pi i(j+Na)y}dy = \frac{1}{\sqrt{N}}\hat{\varphi}_{j+Na}(t)$$

for a an integer. (Here $\hat{\varphi}_n(t)$ is the nth Fourier coefficient of $\varphi(t,y)$.) The Parseval formula then yields

$$\int_0^1 |\varphi(t,y)|^2dy = \sum_{j=0}^{|N|-1}\sum_{a=-\infty}^{\infty}|\mathbf{P}_j^*\varphi(t+a)|^2$$

so

$$\begin{aligned}\langle\varphi,\varphi\rangle &= \textstyle\int_0^1\int_0^1 |\varphi(t,y)|^2dtdy = \int_0^1 \sum_{j=0}^{|N|-1}\sum_{a=-\infty}^{\infty}|\mathbf{P}_j^*\varphi(t+a)|^2dt \\ &= \textstyle\sum_{j=0}^{|N|-1}\int_{-\infty}^{\infty}|\mathbf{P}_j^*\varphi(t)|^2dt = \sum_{j=0}^{|N|-1}(\mathbf{P}_j^*\varphi,\mathbf{P}_j^*\varphi).\end{aligned}$$

Thus this is an inner product preserving mapping of $V' = \oplus_j V'_j$ back into $L_2(R)$. Moreover, it is easy to verify that

$$\langle \mathbf{P}_j\psi,\varphi\rangle = (\psi,\mathbf{P}_j^*\varphi)$$

for $\psi\in L_2(R)$, $\varphi\in V'$, i.e., $\mathbf{P}^* = \sum_j \mathbf{P}_j^*$ is the adjoint of $\mathbf{P}\sum_j\mathbf{P}_j$. Since $\mathbf{P}^*\mathbf{P} = \mathbf{e}$ on $L_2(R)$ it follows that $\mathbf{P}$ is a unitary operator mapping $L_2(R)$ **onto** V' and $\mathbf{P}^* = \mathbf{P}^{-1}$ is a unitary operator mapping V' **onto** $L_2(R)$.

Finally,

$$\begin{aligned}(\mathbf{P}_j\mathbf{T}^N[\mathbf{y}]\psi)(\mathbf{x}) &= e^{2\pi i[j+N(x_3+y_3+x_1y_2)]}\textstyle\sum_{n=-\infty}^{\infty}e^{2\pi i n(x_2+y_2)}\psi(n+x_1+y_1) \\ &= (\tilde{\mathbf{T}}[\mathbf{y}]\mathbf{P}_j\psi)(\mathbf{x})\end{aligned}$$

so $\mathbf{P}_j\mathbf{T}^N[\mathbf{y}] = \tilde{\mathbf{T}}_N[\mathbf{y}]\mathbf{P}_j$ and the unitary reps $\mathbf{T}^N$ and $\tilde{\mathbf{T}}_N|V'_j$ are equivalent for each $j = 0, 1, \cdots, |N| - 1$.

Thus we have shown that The representation space V' splits into a direct sum of $|N|$ subspaces V'_j, and restricted to each of these subspaces the representation is irreducible and equivalent to $\mathbf{T}^N$. Note that $\tilde{\mathbf{T}}_N$ is irreducible only if $N = \pm 1$.

Exercise 5 *Suppose $f \in L_2(R)$ such that $f_{\mathbf{P}} \neq 0$ almost everywhere. Prove that the set $\{e^{2\pi i(m_1x_1+m_2x_2)}f_{\mathbf{P}}/|f_{\mathbf{P}}|,\ m_1, m_2 = \pm 1, \pm 2, \cdots\}$ is an ON basis for the lattice Hilbert space V'. Find an explicit expression for the corresponding ON basis of $L_2(R)$ obtained from the mapping $\mathbf{P}^{-1}$.*

Solution: We have that $f_{\mathbf{P}} \neq 0$ a.e., that it is square integrable on the unit square, and that

$$f_{\mathbf{P}}(a_1 + x_1, a_2 + x_2) = e^{-2\pi i a_1 x_2} f_{\mathbf{P}}(x_1, x_2),$$

Moreover,

$$f_{\mathbf{P}}(x_1, x_2) = \sum_{k=-\infty}^{\infty} e^{2\pi i k x_2} f(x_1 + k).$$

Note also that

$$< f, g >= \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = (f_{\mathbf{P}}, g_{\mathbf{P}}) = \int_0^1 \int_0^1 f_{\mathbf{P}}\overline{g_{\mathbf{P}}}d_1 dx_2$$

and

$$\mathbf{T}^1[y_1, y_2]f_{\mathbf{P}}(x_1, x_2) = e^{2\pi i x_1 y_2} f_{\mathbf{P}}(x - 1 + y_1, x_2 + y_2),$$
$$\mathbf{T}^1[y_1, y_2]f(t) = e^{2\pi i t y_2} f(t + y_1).$$

