Homework Problem Set #1

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Exercise 1 *For $n > 0$, define the functions $f_n \in L^2[0,\infty]$ by*

$$f_n(t) = \begin{cases} \sqrt{n}, & \text{for } n \le t \le \ n + \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

1. *Compute $||f_n - f_m||$. Does the sequence $\{f_n\}$ converge in the L^2 norm?*
2. *Show that $f_n(t)$ converges pointwise in $[0,\infty)$ and find the limit.*
3. *Does the sequence converge pointwise uniformly? Justify your answer.*
4. *Show that $\{f_n\}$ is ON. Is it a basis?*

Exercise 2 *Does the sequence*

$$f_n(x) = \frac{x}{1+nx^2}$$

for $n = 1, 2, \cdots$ converge uniformly on the real line? Justify your answer.

Exercise 3 *For $n > 0$, let*

$$f_n(t) = \begin{cases} 0, & \text{for } -\pi \le t \le \ 0 \\ nt, & \text{for } 0 \le t \le \frac{1}{n} \\ 1, & \frac{1}{n} \le t \le \pi. \end{cases}$$

This sequence belongs to $C[-\pi,\pi]$, i.e., the space of continuous real-valued continuous functions on the interval $[-\pi,\pi]$ with the usual inner product and norm.

1. *Show that* $f_n \to \chi_{(0,\pi]}$ *in the* L^2 *norm, where*

$$\chi_{(0,\pi]}(t) = \begin{cases} 0, & \text{for } -\pi \le t \le 0 \\ 1, & \text{for } 0 < t \le \pi, \end{cases}$$

so that $\{f_n\}$ *is Cauchy in* L^2.

2. *Show that* $||\chi - h|| > 0$ *for every* $h \in C[-\pi, \pi]$. *Conclude that* $C[-\pi, \pi]$ *is not a Hilbert space.*

Exercise 4 *Suppose the function* $\phi(t) \in L^2[-\infty, \infty]$ *satisfies* $\int_{-\infty}^{\infty} \phi(t)\overline{\phi(t-k)}dt = \delta_{0,k}$, *i.e., the integral equals 1 for* $k = 0$ *and vanishes for* $k = 1, 2, \cdots$. *Show that for any fixed integer* j *the functions* $\phi_{jk}(t) = 2^{j/2}w(2^j t - k)$, $(k = 0, \pm 1, \pm 2, \cdots)$ *form an ON set.*

Exercise 5 *Project the function* $f(t) = t^2$ *onto the subspace of* $L^2[0,1]$ *spanned by the functions* $\phi(t), \psi(t), \psi(2t), \psi(2t-1)$, *where*

$$\phi(t) = \begin{cases} 1, & \text{for } 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases} \qquad \psi(t) = \begin{cases} 1, & \text{for } 0 \le t < \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(This is related to the Haar wavelet expansion for f.*)*

Exercise 6 *Let* $L^2[-\infty, \infty, \omega(t)]$ *be the space of square integrable functions on the real line, with respect to the weight function* $\omega(t) = e^{-t^2}$. *The inner product on this space is thus*

$$(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\omega(t)dt.$$

The Hermite polynomials $H_n(t)$, $n = 0, 1, \cdots$ *are the ON set of polynomials on* $L^2[-\infty, \infty, \omega(t)]$, *obtained by applying the Gram-Schmidt process to the monomials* $1, t, t^2, t^3, \cdots$ *and defined uniquely by the requirement that the coefficient of* t^n *in* $H_n(t)$ *is positive. (In fact they form an ON basis for* $L^2[-\infty, \infty, \omega(t)]$.*) Compute the first 4 of these polynomials. NOTE: Later we will show that* $\int_{-\infty}^{\infty} e^{-ist}e^{-t^2}dt = \sqrt{\pi}e^{-s^2/4}$. *You can use this result, if you wish, to simplify the calculations.*

Exercise 7 *Note that in the last problem, $H_0(t), H_2(t)$ contained only even powers of t and $H_1(t), H_3(t)$ contained only odd powers. Can you find a simple proof, using only the uniqueness of the Gram-Schmidt process, of the fact that $H_n(-t) = (-1)^n H_n(t)$ for all n?*

Exercise 8 *Find the $L^2[-\pi, \pi]$ projection of the function $f_1(t) = |t|$ onto the $(2n+1)$-dimensional subspace spanned by the ON set*

$$\{\frac{1}{\sqrt{2\pi}}, \frac{\cos kt}{\sqrt{\pi}}, \frac{\sin kt}{\sqrt{\pi}} : \; k = 1, \cdots, n\}$$

for $n = 1$. Repeat for $n = 2, 3$. Plot these projections along with f_1. (You can use MATLAB, a computer algebra system, a calculator, etc.) repeat the whole exercise for $f_2(t) = t$. Do you see any marked differences between the graphs in the two cases?

Exercise 9 *Use least squares to fit a straight line of the form $y = bx + c$ to the data*

$$\begin{array}{rcllll} x & = & 0 & 1 & 3 & 4 \\ y & = & 0 & 8 & 8 & 20 \end{array}$$

in order to estimate the value of y when $x = 2.0$.

Exercise 10 *Repeat the above problem to find the best least squares fit of the data to a parabola of the form $y = ax^2 + bx + c$. Again, estimate the value of y when $x = 2.0$.*