Now let $g_{\mathbf{P}} \in V'$. We want to expand $g_{\mathbf{P}}$ in the form

$$g_{\mathbf{P}}(x_1, x_2) = \sum_{m_1, m_2} c_{m_1 m_2} e^{2\pi i(m_1 x_1 + m_2 x_2)} f_{\mathbf{P}}/|f_{\mathbf{P}}|(x_1, x_2).$$

Note that the set

$$E_{m_1 m_2}(x_1, x_2) = e^{2\pi i(m_1 x_1 + m_2 x_2)} f_{\mathbf{P}}/|f_{\mathbf{P}}|$$

is ON on V'. Indeed

$$(E_{m_1 m_2}, E_{n_1 n_2}) = \int_0^1 \int_0^1 e^{2\pi i[(m_1 - n_1)x_1 + (m_2 - n_2)x_2]} dx_1 dx_2 = \delta_{m_1 n_1}\delta_{m_2 n_2}.$$

To show that this set is a basis, note that since $g_{\mathbf{P}}\overline{f_{\mathbf{P}}i}/|f_{\mathbf{P}}|$ is square integrable, our problem is equivalent to the expansion

$$g_{\mathbf{P}}\frac{\overline{f_{\mathbf{P}}}}{|f_{\mathbf{P}}|}(x_1,x_2) = \sum_{m_1,m_2} c_{m_1m_2}e^{2\pi i(m_1x_1+m_2x_2)}$$

which we know exists. Indeed

$$c_{n_1n_2} = (g_{\mathbf{P}}\frac{\overline{f_{\mathbf{P}}}}{|f_{\mathbf{P}}|}, e^{2\pi i(n_1x_1+n_2x_2)}) = (g_{\mathbf{P}}, E_{n_1n_2}).$$

Next, note that $\frac{\overline{f_{\mathbf{P}}}}{|f_{\mathbf{P}}|}(x_1,x_2)$ satisfies the twisted periodicity property, so that it belongs to V'. Hence there is a square integrable function $\tilde{f}(t)$ such that

$$\int_0^1 \frac{f_{\mathbf{P}}}{|f_{\mathbf{P}}|}(t,y)dy = \tilde{f}(t), \qquad \mathbf{P}\tilde{f} = \frac{f_{\mathbf{P}}}{|f_{\mathbf{P}}|}.$$

Moreover, for any $g \in L^2(R)$ we have

$$\mathbf{T}^1[-m_2,m_1]g_{\mathbf{P}} = e^{2\pi i(m_1x_1+m_2x_2}g_{\mathbf{P}}, \quad \mathbf{T}^1[-m_2,m_1]g(t) = e^{2\pi i m_1 t}g(t-m_2).$$

Therefore, our ON basis in $L^2(R)$ is

$$f_{m_1m_2}(t) = e^{2\pi i m_1 t}\tilde{f}(t-m_2), \qquad \tilde{f}(t) = \int_0^1 \frac{f_{\mathbf{P}}}{|f_{\mathbf{P}}|}(t,y)dy.$$

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The following MATLAB routines should be useful:

```
function fc=compress( f, r)
% Input the vector f and ratio r: 0<= r <=1.
% The output is the vector fc in which the smallest
% 100r% of the terms f_k, in absolute value, are set
% equal to zero.
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
N=length(f); Nr=floor(N*r);
ff=sort(abs(w));
tol=abs(ww(Nr+1);
fc=(abs(w)>=tol).*w;
```

You can discretize the interval $[0, 2\pi]$ and read in the signal as a vector by using the commands

```
t=linspace(0,2*pi,2^8);
f=exp(-t.^2/10).*(sin(2*t)+2*cos(4*t)+0.4*sin(t).*sin(50*t));
```

If $hatf$ is the FFT of f, you can filter out high frequency components from$hatf$ with a command such as

```
filterhatf=[ hatf(1: m) zeros(1, 2^8-2*m) hatf(2^8-m+1:2^8)]
```

```
function L2error =fftcomp(t,f,r)
% Input: time vector t, signal vector f, compression rate r, (between
% 0 and 1)
Output: graph of f, graph of the compression of f, and the relative L2
error
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
hatf=fft(f);
hatfc=compress(hatf,r);
fc=ifft(hatfc);
plot(t,f,t,fc)
L2error=norm(f-fc,2)/norm(f)
```